

II.10. Electron-Phonon Interaction

Here we first review the concept of non-interaction phonons, and then introduce the diagrammatic representations for phonons, followed by descriptions of their vertex contributions to the electron-phonon interaction. Finally, the formalism for finite-temperature contributions is given in the end of this section.

[The non-interacting phonon system]

To understand how electron-phonon interaction comes about, we first consider the occurrence of phonon modes in solids. Generally speaking, there are longitudinal and transverse modes of phonons; only the longitudinal modes provides the changes in density that leads to the Coulomb interaction between electrons and the ionic background, and are therefore the only relevant modes in our consideration. For this reason, in our discussion of electron-phonon interaction, we may simplify the model by approximating the ionic background by a homogeneous and isotropic elastic medium, and assuming that the background has no shear strength so that it is entirely determined by the adiabatic bulk modulus:

$$B \equiv -\Omega \left(\frac{\partial P}{\partial \Omega} \right)_S, \quad (\text{II.593})$$

where P and Ω denote the pressure and volume, respectively, and the subscript S refers to constant entropy. In this sense, the ionic background is treated as if it were a uniform fluid.

Based on the aforementioned simplifications, we are ready to provide a complete description of the longitudinal phonons. We first introduce the *displacement vector* $\mathbf{d}(\mathbf{x})$ of the medium that characterizes the displacement of each point from its equilibrium position. The change in volume under deformation to lowest order can be expressed in terms of the displacement vector as follows:

$$d\Omega' = dx'dy'dz' = dx dy dz \left(1 + \frac{\partial d_x}{\partial x} + \frac{\partial d_y}{\partial y} + \frac{\partial d_z}{\partial z} + \dots \right) \approx d\Omega (1 + \nabla \cdot \mathbf{d}).$$

Therefore, the change in density becomes

$$\frac{\delta \rho}{\rho_0} = \frac{\delta n}{n_0} = -\nabla \cdot \mathbf{d}, \quad (\text{II.594})$$

where ρ_0 (n_0) denotes the equilibrium mass density (particle density). Moreover, in an elastic medium free of shear strength and vorticity, the displacement field satisfies the general condition

$$\oint_{\text{any path}} \mathbf{d} \cdot d\mathbf{l} = 0 \quad \Leftrightarrow \quad \nabla \times \mathbf{d} = 0. \quad (\text{II.595})$$

The Lagrangian of the elastic medium becomes

$$\mathcal{L} = \frac{1}{2} \int d^3x \left[\rho_0 \left(\frac{\partial d_i}{\partial t} \right) \left(\frac{\partial d_i}{\partial t} \right) - B \left(\frac{\partial d_i}{\partial x_j} \right) \left(\frac{\partial d_i}{\partial x_j} \right) \right]. \quad (\text{II.596})$$

From the Euler-Lagrangian equations and EQ. (II.596) we obtain the equation of motion

$$\frac{1}{u_0^2} \frac{\partial^2 \mathbf{d}}{\partial t^2} - \nabla^2 \mathbf{d} = 0, \quad (\text{II.597})$$

which is a wave equation and u_0 is the sound velocity given by $u_0 \equiv (B/\rho_0)^{1/2}$. Also, the corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int d^3x \left[\rho_0^{-1} \boldsymbol{\pi}^2 + B(\boldsymbol{\nabla} \cdot \mathbf{d})^2 \right], \quad \text{where } \boldsymbol{\pi}(\mathbf{x}, t) = \rho_0 \frac{\partial \mathbf{d}}{\partial t}. \quad (\text{II.598})$$

Introducing the normal mode expansions that satisfies the condition $\boldsymbol{\nabla} \times \mathbf{d} = 0$, we have

$$\mathbf{d}(\mathbf{x}, t) = -i \sum_{\mathbf{k}} \left(\frac{1}{2\rho_0 \omega_{\mathbf{k}} \Omega} \right)^{1/2} \frac{\mathbf{k}}{|\mathbf{k}|} \left(b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} - b_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x} + i\omega t} \right), \quad (\text{II.599})$$

$$\boldsymbol{\pi}(\mathbf{x}, t) = - \sum_{\mathbf{k}} \left(\frac{\rho_0 \omega_{\mathbf{k}}}{2\Omega} \right)^{1/2} \frac{\mathbf{k}}{|\mathbf{k}|} \left(b_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} + b_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{x} + i\omega t} \right), \quad (\text{II.600})$$

where $[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$, and $\omega_{\mathbf{k}} = u_0 k$.

Substituting EQs. (II.599) and (II.600) into EQ. (II.598), we obtain

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \frac{1}{2} \right), \quad (\text{II.601})$$

which represents a system of uncoupled harmonic oscillators.

The Hamiltonian given in EQ. (II.601) provides the basis for investigating the thermodynamics and statistical mechanics of the free phonon system. For instance, the total energy E of the system is given by

$$E = \sum_{\mathbf{k}} \left(n_{\mathbf{k}} + \frac{1}{2} \right) \omega_{\mathbf{k}} = \sum_{\mathbf{k}} \left(\frac{1}{\exp(\beta \omega_{\mathbf{k}}) - 1} + \frac{1}{2} \right) \omega_{\mathbf{k}}, \quad (\text{II.602})$$

where $\beta = T^{-1}$. The momentum sum can be converted into energy integration by the following relation:

$$\begin{aligned} \sum_{\mathbf{k}} &\Rightarrow \frac{\Omega}{(2\pi)^3} \int d^3k = \frac{\Omega}{(2\pi)^3} \int dk 4\pi k^2 = \frac{\Omega}{(2\pi)^3} \int d\left(\frac{\omega}{u_0}\right) 4\pi \left(\frac{\omega}{u_0}\right)^2 = \frac{\Omega}{2\pi^2 u_0^3} \int d\omega \omega^2 \equiv \int d\omega g(\omega), \\ \Rightarrow g(\omega) &= \frac{\Omega}{2\pi^2 u_0^3} \omega^2 \quad \text{and} \quad \omega = \omega_{\mathbf{k}} = u_0 k. \end{aligned} \quad (\text{II.603})$$

In a uniform medium there is no upper limit for the frequency. On the other hand, for a real crystal the wavenumber of propagation cannot exceed the reciprocal lattice constant. Therefore, we may define the upper bound for the phonon frequency as ω_D , the Debye frequency, which satisfies the following relation

$$3N = \int_0^{\omega_D} g(\omega) d\omega = \int_0^{\omega_D} d\omega \frac{\omega^2 \Omega}{2\pi^2 u_0^3} = \frac{\omega_D^3 \Omega}{6\pi^2 u_0^3} \Rightarrow g(\omega) d\omega = \frac{9N \omega^2}{\omega_D^3} d\omega, \quad (\text{II.604})$$

because the total number of degrees of freedom in a crystal of N ions is $3N$. We may further define the Debye temperature by $\Theta_D \equiv (\hbar \omega_D / k_B)$ if we restore \hbar and k_B , and apply EQ. (II.603) to EQ. (II.602), so that the energy associated with phonons becomes

$$E = 9Nk_B T \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} du \left(\frac{u^3}{e^u - 1} \right) + \frac{9Nk_B \Theta_D}{8}, \quad (\text{II.605})$$

where $u \equiv (\hbar\omega/k_B T)$ is a dimensionless variable, and we note that the chemical potential for the phonon system is zero. Thus, the specific heat is given by

$$C_v = \frac{\partial E}{\partial T} = 9Nk_B \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} du \frac{u^4 e^u}{(e^u - 1)^2}. \quad (\text{II.606})$$

Equation (II.606) is the Debye theory of the specific heat, which has two limiting values:

$$C_v = 9Nk_B \left(\frac{T}{\Theta_D} \right)^3 \int_0^\infty du \frac{u^4 e^u}{(e^u - 1)^2} = \frac{12\pi^4}{5} Nk_B \left(\frac{T}{\Theta_D} \right)^3, \quad T \rightarrow 0; \quad (\text{II.607})$$

$$C_v = 3Nk_B, \quad T \rightarrow \infty. \quad (\text{II.608})$$

The low-temperature limit of the specific heat is known as the Debye T^3 -law, and the high-temperature limit is consistent with the classical description of equipartition of energy. The Debye theory of the specific heat provides an excellent one-parameter (the Debye temperature) description for the specific heat of metals, and therefore one can obtain the values of the Debye temperature for different metals by fitting their specific data.

Having discussed the basic properties of non-interacting phonons, we want to develop field-theory description for the phonon propagator and electron-phonon interaction. We first define the phonon field operator by the following expression:

$$\varphi(x) = i \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2\Omega} \right)^{1/2} \left(b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}}t} - b_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_{\mathbf{k}}t} \right). \quad (\text{II.609})$$

Restricting the above expression to longitudinal phonons in the Debye model so that the sum over \mathbf{k} is limited to $|\mathbf{k}| < k_D = \omega_D/c$, the Debye wavelength, and noting that the phonon fields are real, we define the phonon propagator $D(x, x')$ as:

$$D(x, x') \equiv -i \langle T [\varphi(x) \varphi(x')] \rangle. \quad (\text{II.610})$$

Substituting the free-field operator EQ. (II.609) into EQ. (II.610) and noting that there are no phonons in the ground state, we obtain the free phonon propagator:

$$\begin{aligned} D^{(0)}(x) &= -\frac{i}{\Omega} \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2} \right) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}}t} \theta(\omega_D - \omega_{\mathbf{k}}) \quad \text{for } t > 0, \\ &= -\frac{i}{\Omega} \sum_{\mathbf{k}} \left(\frac{\omega_{\mathbf{k}}}{2} \right) e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_{\mathbf{k}}t} \theta(\omega_D - \omega_{\mathbf{k}}) \quad \text{for } t < 0. \end{aligned} \quad (\text{II.611})$$

Therefore, the Fourier transformation of EQ. (II.611) becomes

$$D^{(0)}(\mathbf{k}, \omega) = \frac{\omega_{\mathbf{k}}}{2} \left[\frac{1}{\omega - \omega_{\mathbf{k}} + i\delta} - \frac{1}{\omega + \omega_{\mathbf{k}} - i\delta} \right] \theta(\omega_D - \omega_{\mathbf{k}}) = \frac{\omega_{\mathbf{k}}^2}{\omega^2 - \omega_{\mathbf{k}}^2 + i\delta} \theta(\omega_D - \omega_{\mathbf{k}}). \quad (\text{II.612})$$

Diagrammatically, the free phonon propagator, or equivalently the phonon Green's function, is expressed by a thin dashed line. Similar to the fermion Green's functions, we may also define the retarded and advanced phonon propagators in the coordinate representation as follows:

$$D_R(x, x') \equiv -i\theta(t-t')\langle[\varphi(x), \varphi(x')]\rangle, \quad (\text{II.613})$$

$$D_A(x, x') \equiv i\theta(t'-t)\langle[\varphi(x), \varphi(x')]\rangle. \quad (\text{II.614})$$

[Electron-phonon interaction]

As discussed earlier, the charge polarization associated with longitudinal phonons can induce coupling to electrons. For electron charge density $\rho_{el}(\mathbf{x})$, the corresponding electron-phonon interaction Hamiltonian is given by:

$$\mathcal{H}_{el-ph} = \int d^3\mathbf{x} d^3\mathbf{x}' \frac{\rho_{el}(\mathbf{x}) \delta\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (\text{II.615})$$

where $\delta\rho = -zen_0\nabla \cdot \mathbf{d}$ according to EQ. (II.594), and z denotes the valence of the ions in the crystal. Using EQ. (II.599) and expressing the electron charge density in terms of fermion field operators

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}\lambda} \frac{1}{\Omega^{1/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \eta_\lambda a_{\mathbf{k}\lambda}, \quad (\text{II.616})$$

we rewrite EQ. (II.615) into the following:

$$\begin{aligned} \mathcal{H}_{el-ph} &= \frac{zn_0}{u_0} \sum_{\mathbf{k}\lambda} \sum_{\mathbf{q}} \left(\frac{\omega_{\mathbf{k}}}{2\rho_0\Omega} \right)^{1/2} \frac{4\pi e^2}{q^2} \theta(\omega_D - \omega_{\mathbf{q}}) (a_{\mathbf{k}+\mathbf{q},\lambda}^\dagger a_{\mathbf{k},\lambda} b_{\mathbf{q}} + a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k}+\mathbf{q},\lambda} b_{\mathbf{q}}^\dagger) \\ &= \frac{zn_0}{u_0} \sum_{\mathbf{k}\lambda} \sum_{\mathbf{q}} \left(\frac{\omega_{\mathbf{k}}}{2\rho_0\Omega} \right)^{1/2} U_0(\mathbf{q}) \theta(\omega_D - \omega_{\mathbf{q}}) (a_{\mathbf{k}+\mathbf{q},\lambda}^\dagger a_{\mathbf{k},\lambda} b_{\mathbf{q}} + a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k}+\mathbf{q},\lambda} b_{\mathbf{q}}^\dagger), \end{aligned} \quad (\text{II.617})$$

where $U_0(\mathbf{q})$ denotes the bare Coulomb potential. However, our previous analysis of the degenerate electron gas reveals that the bare Coulomb interaction becomes modified by summing over the ring diagrams due to many-body interaction in the electron gas. Hence, we may replace the bare Coulomb interaction potential $U_0(\mathbf{q})$ by the effective static Coulomb interaction in the Thomas-Fermi approximation, $U_r(\mathbf{q})$, which yields

$$U_r(\mathbf{q}) = \frac{4\pi e^2}{|\mathbf{q}|^2 + q_{TF}^2} \approx \frac{4\pi e^2}{q_{TF}^2} \quad \text{because } |\mathbf{q}| \ll k_F \text{ and } q_{TF}^2 \equiv \frac{4k_F}{\pi a_0}. \quad (\text{II.618})$$

Hence, we find

$$\begin{aligned} \mathcal{H}_{el-ph} &= \frac{4\pi z e^2 n_0}{u_0 (\rho_0)^{1/2} q_{TF}^2} \int d^3\mathbf{x} \psi_\alpha^\dagger(\mathbf{x}) \psi_\alpha(\mathbf{x}) \varphi(\mathbf{x}) \\ &\equiv \gamma \int d^3\mathbf{x} \psi_\alpha^\dagger(\mathbf{x}) \psi_\alpha(\mathbf{x}) \varphi(\mathbf{x}), \end{aligned} \quad (\text{II.619})$$

where

$$\gamma \equiv \frac{4\pi z e^2 n_0}{u_0 (\rho_0)^{1/2} q_{TF}^2} = \frac{z\pi^2 n_0}{mk_F B^{1/2}} \quad (\text{II.620})$$

is the electron-phonon coupling constant. It is worth noting that γ is dependent on the electron mass and the bulk modulus, but is independent of the ion mass.

Diagrammatically, the effect of electron-phonon interaction may be treated in a way similar to the results derived previously for Coulomb interactions among fermions if we simply replace the wavy lines for

Coulomb interactions with dashed lines for the phonon propagators. Therefore, each vertex now represents the occurrence of electron-phonon interaction, as shown in Fig. II.10.1.

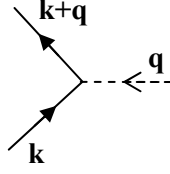


Fig. II.10.1 Illustration of the basic electron-phonon vertex, where solid lines represent the electron propagators and the dashed line indicates the phonon propagator.

Specifically, the electron-electron interaction potential $V(x_1 - x_2)$ mediated by the electron-phonon interaction can be expressed by the following:

$$V(x_1 - x_2) \Rightarrow \gamma^2 D^{(0)}(x_1 - x_2), \quad (\text{II.621})$$

and the corresponding diagram is shown in Fig. II.10.2.

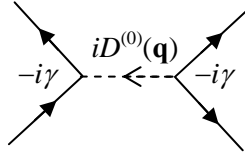


Fig. II.10.2 Illustration of the electron-electron interaction mediated by the electron-phonon interaction with a coupling coefficient γ .

Given the electron-phonon coupling, we consider the correction to the non-interacting phonons. The first non-vanishing correction occurs in the second order with respect to the interaction \mathcal{H}_{el-ph} , as shown in Fig. II.10.3 (a) – (b), and the corresponding expressions are

$$-\gamma^2 i \int d^4 x_1 d^4 x_2 D^{(0)}(x - x_1) G_{\alpha\beta}^{(0)}(x_1 - x_2) G_{\beta\alpha}^{(0)}(x_2 - x_1) D^{(0)}(x_2 - x') \quad \text{for (a),} \quad (\text{II.622})$$

$$\text{and} \quad +\gamma^2 i \int d^4 x_1 d^4 x_2 D^{(0)}(x - x_1) G_{\alpha\alpha}^{(0)}(0) G_{\beta\beta}^{(0)}(0) D^{(0)}(x_2 - x') \quad \text{for (b).} \quad (\text{II.623})$$

However, the disconnected graph in Fig. II.10.3 in fact does not contribute to the electron-electron interaction potential, which may be understood by the following simple consideration. If we inspect in EQ. (II.623) the integration of $D^{(0)}(x - x_1)$ over x_1 , we find that

$$D^{(0)}(x - x_1) \sim \langle T[\varphi(x) \nabla_{\mathbf{x}_1} \cdot \mathbf{d}(x_1)] \rangle \sim \nabla_{\mathbf{x}_1} \cdot \langle T[\varphi(x) \mathbf{d}(x_1)] \rangle,$$

so that

$$\int d^3 x_1 D^{(0)}(x - x_1) \sim \int d^3 x_1 \nabla_{\mathbf{x}_1} \cdot \langle T[\varphi(x) \mathbf{d}(x_1)] \rangle \sim \oint d^2 x_1 \hat{\mathbf{n}} \cdot \langle T[\varphi(x) \mathbf{d}(x_1)] \rangle \rightarrow 0.$$



Fig. II.10.3 Second-order corrections to the phonon propagator: **(a)** connected polarization correction; **(b)** disconnected correction, which can be shown to vanish identically.

We summarize in the following the general rules used to calculate the corrections of order $2n$ (with respect to \mathcal{H}_{el-ph}) to the electron and phonon Green's functions:

- 1) Form all connected, topologically non-equivalent diagrams with $2n$ vertices;
- 2) With each solid line associate with a free-particle Green's function $G_{\alpha\beta}^{(0)}(x-x')$, and with each dashed line associate a function $D^{(0)}(x-x')$;
- 3) Integrate over the coordinates of all vertices and sum over the corresponding spin variables;
- 4) Multiply the resulting expression by $\gamma^{2n} i^n (-1)^F$, where F is the number of closed loops formed by the fermion $G^{(0)}$ -lines.

As an example, consider the diagram in Fig. II.10.4. Using the rules given above, we obtain the corresponding expression for the diagram as follows:

$$\begin{aligned} & \gamma^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 D^{(0)}(x-x_1) G_{\gamma_1\gamma_2}^{(0)}(x_1-x_2) D^{(0)}(x_2-x_3) G_{\gamma_2\gamma_4}^{(0)}(x_2-x_4) \\ & \times G_{\gamma_4\gamma_3}^{(0)}(x_4-x_3) G_{\gamma_3\gamma_1}^{(0)}(x_3-x_1) D^{(0)}(x_4-x'). \end{aligned} \quad (\text{II.624})$$

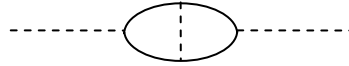


Fig. II.10.4 A fourth order correction to the phonon propagator.

Next, we consider the equivalent electron-electron interaction due to electron-phonon coupling in the momentum space. From EQ. (II.612), the Fourier transform of the interaction potential is

$$V(\mathbf{q}, \omega) = \gamma^2 \frac{\omega_{\mathbf{q}}^2}{\omega^2 - \omega_{\mathbf{q}}^2 + i\delta} \theta(\omega_D - \omega_{\mathbf{q}}). \quad (\text{II.625})$$

In the static limit, the equivalent electron-electron interaction potential becomes $V(\mathbf{q}, 0) = -\gamma^2 \theta(\omega_D - \omega_{\mathbf{q}})$. If we assume that the static potential holds for all \mathbf{q} , then the equivalent interaction potential becomes an *attractive* delta function:

$$V(\mathbf{x}) = -\gamma^2 \delta(\mathbf{x}). \quad (\text{II.626})$$

On the other hand, for energy transfer greater than all of the relevant phonon modes, $\omega > \omega_D$, we find from EQ. (II.625) that

$$V(\mathbf{q}, \omega > \omega_D) > 0, \quad (\text{II.627})$$

implying that the interaction is always repulsive. Therefore, the equivalent interaction potential between two electrons can only be attractive if both electrons lie within an energy shell ω_D below the Fermi surface and become excited to unoccupied states within an energy shell ω_D above the Fermi surface. (*N.B.!* for most metals $\omega_D \sim 10^2$ meV and the Fermi energy $\varepsilon_F \sim 1$ eV so that $\omega_D \ll \varepsilon_F$). This attractive interaction between particles near the Fermi surface as a consequence of electron-phonon interaction has important implications on the occurrence of superconductivity in metallic systems.

Finally, we summarize the general rules used to calculate the corrections of order $2n$ (with respect to \mathcal{H}_{el-ph}) to the electron and phonon Green's functions in momentum space:

- 1) Form all connected, topologically non-equivalent diagrams with $2n$ vertices;
- 2) With each solid line associate with a free-particle Green's function

$$G^{(0)}(k) = \lim_{\delta \rightarrow 0^+} \frac{1}{\omega - (\varepsilon_k - \mu) + i\delta \operatorname{sgn}(\varepsilon_k - \mu)},$$

and with each dashed line associate a function

$$D^{(0)}(q) = \lim_{\delta \rightarrow 0^+} \frac{\omega_{\mathbf{q}}^2}{\omega^2 - \omega_{\mathbf{q}}^2 + i\delta};$$

- 3) Integrate over n independent momenta;
- 4) Multiply the resulting expression by $\gamma^{2n} (2\pi)^{-4n} i^n (-1)^F$, where F is the number of closed loops formed by the fermion $G^{(0)}$ -lines.

[Dyson's equation and the vertex part]

Given the electron-phonon interaction as corrections to the electron and phonon Green's functions, we can express these corrections diagrammatically with Dyson's equation.

In the case of electron Green's functions, the Dyson's equation for the total Green's function G with a proper self-energy Σ^* correction is

$$G = G^{(0)} + G^{(0)} \Sigma^* G \quad \Leftrightarrow \quad G^{-1} = (G^{(0)})^{-1} - \Sigma^*. \quad (\text{II.628})$$

The simplest diagram for the self-energy correction due to electron-phonon coupling is given by the diagram in Fig. II.10.5 (a), which is the skeleton diagram for the generalized diagram of Σ^* expressed by the graph in Fig. II.10.5 (b). The shaded area on the right of the graph in Fig. II.10.5 (b) represents all diagrams with three external points, one associated with a phonon and two with electrons. This shaded triangle is the vertex part, and is denoted by $\Gamma(k, k-q; q)$. Thus, the proper self-energy can be explicitly given by the vertex part and the electron and phonon Green's functions:

$$\Sigma^*(k) = i\gamma \int \frac{d^4q}{(2\pi)^4} G(k-q) D(q) \Gamma(k-q, k; q), \quad (\text{II.629})$$

where we have set $G_{\alpha\beta} = G\delta_{\alpha\beta}$.



Fig. II.10.5 Corrections to the proper self-energy of the electron Green's function due to electron-phonon coupling: **(a)** the skeleton diagram for the lowest order correction; **(b)** the generalized diagram consisting of the vertex part.

Substituting EQ. (II.629) into EQ. (II.628), we obtain the Dyson's equation for the electron Green's function:

$$[\omega - (\varepsilon_k - \mu)]G(k) - i \int \frac{d^4q}{(2\pi)^4} G(k-q) D(q) \Gamma(k-q, k; q) G(k) = 1. \quad (\text{II.630})$$

Similarly, the proper self-energy part of the phonon propagator due to electron-phonon coupling is associated with the polarization insertion, and is denoted by Π^* . The corresponding skeleton diagram and the generalized diagram containing the vertex part are shown in Fig. II.10.6 (a) – (b), and the Dyson's equation for the phonon Green's function is given by

$$D = D^{(0)} + D^{(0)}\Pi^*D \Leftrightarrow D^{-1} = (D^{(0)})^{-1} - \Pi^*. \quad (\text{II.631})$$

From Fig. II.10.6 (b), the proper self-energy of phonon propagator is given by

$$\Pi^*(k) = -2i \int \frac{d^4q}{(2\pi)^4} G(q) G(q-k) \Gamma(q, q-k; k), \quad (\text{II.632})$$

where the factor 2 takes care of the spin degeneracies of electrons. Inserting EQ. (II.632) into EQ. (II.631), we obtain an explicit form for the Dyson's equation of phonons:

$$\omega_k^{-2} (\omega^2 - \omega_k^2) D(k) + 2i \int \frac{d^4q}{(2\pi)^4} G(q) G(q-k) \Gamma(q, q-k; k) D(k) = 1. \quad (\text{II.633})$$



Fig. II.10.6 Corrections to the proper self-energy of the phonon Green's function due to electron-phonon coupling: **(a)** the skeleton diagram for the lowest order correction; **(b)** the generalized diagram consisting of the vertex part.

In general, Dyson's equation can be obtained directly from the equations of motion for the Heisenberg operators of fermions, which you are asked to prove in Problem Set 3. The final result for a fermion system under an interaction Hamiltonian \mathcal{H}_{int} is:

$$\left[i \frac{\partial}{\partial t} + \frac{\nabla_x^2}{2m} + \mu \right] G_{\alpha\beta}(x, x') = \delta(x - x') \delta_{\alpha\beta} - i \langle 0 | T \left[\psi_{H\alpha}(x), \mathcal{H}_{\text{int}} \right] \psi_{H\beta}^\dagger(x') | 0 \rangle. \quad (\text{II.634})$$

Taking $\mathcal{H}_{\text{int}} = \mathcal{H}_{el-ph}$ as given in EQ. (II.619), we find that the last term in EQ. (II.634) becomes

$$-i\gamma \langle 0 | T \left[\psi_{H\alpha}(x) \psi_{H\beta}^\dagger(x') \varphi_H(x) \right] | 0 \rangle. \quad (\text{II.635})$$

We may therefore associate the quantity

$$G_{\alpha\beta}(x_1, x_2; x_3) = \langle 0 | T \left[\psi_{H\alpha}(x_1) \psi_{H\beta}^\dagger(x_2) \varphi_H(x_3) \right] | 0 \rangle \equiv \left\langle T \left[\psi_{H\alpha}(x_1) \psi_{H\beta}^\dagger(x_2) \varphi_H(x_3) \right] \right\rangle \quad (\text{II.636})$$

with a set of Feynman diagrams with one external phonon line and two external electron lines. The simplest of these diagrams is shown in Fig. II.10.7 (a), which corresponds to the following quantity

$$-\gamma \delta_{\alpha\beta} \int d^4 y G^{(0)}(x_1 - y) G^{(0)}(y - x_2) D^{(0)}(y - x_3). \quad (\text{II.637})$$

The generalized diagram for EQ. (II.636) is shown in Fig. II.10.7 (b), which corresponds to the following expression:

$$\begin{aligned} G_{\alpha\beta}(x_1, x_2; x_3) &= \delta_{\alpha\beta} G(x_1, x_2; x_3) \\ &= -\delta_{\alpha\beta} \int d^4 x'_1 d^4 x'_2 d^4 x'_3 G(x_1 - x'_1) G(x'_2 - x_2) D(x'_3 - x_3) \Gamma(x'_1, x'_2; x'_3). \end{aligned} \quad (\text{II.638})$$



Fig. II.10.7 The Feynman diagrams associated with electron phonon interaction: **(a)** the first-order perturbation theory approximation; **(b)** the generalized expression containing all diagrams associated with the vertex part.

The vertex part Γ represents the vertex part for electron-phonon interaction, which corresponds to the set of all diagrams with three external points associated with one phonon and two electrons. The Fourier transform of Γ may be written in the form

$$\Gamma(k, k - q; q) (2\pi)^4 \delta(k - k' - q) = \int d^4 x_1 d^4 x_2 d^4 x_3 \Gamma(x_1, x_2; x_3) e^{-ikx_1 + ik'x_2 + iqx_3}. \quad (\text{II.639})$$

The relation between the Fourier components of Γ and G is given by:

$$G(k, k - q; q) = -G(k) G(k - q) D(q) \Gamma(k, k - q; q). \quad (\text{II.640})$$

To calculate the vertex part Γ under electron-phonon interaction, we need to find all compact diagrams and associate the corresponding analytical expressions with them according to the rules prescribed earlier. Keeping in mind that the electron-phonon vertex part consists of three external points associated with one phonon and two electrons, we find that the diagrams consist of contributions from terms of order γ , γ^3 , γ^5 , etc. Some representative examples of such contributions up to order γ^7 are illustrated in Fig. II.10.8, and a complete set of contributions up to order γ^5 are given in Fig. II.10.9.

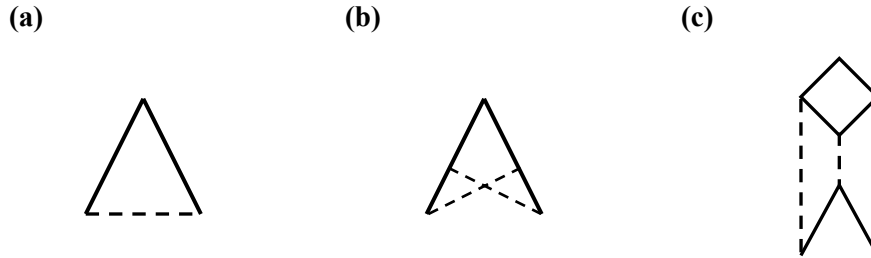


Fig. II.10.8 Representative vertex contributions due to the electron-phonon interaction: **(a)** to order γ^3 ; **(b)** to order γ^5 ; and **(c)** to order γ^7 .

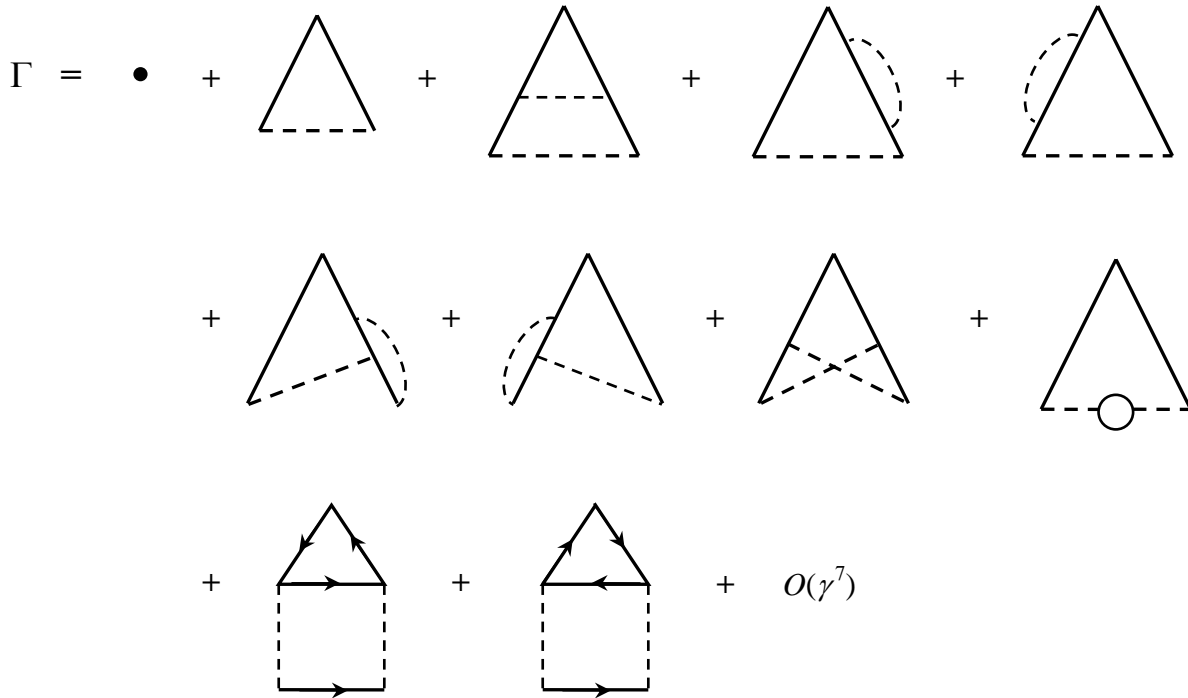


Fig. II.10.9 Vertex contributions due to the electron-phonon interaction, up to order γ^5 .

The vertex contributions exemplified in Figs. II.10.8 and II.10.9 seem tedious and difficult to compute exactly. Interestingly, however, the exact proper vertex can be greatly simplified by a remarkable theorem known as the *Migdal theorem*, which states that the exact vertex in the electron-phonon system satisfies a simple relation:

$$\Gamma = \gamma \left\{ 1 + O \left[\left(\frac{m}{M} \right)^{1/2} \right] \right\}, \quad (\text{II.641})$$

where (m/M) is the ratio of the electron mass to the ion mass. Therefore, EQ. (II.641) allows us to replace the vertex by the point value γ to an excellent approximation. Here we only mention this important result without going through the proof of the theorem. For details of the proof, you may consult the original paper by A. B. Migdal, *Sov. Phys.-JETP* **7**, 996 (1958). In addition, you may refer to Section 47 of the book by Fetter & Walecka for proof to the second-order vertex correction.

The aforementioned discussion suggests that attractive electron-phonon interaction can always occur in metals, which in turn implies (according to the BCS theory, a topic of our later discussion) that superconductivity will always occur at sufficiently low temperatures, *i.e.* for $T < T_c$ where T_c denotes a critical temperature. On the other hand, in the “normal state” of the metal where $T_c \ll T \ll \omega_D$, our derivation of the electron and phonon Green’s functions must be modified to incorporate the finite temperature effect. In the following section we discuss the finite-temperature Green’s functions for electron-phonon interaction.

[Electron-phonon interactions at finite temperatures]

To deal with problems at finite temperatures, we need to introduce temperature Green’s function. As discussed in Part II.7, we may transcribe the Green’s functions using the “imaginary time” trick so that we write $it \Rightarrow \tau$, and τ varies from 0 to T^{-1} . The temperature Green’s function is therefore defined as

$$\begin{aligned} g_{\alpha\beta}(\mathbf{x}_1, \tau_1; \mathbf{x}_2, \tau_2) &= -\text{Tr} \left\{ e^{(\Omega_G + \mu N - \mathcal{H})/T} T_\tau \left[\psi_{H\alpha}(\mathbf{x}_1, \tau_1) \bar{\psi}_{H\beta}(\mathbf{x}_2, \tau_2) \right] \right\}, \\ &= -\left\langle T_\tau \left[\psi_{H\alpha}(\mathbf{x}_1, \tau_1) \bar{\psi}_{H\beta}(\mathbf{x}_2, \tau_2) \right] \right\rangle, \end{aligned} \quad (\text{II.642})$$

where Ω_G is the grand potential, T_τ is the imaginary time ordering operator, and the Heisenberg field operators ψ_H and $\bar{\psi}_H$ for electrons and φ_H for phonons at $T \neq 0$ are given by:

$$\psi_{H\alpha}(\mathbf{x}, \tau) = e^{(\mathcal{H} - \mu N)\tau} \psi_\alpha(\mathbf{x}) e^{-(\mathcal{H} - \mu N)\tau}, \quad (\text{II.643})$$

$$\bar{\psi}_{H\alpha}(\mathbf{x}, \tau) = e^{(\mathcal{H} - \mu N)\tau} \psi_\alpha^\dagger(\mathbf{x}) e^{-(\mathcal{H} - \mu N)\tau}, \quad (\text{II.644})$$

$$\varphi_H(\mathbf{x}, \tau) = e^{\mathcal{H}\tau} \varphi(\mathbf{x}) e^{-\mathcal{H}\tau}. \quad (\text{II.645})$$

We note that the field operators ψ_H and $\bar{\psi}_H$ are no longer Hermitian conjugates of each other. Moreover, given that the temperature Green’s function is a function of the imaginary time difference $\tau_1 - \tau_2 = \tau$, it is defined in the interval from $-1/T$ to $1/T$. On the other hand, from EQ. (II.642) we have

$$g(\tau < 0) = -g \left(0 < \tau + \frac{1}{T} < \frac{1}{T} \right) \quad \text{for fermions,} \quad (\text{II.646})$$

$$= g \left(0 < \tau + \frac{1}{T} < \frac{1}{T} \right) \quad \text{for bosons,} \quad (\text{II.647})$$

and the phonon temperature Green’s function

$$\mathcal{D}(\tau < 0) = \mathcal{D}\left(\tau + \frac{1}{T}\right). \quad (\text{II.648})$$

In addition, noting that the phonon temperature Green's function is real, we have

$$\mathcal{D}(\tau < 0) = \mathcal{D}^*(\tau < 0) = \mathcal{D}(\tau > 0) = \mathcal{D}(-\tau). \quad (\text{II.649})$$

Next, we consider the temperature Green's functions for free particles, which play an important role in the perturbation theory. In the case of free particles, the total Hamiltonian \mathcal{H} is equal to the non-interacting Hamiltonian \mathcal{H}_0 , and the field operators in EQs. (II.643) and (II.644) are given by

$$\psi_\alpha(\mathbf{x}_1) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}_1} a_{\mathbf{k}_1\alpha} e^{i\mathbf{k}_1 \cdot \mathbf{x}_1}, \quad \psi_\beta^\dagger(\mathbf{x}_2) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}_2} a_{\mathbf{k}_2\beta}^\dagger e^{-i\mathbf{k}_2 \cdot \mathbf{x}_2}. \quad (\text{II.650})$$

Consequently, we obtain

$$\begin{aligned} g_{\alpha\beta}^{(0)}(\mathbf{x}_1, \mathbf{x}_2; \tau > 0) &= -\frac{1}{\Omega} \sum_{\mathbf{k}_1, \mathbf{k}_2} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 - \mathbf{k}_2 \cdot \mathbf{x}_2)} \text{Tr} \left\{ e^{(\Omega_{G_0} + \mu N - \mathcal{H}_0)/T} e^{(\mathcal{H}_0 - \mu N)\tau} a_{\mathbf{k}_1\alpha} e^{-(\mathcal{H}_0 - \mu N)\tau} a_{\mathbf{k}_2\beta}^\dagger \right\} \\ &= -\frac{1}{\Omega} \sum_{\mathbf{k}_1, \mathbf{k}_2} e^{i(\mathbf{k}_1 \cdot \mathbf{x}_1 - \mathbf{k}_2 \cdot \mathbf{x}_2) - [\varepsilon_0(\mathbf{k}_1) - \mu]\tau} \text{Tr} \left\{ e^{(\Omega_{G_0} + \mu N - \mathcal{H}_0)/T} a_{\mathbf{k}_1\alpha} a_{\mathbf{k}_2\beta}^\dagger \right\} \\ \Rightarrow g_{\alpha\beta}^{(0)}(\mathbf{x}_1 - \mathbf{x}_2; \tau > 0) &= -\delta_{\alpha\beta} \frac{1}{\Omega} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2) - [\varepsilon_0(\mathbf{k}) - \mu]\tau} \langle a_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}^\dagger \rangle. \end{aligned} \quad (\text{II.651})$$

The quantity $\langle a_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}^\dagger \rangle$ can be expressed in terms of the equilibrium occupation number $n(\mathbf{k})$ at a finite temperature T :

$$\begin{aligned} \langle a_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}^\dagger \rangle &= 1 - n(\mathbf{k}), & n(\mathbf{k}) &= \left\{ e^{[\varepsilon_0(\mathbf{k}) - \mu]/T} + 1 \right\}^{-1} && \text{for fermions,} \\ \langle a_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha}^\dagger \rangle &= 1 + n(\mathbf{k}), & n(\mathbf{k}) &= \left\{ e^{[\varepsilon_0(\mathbf{k}) - \mu]/T} - 1 \right\}^{-1} && \text{for bosons.} \end{aligned} \quad (\text{II.652})$$

Taking the volume Ω to infinity, we may rewrite the free-particle temperature Green's function into:

$$g_{\alpha\beta}^{(0)}(\mathbf{x}; \tau > 0) = -\delta_{\alpha\beta} \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x} - [\varepsilon_0(\mathbf{k}) - \mu]\tau} [1 \mp n(\mathbf{k})], \quad (\text{II.653})$$

with the upper (lower) sign for fermions (bosons). Moreover,

$$\begin{aligned} g_{\alpha\beta}^{(0)}(\mathbf{x}; \tau < 0) &= \mp g_{\alpha\beta}^{(0)}\left(\mathbf{x}; \tau + \frac{1}{T}\right) \\ &= \pm \delta_{\alpha\beta} \frac{1}{(2\pi)^3} \int d^3\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x} - [\varepsilon_0(\mathbf{k}) - \mu]\tau} n(\mathbf{k}). \end{aligned} \quad (\text{II.654})$$

Similarly, the phonon Green's function may be derived from the free-phonon field operator

$$\varphi(\mathbf{x}) = \frac{i}{\sqrt{\Omega}} \sum_{\mathbf{q}} \left(\frac{\omega_{\mathbf{q}}}{2}\right)^{1/2} (b_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} - b_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{x}}), \quad (\text{II.655})$$

so that

$$\mathcal{D}^{(0)}(\mathbf{x}, \tau) = -\frac{1}{2(2\pi)^3} \int d^3\mathbf{q} \left\{ [N(\mathbf{q}) + 1] e^{i\mathbf{q}\cdot\mathbf{x} - \omega_0(\mathbf{q})|\tau|} + N(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x} + \omega_0(\mathbf{q})|\tau|} \right\}, \quad (\text{II.656})$$

where

$$N(\mathbf{q}) = \left\{ e^{\omega_0(\mathbf{q})/T} - 1 \right\}^{-1}.$$

Apparently $\mathcal{D}^{(0)}$ is an even function of τ according to EQs. (II.647) and (II.656).

Diagrammatically, we may apply similar technique of Feynman rules for $T = 0$ to the particle and phonon temperature Green's functions. In coordinate space, the following formulae are satisfied:

$$\left\langle T_\tau \left[\psi_\alpha(\mathbf{x}_1, \tau_1) \bar{\psi}_\beta(\mathbf{x}_2, \tau_2) \right] \right\rangle = -g_{\alpha\beta}^{(0)}(\mathbf{x}_1 - \mathbf{x}_2; \tau_1 - \tau_2), \quad (\text{II.657})$$

$$\left\langle T_\tau \left[\bar{\psi}_\beta(\mathbf{x}_2, \tau_2) \psi_\alpha(\mathbf{x}_1, \tau_1) \right] \right\rangle = \pm g_{\alpha\beta}^{(0)}(\mathbf{x}_1 - \mathbf{x}_2; \tau_1 - \tau_2), \quad (+: \text{fermions}, -: \text{bosons}) \quad (\text{II.658})$$

$$\left\langle T_\tau \left[\phi(\mathbf{x}_1, \tau_1) \phi(\mathbf{x}_2, \tau_2) \right] \right\rangle = -\mathcal{D}^{(0)}(\mathbf{x}_1 - \mathbf{x}_2; \tau_1 - \tau_2). \quad (\text{II.659})$$

Given the electron-phonon interaction Hamiltonian for $T > 0$:

$$\mathcal{H}_{el-ph}(\tau) = \gamma \int d^3\mathbf{x} \bar{\psi}_\alpha(\mathbf{x}, \tau) \psi_\alpha(\mathbf{x}, \tau) \phi(\mathbf{x}, \tau), \quad (\text{II.660})$$

we can calculate the second-order correction to the phonon Green's function in ways similar to the case for $T = 0$, and we find

$$\gamma^2 \int d^4z_1 d^4z_2 \mathcal{D}^{(0)}(x - z_1) g_{\alpha\beta}^{(0)}(z_1 - z_2) g_{\beta\alpha}^{(0)}(z_2 - z_1) \mathcal{D}^{(0)}(z_2 - x'), \quad (\text{II.661})$$

which is similar to the result in EQ. (II.622) for the diagram shown in Fig. II.10.3 (a).

The general rules for calculating the corrections of order $2n$ (with respect to \mathcal{H}_{el-ph}) to the electron and phonon Green's functions at $T > 0$ and in coordinate space may be summarized as follows:

- 1) Form all connected, topologically non-equivalent diagrams with $2n$ vertices;
- 2) With each solid line associate with a free-particle Green's function $g_{\alpha\beta}^{(0)}(x - x')$, and with each dashed line associate a function $\mathcal{D}^{(0)}(x - x')$;
- 3) Integrate over the coordinates of all vertices with respect to both \mathbf{x} and τ , and sum over the corresponding spin variables;
- 4) Multiply the resulting expression by $\gamma^{2n} (-1)^{n+F}$, where F is the number of closed loops formed by the fermion $g^{(0)}$ -lines.

In reality, the aforementioned diagrammatic techniques in coordinate space are not practical for calculating Green's function corrections at finite temperatures, because the imaginary time varies from 0 to $1/T$ rather than to infinity. The situation can be much simplified by expanding all quantities depending on τ in Fourier series relative to the imaginary time difference τ . Noting that τ is defined in an interval between $-1/T$ and $1/T$, we obtain

$$g(\tau) = T \sum_n e^{-i\omega_n \tau} g(\omega_n), \quad (\text{II.662})$$

and
$$g(\omega_n) = \frac{1}{2} \int_{-1/T}^{1/T} d\tau e^{i\omega_n \tau} g(\tau), \quad \omega_n = n\pi T. \quad (\text{II.663})$$

Using EQs. (II.646) and (II.647), we may rewrite EQ. (II.663) into the following:

$$\begin{aligned} g(\omega_n) &= \frac{1}{2} \int_0^{1/T} d\tau e^{i\omega_n \tau} g(\tau) + \frac{1}{2} \int_{-1/T}^0 d\tau e^{i\omega_n \tau} g(\tau) \\ &= \frac{1}{2} \int_0^{1/T} d\tau e^{i\omega_n \tau} g(\tau) + \frac{1}{2} \int_0^{1/T} d\tau e^{i\omega_n \tau} e^{i\omega_n/T} g\left(\tau + \frac{1}{T}\right) \\ &= \frac{1}{2} (1 \mp e^{i\omega_n/T}) \int_0^{1/T} d\tau e^{i\omega_n \tau} g(\tau) \\ &= \int_0^{1/T} d\tau e^{i\omega_n \tau} g(\tau), \quad \omega_n = (2n+1)\pi T \text{ for fermions,} \end{aligned} \quad (\text{II.664})$$

$$= \int_0^{1/T} d\tau e^{i\omega_n \tau} g(\tau), \quad \omega_n = 2n\pi T \text{ for bosons.} \quad (\text{II.665})$$

Taking the Fourier transformation of the coordinates, we find

$$g_{\alpha\beta}^{(0)}(\mathbf{k}; \tau > 0) = -\delta_{\alpha\beta} e^{-[\varepsilon_0(\mathbf{k}) - \mu]\tau} [1 - n(\mathbf{k})], \quad (\text{II.666})$$

so that

$$\begin{aligned} g_{\alpha\beta}^{(0)}(\mathbf{k}; \omega_n) &= -\delta_{\alpha\beta} [1 - n(\mathbf{k})] \int_0^{1/T} d\tau e^{i\omega_n \tau} e^{-[\varepsilon_0(\mathbf{k}) - \mu]\tau}, \\ &= \frac{1}{i\omega_n - \varepsilon_0(\mathbf{k}) + \mu}, \quad \omega_n = (2n+1)\pi T \text{ for fermions,} \end{aligned} \quad (\text{II.667})$$

$$= \frac{1}{i\omega_n - \varepsilon_0(\mathbf{k}) + \mu}, \quad \omega_n = 2n\pi T \text{ for bosons.} \quad (\text{II.668})$$

Similarly, for phonon propagators, we find that

$$\mathcal{D}^{(0)}(\mathbf{k}; \omega_n) = -\frac{\omega_{\mathbf{k}}^2}{\omega_n^2 + \omega_{\mathbf{k}}^2}, \quad \omega_n = 2n\pi T. \quad (\text{II.669})$$

In general, there are even numbers of fermion lines at each vertex, so that the integration over τ involves the following integral

$$\begin{aligned} \int_0^{1/T} d\tau e^{i\tau \sum \omega_n} &= \frac{1}{T} \quad \text{for } \sum \omega_n = 0. \\ &= 0 \quad \text{for } \sum \omega_n \neq 0. \end{aligned} \quad (\text{II.670})$$

Consequently, the condition for summing over imaginary time in temperature Green's functions is similar to that for conservation of energy at a vertex in the case of Green's functions at $T = 0$.

As an example for diagrammatic analysis of temperature Green's functions in momentum space, we consider the diagram in Fig. II.10.10 (a) for the Green's function correction $\delta g^{(1)}$ under a two-particle interaction potential $V(z_1 - z_2)$. According to the rules given for coordinate space, we find

$$\delta g_{\alpha\beta}^{(1)}(x - x') = -\int d^4 z_1 d^4 z_2 g_{\alpha\gamma_1}^{(0)}(x - z_1) g_{\gamma_1\gamma_2}^{(0)}(z_1 - z_2) V(z_1 - z_2) g_{\gamma_2\beta}^{(0)}(z_2 - x'). \quad (\text{II.671})$$

Taking the Fourier transform of $\delta g^{(1)}$, we have

$$\delta g^{(1)}(\mathbf{k}, \omega_n) = \frac{1}{2} \int d(\mathbf{x} - \mathbf{x}') \int_{-1/T}^{1/T} d(\tau_x - \tau_{x'}) \delta g^{(1)}(x - x') e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') + i\omega_n(\tau_x - \tau_{x'})}, \quad (\text{II.672})$$

$$V(\mathbf{x}, \tau) = \frac{T}{(2\pi)^3} \sum_{\omega_{n4}} \int d^3\mathbf{q} e^{i\mathbf{q} \cdot \mathbf{x} - i\omega_{n4}\tau} V(\mathbf{q}, \omega_{n4}). \quad (\text{II.673})$$

Noting that

$$T \sum_{n=-\infty}^{\infty} e^{i2n\pi T\tau} = \delta(\tau), \quad (\text{II.674})$$

we have

$$\begin{aligned} \delta g_{\alpha\beta}^{(1)}(\mathbf{k}, \omega_n) &= -\frac{1}{2} \left[\frac{T}{(2\pi)^3} \right]^4 \sum_{\omega_{n1}} \sum_{\omega_{n2}} \sum_{\omega_{n3}} \sum_{\omega_{n4}} \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 d^3\mathbf{q} g_{\alpha\gamma_1}^{(0)}(\mathbf{k}_1, \omega_{n1}) g_{\gamma_1\gamma_2}^{(0)}(\mathbf{k}_2, \omega_{n2}) g_{\gamma_2\beta}^{(0)}(\mathbf{k}_3, \omega_{n3}) V(\mathbf{q}, \omega_{n4}) \\ &\quad \times \int d(\mathbf{x} - \mathbf{x}') \int d^3\mathbf{z}_1 \int d^3\mathbf{z}_2 \int_{-1/T}^{1/T} d(\tau_x - \tau_{x'}) \int_{-1/T}^{1/T} d\tau_1 \int_{-1/T}^{1/T} d\tau_2 e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') + i\omega_n(\tau_x - \tau_{x'})} \\ &\quad \times e^{i\mathbf{k}_1 \cdot (\mathbf{x} - \mathbf{z}_1) - i\omega_{n1}(\tau_x - \tau_1)} e^{i\mathbf{k}_2 \cdot (\mathbf{z}_1 - \mathbf{z}_2) - i\omega_{n2}(\tau_1 - \tau_2)} e^{i\mathbf{k}_3 \cdot (\mathbf{z}_2 - \mathbf{x}') - i\omega_{n3}(\tau_2 - \tau_{x'})} e^{i\mathbf{q} \cdot (\mathbf{z}_1 - \mathbf{z}_2) - i\omega_{n4}(\tau_1 - \tau_2)} \\ &= -\left[\frac{T}{(2\pi)^3} \right]^4 \sum_{\omega_{n1}} \sum_{\omega_{n2}} \sum_{\omega_{n3}} \sum_{\omega_{n4}} \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 d^3\mathbf{q} g_{\alpha\gamma_1}^{(0)}(\mathbf{k}_1, \omega_{n1}) g_{\gamma_1\gamma_2}^{(0)}(\mathbf{k}_2, \omega_{n2}) g_{\gamma_2\beta}^{(0)}(\mathbf{k}_3, \omega_{n3}) V(\mathbf{q}, \omega_{n4}) \\ &\quad \times \left[\frac{(2\pi)^3}{T} \right]^3 \delta(\mathbf{k} - \mathbf{k}_1) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{q}) \delta(\mathbf{k}_2 - \mathbf{k}_3 + \mathbf{q}) \delta_{\omega_n - \omega_{n1}} \delta_{\omega_{n1} - \omega_{n2} - \omega_{n4}} \delta_{\omega_{n2} - \omega_{n3} + \omega_{n4}} \\ &= -\frac{T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 g_{\alpha\gamma_1}^{(0)}(\mathbf{k}, \omega_n) g_{\gamma_1\gamma_2}^{(0)}(\mathbf{k}_1, \omega_{n1}) g_{\gamma_2\beta}^{(0)}(\mathbf{k}, \omega_n) V(\mathbf{k} - \mathbf{k}_1, \omega_n - \omega_{n1}). \end{aligned} \quad (\text{II.675})$$

Inserting EQs. (II.667) and (II.668) into EQ. (II.675), we obtain

$$\delta g_{\alpha\beta}^{(1)}(\mathbf{k}, \omega_n) = -\frac{\delta_{\alpha\beta}}{[i\omega_n - \varepsilon_0(\mathbf{k}) + \mu]^2} \frac{T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 \frac{V(\mathbf{k} - \mathbf{k}_1, \omega_n - \omega_{n1})}{i\omega_{n1} - \varepsilon_0(\mathbf{k}_1) + \mu}, \quad (\text{II.676})$$

and the diagrammatic representation for EQ. (II.676) is shown in Fig. II.10.10 (b).

Similarly, the correction to the temperature Green's function shown in Fig. II.10.10 (c) is given by

$$\pm \frac{\delta_{\alpha\beta}}{[i\omega_n - \varepsilon_0(\mathbf{k}) + \mu]^2} V(0,0)(2s+1) \frac{T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 \frac{e^{i\omega_{n1}\tau}}{i\omega_{n1} - \varepsilon_0(\mathbf{k}_1) + \mu}, \quad (\text{II.677})$$

where the plus (negative) sign corresponds to the correction to fermions (bosons).

As another example, the generalized two-particle interaction that involves the vertex contribution as illustrated in Fig. II.10.10 (d) is given by:

$$-\frac{1}{[i\omega_n - \varepsilon_0(\mathbf{k}) + \mu]^2} \frac{T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 \Gamma_{\alpha\gamma, \gamma\beta}^{(0)}(\mathbf{k}, \omega_n, \mathbf{k}_1, \omega_{n1}; \mathbf{k}_1, \omega_{n1}, \mathbf{k}, \omega_n) \frac{1}{i\omega_{n1} - \varepsilon_0(\mathbf{k}_1) + \mu}, \quad (\text{II.678})$$

where the vertex part generally takes the form $\Gamma_{\alpha\gamma,\gamma\beta}^{(0)}(k_1, k_2; k_3, k_1 + k_2 - k_3)$ that ensures momentum and energy conservation.

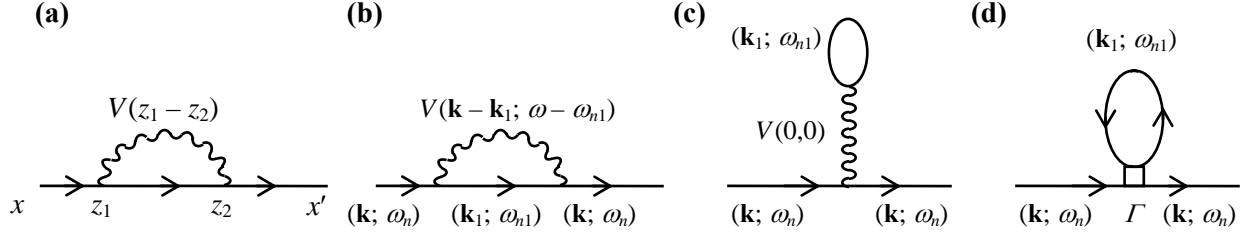


Figure II.10.10 (a) Temperature Green's function correction associated with two-particle interaction to first order in coordinate space; (b) similar diagram to (a) in momentum space; (c) another first-order diagram in momentum space; (d) generalized vertex correction to the temperature Green's function.

Next, we consider the case for electron-phonon interaction. As discussed before, only contributions associated with even order of \mathcal{H}_{el-ph} are non-zero. For an arbitrary diagram of order $2n$ in \mathcal{H}_{el-ph} , there are $(2n+1)$ internal electron lines, n internal phonon lines, and $2n$ vertices. Hence, there are n independent integrations. The general rules for calculating a diagram of order $2n$ relative to \mathcal{H}_{el-ph} are summarized below:

- 1) Form all connected, topologically non-equivalent diagrams with $2n$ vertices;
- 2) With each solid line associate with a free-particle Green's function

$$g^{(0)}(k) = \frac{1}{i\omega_n - \varepsilon_0(\mathbf{k}) + \mu}$$

and with two solid external lines associate a quantity

$$\frac{\delta_{\alpha\beta}}{[i\omega_n - \varepsilon_0(\mathbf{k}) + \mu]^2}.$$

- 3) With each phonon (dashed) line associate a function

$$\mathcal{D}^{(0)}(q) = -\frac{\omega_{\mathbf{q}}^2}{\omega_n^2 + \omega_{\mathbf{q}}^2};$$

- 4) Integrate over n independent momenta;
- 5) Multiply the resulting expression by

$$\gamma^{2n} \frac{T^n}{(2\pi)^{3n}} (-1)^n (2s+1)^F (\mp 1)^F,$$

where F is the number of closed loops formed by the fermion $g^{(0)}$ -lines.

For example, we consider the second-order correction to the phonon propagator at $T > 0$, as shown in Fig. II.10.11. Applying the general rules outlined above, we obtain

$$\pm \left[\frac{\omega_q^2}{\omega_n^2 + \omega_q^2} \right]^2 \gamma^2 (2s+1) \frac{T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k} \frac{1}{i\omega_{n1} - \varepsilon_0(\mathbf{k}) + \mu} \frac{1}{i(\omega_{n1} - \omega_n) - \varepsilon_0(\mathbf{k}-\mathbf{q}) + \mu}$$

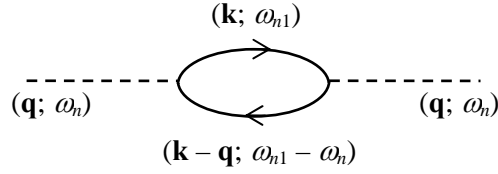


Figure II.10.11 Second-order correction to the phonon propagator at $T > 0$, in momentum space.

In general, the correction to the temperature Green's function g can be obtained from the expression for the correction to the Green's function G at $T = 0$ by replacing the frequencies ω in G by $i\omega_n$ and changing all integrals over ω to sums over ω_n :

$$\frac{1}{2\pi} \int d\omega \dots \Rightarrow iT \sum_{\omega_n} \dots$$

Finally, we note that the Dyson's equations for the temperature Green's functions of electrons and phonons are similar to those for the Green's functions at $T = 0$. The Dyson's equations for corrections associated with electron-phonon interactions are given below and also illustrated in Figs. II.10.12 (a) and (b) for electrons and phonons, respectively:

$$g_{\alpha\beta}^{-1}(\mathbf{k}, \omega_n) = [i\omega_n - \varepsilon_0(\mathbf{k}) + \mu] \delta_{\alpha\beta} + \frac{\gamma T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 g_{\alpha\beta}(\mathbf{k}_1, \omega_{n1}) \mathcal{D}(\mathbf{k}_1 - \mathbf{k}, \omega_{n1} - \omega_n) \Gamma(\mathbf{k}, \mathbf{k}_1; \omega_n, \omega_{n1}) \tag{II.679}$$

$$\mathcal{D}^{-1}(\mathbf{q}, \omega_n) = -\omega_q^{-2} [\omega_n^2 + \omega_q^2] - \frac{\gamma T}{(2\pi)^3} \sum_{\omega_{n1}} \int d^3\mathbf{k}_1 g_{\alpha\beta}(\mathbf{k}_1, \omega_{n1}) g_{\beta\alpha}(\mathbf{k}_1 - \mathbf{q}, \omega_{n1} - \omega_n) \Gamma(\mathbf{k}_1, \mathbf{k}_1 - \mathbf{q}; \omega_{n1}, \omega_{n1} - \omega_n) \tag{II.680}$$

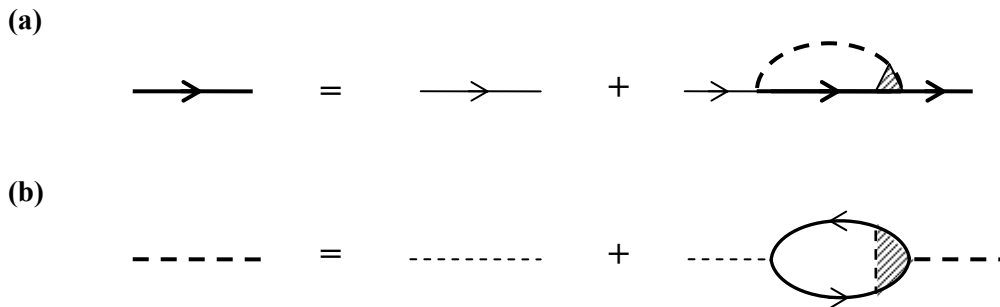


Fig. II.10.12 Dyson's equations for temperature Green's functions of (a) electrons and (b) phonons.

Further Readings:

1. Zee, “Quantum Theory in a Nutshell”: Parts I.1 – I.3, I.7, I.8, II.1 – II.5.
2. Peskin and Schroeder, “An Introduction to Quantum Field Theory”, Chapters 2 – 4.
3. Ryder, “Quantum Field Theory”, Chapters 4 – 6.
4. Fetter and Walecka, “Quantum Theory of Many-Particle Systems”, Sections 7 – 10, 13 – 15, 44 – 47.
5. Abrikosov, Gorkov, and Dzyaloshinski, “Methods of Quantum Field Theory in Statistical Physics”, Chapters 2 and 3.
6. Doniach and Sondheimer, “Green’s Functions for Solid State Physicists”, Chapters 3, 5 and 6.