PART II. Quantum Field Theory for Many-Body Systems

Quantum field theory (QFT) can be thought of as taking quantum mechanics of a system to the limit with an infinite number of degrees of freedom, and is therefore a very useful tool in describing many-body systems and has become indispensable in modern physics. In general, the application of QFT to different branches of physics often involves quite different techniques and approaches. For instance, the relevant QFT considered in particle physics is generally relativistic with the assumption of Lorentz invariance. On the other hand, most phenomena of interest to condensed matter physicists involve QFT in the non-relativistic limit, and time-dependent perturbation theory in the interaction picture is often employed. The objective of Part II is to familiarize you with the basic notations of quantum field theory, first in the relativistic limit and then in the non-relativistic limit, with emphasis of the latter on Green's function (or propagator) techniques and on the related application to many-body interactions and linear response in condensed matter systems.

We begin with a brief review of canonical formalism for QFT in Part II.1, followed by a review of the path integral formalism in Part II.2. The path integral formalism will be the primary choice in our subsequent development of QFT mostly for its elegance and heuristic clarity to go from quantum mechanics to quantum field theory, and also for its preservation of the Lorentz invariance that is much more convenient than the canonical approach for dealing with relativistic properties. The rules and applications of Feynman diagrams in the relativistic limit for bosons are discussed in Part II.3. Brief reviews of the Dirac equation and spinor fields are given in Part II.4, followed by Part II.5 that considers the Grassmann algebra and Feynman diagrams for fermions. However, Part II.4 and Part II.5 are only provided as references and shall not be covered in the lectures because I assume that most of you already have adequate background in relativistic QFT. If you have not learnt these topics before, you are encouraged to read the self-contained notes. These notes will become useful when we discuss topological defects and the Chern-Simons theory for the fractional quantum Hall states later in this course. The non-relativistic limit of QFT is derived from the relativistic QFT in Part II.6, and non-relativistic Green's function techniques commonly used in condensed matter physics are introduced in Part II.7. Important applications of the Green's function techniques to many-body interactions in condensed matter systems, such as the Hartree-Fock and the random phase approximations, are discussed in Part II.8. Studies of the linear response and dissipation-fluctuation theories using the Green's function techniques are covered in Part II.9 and Part II.10, respectively. Throughout Part II we assume Lorentz invariance and zero temperatures, $T = 0$, and we take $\hbar = 1$ and $c = 1$ for simplicity. The finite-temperature conditions will be considered later in the course.

II.1. The Canonical Formalism

We first consider the canonical formalism because of its historical and continuing importance. As you have learnt in quantum mechanics, the classical Lagrangian $\mathcal{L}$ for a single particle moving in space time is given by

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{q}}^2 - \mathcal{V}(\mathbf{q}), \tag{II.1}$$

where $\mathcal{V}(\mathbf{q})$ is the potential energy, and we have sent the mass to 1 for convenience. The canonical momentum is defined as

$$\mathbf{p} \equiv \delta\mathcal{L} / \delta\dot{\mathbf{q}} = \dot{\mathbf{q}}, \tag{II.2}$$

and the Hamiltonian is therefore expressed by

$$\mathcal{H} = \mathbf{p}\dot{\mathbf{q}} - \mathcal{L} = (\mathbf{p}^2 / 2) + \mathcal{V}(\mathbf{q}). \tag{II.3}$$

In quantum mechanics, both **p** and **q** are taken as operators, and they satisfy the canonical commutation relation [**p**, **q**] = $-i$. Using the results derived in Part I, the time evolution of the operators is described by the relations:

$$\frac{d\mathbf{p}}{dt} = i\left[\mathcal{H}, \mathbf{p}\right] = -\mathcal{V}'(\mathbf{q}) \quad \text{and} \quad \frac{d\mathbf{q}}{dt} = i\left[\mathcal{H}, \mathbf{q}\right] = \mathbf{p}, \tag{II.4}$$

so that operators constructed out of **p** and **q** evolve with time according to the Heisenberg picture $O_H(t) = \exp(i\mathcal{H}t)O_H(0)\exp(-i\mathcal{H}t)$, where $\mathcal{H}$ has been taken as time-independent, $O_H(0) = O_s$ as in EQ. (I.91), and the operator equation of motion is:

$$\ddot{\mathbf{q}} = -\mathcal{V}'(\mathbf{q}). \tag{II.5}$$

Similar to our derivation of the bosonic fields in the case of free electromagnetic fields and plasmons in Part I, we introduce new creation and annihilation operators a and $a^\dagger$ as a linear combination of **p** and **q**, and the operators satisfy the commutation relation $[a, a^\dagger] = 1$. Let's define $a = \alpha\,\mathbf{q} + i\beta\,\mathbf{p}$ where α and β are both real numbers, so that we have $\alpha\beta = ½$. We also note that the dimension of **q** differs from that of **p** by frequency ω according to EQ. (II.4). Consequently, we take $\alpha = (\omega/2)^{1/2}$ and $\beta = 1/(2\omega)^{1/2}$, which yields

$$\frac{da}{dt} = i\left[\mathcal{H}, a\right] = i\left[\mathcal{H}, \frac{1}{\sqrt{2\omega}}(\omega\,\boldsymbol{q} + i\,\boldsymbol{p})\right] = i\left[\mathcal{H}, \sqrt{\frac{\omega}{2}}\,\boldsymbol{q}\right] - \left[\mathcal{H}, \frac{1}{\sqrt{2\omega}}\,\boldsymbol{p}\right] = -i\sqrt{\frac{\omega}{2}}\left[i\,\boldsymbol{p} + \frac{\mathcal{V}'(\boldsymbol{q})}{\omega}\right]. \tag{II.6}$$

The ground state of the system $|0\rangle$ is defined by the relation $a|0\rangle = 0$.

For a special case with $\mathcal{V}'(\mathbf{q}) = \omega^2 q$, which corresponds to an effective motion of a simple harmonic oscillator (such as the cases of photons and plasmons discussed in Part I), we have $(da/dt) = -i\omega a$ as in EQ. (I.76), and the Hamiltonian in EQ. (II.3) becomes

$$\mathcal{H} = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(\hat{n} + \frac{1}{2}\right), \tag{II.7}$$

where $\hat{n} = a^\dagger a$ is the number operator.

To generalize the single particle problem to a many-particle system, we consider the Lagrangian

$$\mathcal{L} = \sum_{a=1}^{N} \frac{1}{2}(\dot{\mathbf{q}}_a)^2 - \mathcal{V}(\mathbf{q}_1, \mathbf{q}_2 ..., \mathbf{q}_N). \tag{II.8}$$

Using $\mathbf{p}_a \equiv \delta\mathcal{L}/\delta\dot{\mathbf{q}}_a = \dot{\mathbf{q}}_a$, we have $[\mathbf{p}_a, \mathbf{q}_b] = -i\delta_{ab}$. Next, we take the continuum limit to generalize the many-particle system to quantum field theory for a D-dimensional space. Using the substitutions $q \to \varphi$ and $\mathcal{V}(q) \to m^2\varphi^2 + u(\varphi)$, the Lagrangian becomes:

$$\mathcal{L} = \int d^D x\left[\frac{1}{2}\left(\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right) - u(\varphi)\right] = \int d^D x\left[\frac{1}{2}\left((\partial\varphi)^2 - m^2\varphi^2\right) - u(\varphi)\right] \equiv \int d^D x\, L(\varphi), \tag{II.9}$$

where the anharmonic term is given by $u(\varphi)$, $\varphi(\mathbf{x}, t)$ denotes a scalar field, and we have used the notation $\partial^2 \equiv \partial_t^2 - \nabla^2$. The canonical momentum density $\pi(\mathbf{x}, t)$ conjugate to $\varphi(\mathbf{x}, t)$ is:

$$\pi(\mathbf{x},t) \equiv \frac{\delta L}{\delta\dot{\varphi}(\mathbf{x},t)} = \partial_0\varphi(\mathbf{x},t), \text{ and } \left[\pi(\mathbf{x},t),\varphi(\mathbf{x}',t)\right] = \left[\partial_0\varphi(\mathbf{x},t),\varphi(\mathbf{x}',t)\right] = -\delta^{(D)}(\mathbf{x}-\mathbf{x}'). \quad \text{(II.10)}$$

Moreover, $\left[\pi(\mathbf{x},t),\ \pi(\mathbf{x}',t)\right] = \left[\varphi(\mathbf{x},t),\ \varphi(\mathbf{x}',t)\right] = 0$. The Hamiltonian for our quantum field theory becomes:

$$\mathcal{H} = \int d^D x\left[\pi(\mathbf{x},t)\,\partial_0\varphi(\mathbf{x},t) - L\right] = \int d^D x\left[\frac{1}{2}\left(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right) + u(\varphi)\right]. \quad \text{(II.11)}$$

In the case of the harmonic oscillators, the anharmonic term $u(\varphi)$ vanishes, and the equation of motion for the scalar field $\varphi(\mathbf{x},t)$ satisfies the relation

$$\left(\partial^2 + m^2\right)\varphi = 0, \quad \text{(II.12)}$$

which is known as *the Klein-Gordon equation*. Moreover, the field can be rewritten in terms of its Fourier expansion as follows:

$$\varphi(\mathbf{x},t) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right) + a^{\dagger}(\mathbf{k})\exp\left(i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)\right], \quad \text{(II.13)}$$

where $\omega_{\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2}$, $[\,a(\mathbf{k})\,,a^{\dagger}(\mathbf{k}')\,] = \delta^{(D)}(\mathbf{k}-\mathbf{k}')$, $[\,a(\mathbf{k})\,,a(\mathbf{k}')\,] = [\,a^{\dagger}(\mathbf{k}),a^{\dagger}(\mathbf{k}')\,] = 0$, $a(\mathbf{k})$ and $a^{\dagger}(\mathbf{k}')$ are the annihilation and creation operators for the field, respectively. We further note that $a(\mathbf{k})\,|0\rangle = 0$ for all $\mathbf{k}$-values, where $|0\rangle$ denotes either vacuum or the ground state.

Having developed the canonical expressions for the scalar field $\varphi(\mathbf{x},t)$, let us evaluate the quantity $\left\langle 0\middle|\varphi(\mathbf{x},t>0)\,\varphi(0,0)\middle|0\right\rangle$. Using EQ. (II.13) and $a(\mathbf{k})\,|0\rangle = 0$, we find that

$$\begin{aligned}
\langle 0|\,\varphi(\mathbf{x},t)\,\varphi(0,0)\,|\,0\rangle &= \langle 0|\int\frac{d^D k'}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}'}}}\int\frac{d^D k}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}}}} \\
&\quad\times\left[a(\mathbf{k})\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right) + a^{\dagger}(\mathbf{k})\exp\left(i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)\right]\left[a(\mathbf{k}') + a^{\dagger}(\mathbf{k}')\right]|\,0\rangle \\
&= \langle 0|\iint\frac{d^D k'}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}'}}}\frac{d^D k}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})\,a^{\dagger}(\mathbf{k}')\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)\right]|\,0\rangle \\
&= \langle 0|\iint\frac{d^D k}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}}}}\frac{d^D k'}{\sqrt{(2\pi)^D 2\omega_{\mathbf{k}'}}}\left[\delta^{(D)}(\mathbf{k}-\mathbf{k}')\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)\right]|\,0\rangle \\
&= \langle 0|\int\frac{d^D k}{(2\pi)^D 2\omega_{\mathbf{k}}}\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)|\,0\rangle = \int\frac{d^D k}{(2\pi)^D 2\omega_{\mathbf{k}}}\exp\left(-i(\omega_{\mathbf{k}}t - \mathbf{k}\bullet\mathbf{x})\right)\cdot
\end{aligned} \quad \text{(II.14)}$$

Therefore we can define a time-ordered product:

$$\hat{T}\left[\varphi(x)\,\varphi(y)\right] = \theta(x-y)\,\varphi(x)\,\varphi(y) + \theta(y-x)\,\varphi(y)\,\varphi(x), \quad \text{(II.15)}$$

so that

$$\langle 0|\,\hat{T}\left[\varphi(x)\,\varphi(0)\right]|\,0\rangle = \int\frac{d^D k}{(2\pi)^D 2\omega_{\mathbf{k}}}\left[\theta(t)\exp\left(-i(\omega_{\mathbf{k}}t - \boldsymbol{k}\bullet\boldsymbol{x})\right) + \theta(-t)\exp\left(+i(\omega_{\mathbf{k}}t - \boldsymbol{k}\bullet\boldsymbol{x})\right)\right], \quad \text{(II.16)}$$

and $\langle 0|\hat{T}\left[\varphi(x)\varphi(0)\right]|0\rangle = iD(x)$, where $D(x)$ is the *free propagator* for the scalar field to go from 0 to x in space-time -- Here we have used x to denote the spacetime coordinate $(t,\mathbf{x})$. We shall derive the free propagator again in the next section using the path integral formalism.

It is instructive to calculate the "vacuum energy" associated with the Hamiltonian of a scalar field $\varphi(\mathbf{x},t)$ in the Gaussian theory:

$$\langle 0|\mathcal{H}|0\rangle = \frac{1}{2}\int d^D x\ \langle 0|\boldsymbol{\pi}^2 + (\nabla\varphi)^2 + m^2\varphi^2|0\rangle\ . \tag{II.17}$$

Using the fact that the system observes translational invariance and the result in EQ. (II.14), we can easily obtain the φ^2 term in EQ. (II.17) as follows:

$$\frac{m^2}{2}\int d^D x\ \langle 0|\varphi(\mathbf{x},t)\,\varphi(\mathbf{x},t)|0\rangle = \frac{m^2}{2}\int d^D x\ \langle 0|\varphi(0,0)\,\varphi(0,0)|0\rangle = \frac{m^2}{2}\int d^D x\ \left[\int \frac{d^D k}{(2\pi)^D 2\omega_{\mathbf{k}}}\right]$$
$$\equiv \frac{m^2}{2}V\left[\int \frac{d^D k}{(2\pi)^D 2\omega_{\mathbf{k}}}\right],$$

where V denotes the volume. Similarly, using EQ. (II.14) and noting that the π^2 term in EQ. (II.17) is the same as $(\partial_0\varphi)^2$ and so can be written as $(\omega_k\,\varphi)^2$, whereas the $(\nabla\varphi)^2$ term can be expressed by $(k\varphi)^2$, we obtain

$$\langle 0|\mathcal{H}|0\rangle = V\int \frac{d^D k}{(2\pi)^D (2\omega_{\mathbf{k}})}\left[\frac{1}{2}\left(\omega_{\mathbf{k}}^{\ 2} + \mathbf{k}^2 + m^2\right)\right] = V\int \frac{d^D k}{(2\pi)^D}\left(\frac{1}{2}\hbar\omega_{\mathbf{k}}\right) \tag{II.18}$$

upon restoring the Planck constant. Equation (II.18) simply represents the zero-point energy of harmonic oscillators in vacuum of volume V. In practice, it is sufficient to consider the energy of a physical system in the context of its value relative to the "vacuum energy" ($\mathcal{E} - \langle 0|\mathcal{H}|0\rangle$) rather than of its absolute value.

It is also instructive to calculate the quantity $\langle \mathbf{k}'|\mathcal{H}|\mathbf{k}\rangle$, where $|\mathbf{k}\rangle = a^\dagger(\mathbf{k})|0\rangle$. We recall the definition of the Hamiltonian in EQ. (II.11) with $u(\varphi) = 0$, so that we first consider the integration

$$\int d^D x\ \varphi(\mathbf{x})^2 = \int d^D x \iint \frac{d^D q}{\sqrt{(2\pi)^D 2\omega_{\mathbf{q}}}}\frac{d^D q'}{\sqrt{(2\pi)^D 2\omega_{\mathbf{q}'}}}\left[a(\mathbf{q})\,a^\dagger(\mathbf{q}')\exp\left(-i(\omega_{\mathbf{q}}t - \mathbf{q}\bullet\mathbf{x})\right)\exp\left(i(\omega_{\mathbf{q}'}t - \mathbf{q}'\bullet\mathbf{x})\right) + h.c.\right]$$
$$= \int \frac{d^D q}{2\omega_{\mathbf{q}}}\left[a(\mathbf{q})\,a^\dagger(\mathbf{q}) + a^\dagger(\mathbf{q})\,a(\mathbf{q})\right] = \int \frac{d^D q}{2\omega_{\mathbf{q}}}\left[2a^\dagger(\mathbf{q})\,a(\mathbf{q}) + \delta^{(D)}(0)\right].$$

Therefore $\langle \mathbf{k}'|\mathcal{H}|\mathbf{k}\rangle$ becomes:

$$\langle \mathbf{k}'|\mathcal{H}|\mathbf{k}\rangle = \langle 0|a(\mathbf{k}')\left[\int \frac{d^D q}{2\omega_{\mathbf{q}}}\left(\omega_{\mathbf{q}}^{\ 2}\right)\left(2a^\dagger(\mathbf{q})\,a(\mathbf{q}) + \delta^{(D)}(0)\right)\right]a^\dagger(\mathbf{k})|0\rangle\,. \tag{II.19}$$

The delta function term in EQ. (II.19) simply gives the vacuum energy, which can be subtracted off if we are only interested in the expectation value of $\langle \mathbf{k}'|\mathcal{H}|\mathbf{k}\rangle$ relative to vacuum. Therefore EQ. (II.19) is simplified to:

$$\langle \mathbf{k}'|\mathcal{H}|\mathbf{k}\rangle = \int d^D q\ (\omega_{\mathbf{q}})\langle 0|a(\mathbf{k}')\,a^\dagger(\mathbf{q})\,a(\mathbf{q})\,a^\dagger(\mathbf{k})|0\rangle = \int d^D q\ (\omega_{\mathbf{q}})\langle 0|a(\mathbf{k}')\,a^\dagger(\mathbf{q})\left[\delta^{(D)}(\mathbf{k}-\mathbf{q}) + a^\dagger(\mathbf{k})\,a(\mathbf{q})\right]|0\rangle$$
$$= (\omega_{\mathbf{k}})\langle 0|a(\mathbf{k}')\,a^\dagger(\mathbf{k})\,|0\rangle = (\omega_{\mathbf{k}})\langle 0|\left[\delta^{(D)}(\mathbf{k}-\mathbf{k}') + a^\dagger(\mathbf{k})\,a(\mathbf{k}')\right]|0\rangle = \delta^{(D)}(\mathbf{k}-\mathbf{k}')\,\omega_{\mathbf{k}}\,. \tag{II.20}$$

We note that in EQ. (II.20) we have used the commutation relation $[a(\mathbf{k}'), a^\dagger(\mathbf{k})] = \delta^{(D)}(\mathbf{k}-\mathbf{k}')$ and the condition $a(\mathbf{k})|0\rangle = 0$. Equation (II.20) implies that the energy of a particle of momentum **k** is $\omega_\mathbf{k}$ relative to vacuum.

Finally, we close our canonical discussion of the bosonic fields by commenting on the issue of symmetry and introducing *Noether's theorem* that describes the conservation principles of a system based on its symmetry. We note that Noether's theorem plays an important role in field theory and particle theory, because it accounts for conservation of energy, momentum, angular momentum, and other quantum numbers associated with particles, such as the charge, parity, isospin, color, etc. For instance, so far we have constrained the action associated with various bosonic fields by Lorentz invariance, which has the $SO(3,1)$ symmetry in spacetime. [You may refer to supplementary notes Supplement_1 for general properties of groups and their representations, and Supplement_2 for properties of continuous groups $SO(N)$ and $SU(N)$.] Apparently there are other types of symmetry that can be associated with a physical system. As a simple example, consider the Lagrangian of a scalar field φ in EQ. (II.9) that involves terms quadratic in φ and $(\partial\varphi)$, implying invariance under the transformation $\varphi \to (-\varphi)$, which is associated with conservation of the parity (P), an internal symmetry of the system. In general, an action may be invariant under either an internal, isospin symmetry transformation of the fields (such as the rotation group $O(N)$, or the $SU(N)$ group for mixing N quarks among themselves), or under some spacetime symmetry (such as the Lorentz and Poincaré groups), and the symmetry of the system restricts the form of the theory. Therefore, if the action of a system is time invariant, the energy of the system is conserved. On the other hand, if the action is translational invariant, the linear momentum is conserved. In the case of rotational invariance, there is conservation of the angular momentum.

As an explicit example, let's consider a system consisting of N scalar fields $(\varphi_1, \varphi_2, \ldots, \varphi_N)$ that obey the $SO(N)$ symmetry. The theory of such a system is invariant under the transformation $\varphi_a \to R_{ab}\varphi_b$, where the matrix $\{R_{ab}\}$ is a representation of the $SO(N)$ group. We also know that for $SO(N)$ symmetry the scalar product $\varphi \bullet \varphi = \varphi_a \varphi_a \equiv \varphi^2$ with $\varphi \equiv (\varphi_1, \varphi_2, \ldots, \varphi_N)$ is invariant, and that there are $N(N-1)/2$ generators in the group, so that any transformation of any of the scalar fields can be expressed in terms of a linear combination of the product of the generators and the scalar fields. We may write the representation in the form $R = \exp(\boldsymbol{\theta} \bullet \boldsymbol{\tau})$, where $\boldsymbol{\theta} \bullet \boldsymbol{\tau} = \theta^A \tau^A$ is a real anti-symmetric matrix, and τ^A denotes the $N(N-1)/2$ generators. Consequently, for an infinitesimal transformation $\varphi_a \to R_{ab}\,\varphi_b = \varphi_a + \delta\varphi_a \approx (1 + \theta^A \tau^A)_{ab}\,\varphi_b$, so that $\delta\varphi_a \approx (\theta^A \tau^A)_{ab}\,\varphi_b$. We expect that there are $N(N-1)/2$ conserved physical quantities associated with the system of $SO(N)$ symmetry.

More generally, for a system of certain continuous symmetry, if its action is invariant under an infinitesimal transformation of the symmetry operation and if the transformation can be expressed as a matrix multiplied by an infinitesimal parameter, Noether's theorem asserts that there are conserved currents associated with the generators of the symmetry group. However, Noether's theorem cannot account for all conserved quantities. For instance, the conservation of topological objects in nature has nothing to do with Noether's theorem, which is a topic of our later study regarding topological field theory. The explicit proof for Noether's theorem is given below.

[Noether's theorem]

The Noether's theorem states that for a system obeying the symmetry operation of a continuous group, there is a conserved current associated with each generator of the continuous symmetry.

Proof: For the Lagrangian ($\mathcal{L}$) of a system possessing some type of continuous symmetry (such as the $SO(N)$ or $SU(N)$ symmetries), we consider infinitesimal variations in the fields φ_a ($a = 1, 2, \ldots N$) that do not change $\mathcal{L}$, so that

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\delta(\partial_\mu\varphi_a) = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\partial_\mu(\delta\varphi_a) = 0\,. \qquad \text{(II.21)}$$

Using the Euler-Lagrangian equation of motion $\delta\mathcal{L}/\delta\varphi_a = \partial_\mu(\delta\mathcal{L}/\delta\partial_\mu\varphi_a)$, we can rewrite EQ. (II.21) into:

$$\delta\mathcal{L} = \left[\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\right]\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\partial_\mu(\delta\varphi_a) = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\delta\varphi_a\right] = 0\,. \qquad \text{(II.22)}$$

Therefore we can define the conservation principle of the system. Namely, $\partial_\mu J^\mu = 0$, where the conserved current J^μ is defined as:

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi_a)}\delta\varphi_a\,. \qquad \text{(II.23)}$$

Noting that $\delta\varphi_a \approx (\theta^A\tau^A)_{ab}\,\varphi_b$ and θ^A is arbitrary so may be set to 1, we find the conserved currents for the free-field theory as:

$$J_\mu^A = \left(\partial_\mu\varphi_a\right)\left(\tau^A\right)_{ab}\varphi_b\,, \qquad \text{(II.24)}$$

one for each generator of the continuous symmetry group.

From the conserved current, we can also establish a conserved charge $Q_A \equiv \int d^3x\, J_A^0$:

$$0 = \int d^3x\,\partial_\mu J_A^\mu = \int d^3x\,\partial_0 J_A^0 + \int d^3x\,\partial_i J_A^i = \frac{d}{dt}\int d^3x\, J_A^0 + \int_{surface} dS_i J_A^i = \frac{d}{dt}Q_A(t)\,, \qquad \text{(II.25)}$$

if we assume that the surface current vanishes rapidly at infinity. Consequently, we find that the symmetry of the action implies the conservation of currents and charge. In addition, from EQ. (II.23) we find that

$$Q \equiv \int d^3x\, J^0 = \int d^3x\,\frac{\delta L}{\delta(\partial_0\varphi_a)}\delta\varphi_a = \int d^3x\,\pi_a\delta\varphi_a\,. \qquad \text{(II.26)}$$

Using the relation $\left[\pi_a(\mathbf{x}), \varphi_a(\mathbf{x}')\right] = -i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')$ and EQ. (II.26), we obtain

$$i[Q, \varphi_a] = i\int d^3x\,[\pi_a\delta\varphi_a, \varphi_a] = \delta\varphi_a\,. \qquad \text{(II.27)}$$

Equation (II.27) implies that the charge operator generates the corresponding transformation on the fields of a system with a continuous symmetry such as $SO(N)$. In the special case of $SO(2)$, which corresponds to a complex field φ in $SO(2) \cong U(1)$ theory, we can define a generator $e^{i\theta}$, so that $\varphi \to e^{i\theta}\varphi = (\varphi+\delta\varphi) \approx (1+i\theta)\varphi$, $\delta\varphi \approx i\theta\varphi$, and for arbitrary θ (which can be set to 1) we have $[Q, \varphi] = -i\delta\varphi = \varphi$. Moreover, using $[Q, \varphi] = \varphi$ we obtain $e^{i\theta Q}\varphi e^{-i\theta Q} = e^{i\theta}\varphi$.

II.2. Path Integral Formalism

In this section, we consider the path integral formalism in QFT. To begin, we consider the propagation of a quantum system, governed by a Hamiltonian $\mathcal{H}$, from a point q_I to a point q_F in time t. The amplitude for the propagation is given by $\langle q_F | \exp(-i\mathcal{H}t) | q_I \rangle$, where we have used the Dirac bra and ket notations, and $\exp(-i\mathcal{H}t)$ is a unitary operator. If we divide the time t into N segments and define $\delta t = t/N$, and recall that $|q\rangle$ forms a complete set of states so that $\int dq\, |q\rangle \langle q| = 1$, we can rewrite $\langle q_F | \exp(-i\mathcal{H}t) | q_I \rangle$ into the following:

$$\left\langle q_F \,|\, e^{-i\mathcal{H}t} \,|\, q_I \right\rangle$$
$$= \left(\prod_{j=1}^{N-1} \int dq_j \right) \left\langle q_F \,|\, e^{-i\mathcal{H}\delta t} \,|\, q_{N-1} \right\rangle \left\langle q_{N-1} \,|\, e^{-i\mathcal{H}\delta t} \,|\, q_{N-2} \right\rangle \ldots \left\langle q_2 \,|\, e^{-i\mathcal{H}\delta t} \,|\, q_1 \right\rangle \left\langle q_1 \,|\, e^{-i\mathcal{H}\delta t} \,|\, q_I \right\rangle. \qquad \text{(II.28)}$$

The simplest case is the free-particle Hamiltonian $\mathcal{H} = \mathbf{p}^2/(2m)$, where **p** is the momentum operator. Noting that $\mathbf{p}\,|p\rangle = p\,|p\rangle$ and $\int (dp/2\pi)\, |p\rangle \langle p| = 1$, we can compute $\langle q_{j+1} |\exp(-i\mathcal{H}\delta t)| q_j \rangle$ as follows:

$$\left\langle q_{j+1} \right| \exp\left(-i\delta t \frac{\mathbf{p}^2}{2m} \right) \left| q_j \right\rangle = \int \frac{dp}{2\pi} \left\langle q_{j+1} \right| \exp\left(-i\delta t \frac{\mathbf{p}^2}{2m} \right) \left| p \right\rangle \left\langle p \middle| q_j \right\rangle = \int \frac{dp}{2\pi} e^{-i\delta t\,(p^2/2m)} \left\langle q_{j+1} \middle| p \right\rangle \left\langle p \middle| q_j \right\rangle$$
$$= \int \frac{dp}{2\pi} \exp\left(-i\delta t \frac{p^2}{2m} \right) \exp\left[-ip\left(q_{j+1} - q_j \right) \right] = \left(\frac{-im}{2\pi\delta t} \right)^{\frac{1}{2}} \exp\left(i\delta t \left(\frac{m}{2} \right) \left[\frac{(q_{j+1} - q_j)}{\delta t} \right]^2 \right). \qquad \text{(II.29)}$$

In the above derivation we have used the identities $\langle q \,|\, p \rangle = e^{ipq}$ and

$$\int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a} \right)^{1/2} \qquad \text{(II.30)}$$

Therefore $\langle q_I \,|\, \exp(-i\mathcal{H}t)\, |q_F\rangle$ becomes:

$$\left\langle q_F \left| e^{-i\mathcal{H}t} \right| q_I \right\rangle = \left(\frac{-im}{2\pi\delta t} \right)^{\frac{N}{2}} \left(\prod_{j=1}^{N-1} \int dq_j \right) \exp\left(i\delta t \left(\frac{m}{2} \right) \sum_{j=0}^{N-1} \left[\frac{(q_{j+1} - q_j)}{\delta t} \right]^2 \right), \qquad \text{(II.31)}$$

where we have taken $q_I = q_0$ and $q_F = q_N$. In the $\delta t \to 0$ limit, $[(q_{j+1} - q_j)/\delta t]^2 \to \dot{q}^2$, and $\delta t \sum_j \to \int_0^t dt$. If we further define the integral over paths as

$$\int Dq(t) = \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t} \right)^{\frac{N}{2}} \left(\prod_{j=1}^{N-1} \int dq_j \right), \qquad \text{(II.32)}$$

we arrive at the path integral representation for free particles:

$$\left\langle q_F \,|\, e^{-i\mathcal{H}t} \,|\, q_I \right\rangle = \int Dq(t') \exp\left(i\int_0^t dt' \frac{1}{2} m\dot{q}^2 \right). \qquad \text{(II.33)}$$

The above result can be generalized to a Hamiltonian for a particle in a potential $\mathcal{V}(\mathbf{q})$, so that $\mathcal{H} = \mathbf{p}^2/(2m) + \mathcal{V}(\mathbf{q})$, and

$$\left\langle q_F \,|\, e^{-i\mathcal{H}t} \,|\, q_I \right\rangle = \int Dq(t')\exp\left(i\int_0^t dt'\left[\frac{1}{2}m\dot{q}^2 - \mathcal{V}(q)\right]\right) = \int Dq(t')\exp\left(i\int_0^t dt'\mathcal{L}(\dot{q},q)\right), \tag{II.34}$$

where $\mathcal{L}(\dot{q},q)$ denotes the Lagrangian of the system.

In most of the problems that interest us, we want to consider the amplitude between an initial state $|I\rangle$ and a final state $|F\rangle$ rather than between the initial and the final positions $|q_I\rangle$ and $|q_F\rangle$. We therefore rewrite EQ. (II.34) into the following form:

$$\left\langle F|\, e^{-i\mathcal{H}t}\, |I\right\rangle = \int dq_F \int dq_I \left\langle F\,|\, q_F\right\rangle\left\langle q_F|\, e^{-i\mathcal{H}t}\,|\, q_I\right\rangle\left\langle q_I|\, I\right\rangle = \int dq_F \int dq_I\, \Psi_F(q_F)^*\left\langle q_F|\, e^{-i\mathcal{H}t}\,|\, q_I\right\rangle \Psi_I(q_I)\,, \tag{II.35}$$

where Ψ_I and Ψ_F denote the Schrödinger's wavefunctions for the initial and final states, respectively.

Next, we consider the amplitude $Z \equiv \langle 0|\exp(-i\mathcal{H}t)\,|0\rangle \equiv \int Dq\, e^{iS(q)}$ evaluated at the ground state of an N-particle system, where $S(q)$ denotes the action

$$S(q) = \int_0^t dt'\left[\sum_{a=1}^{N}\frac{1}{2}m_a\dot{q}_a{}^2 - \mathcal{V}(q_1,q_2,\ldots,q_N)\right] = \int_0^t dt'\,\mathcal{L}\left[\dot{q}_a(t'),q_a(t')\right], \tag{II.36}$$

and $\mathcal{V}(q_1, q_2,\ldots, q_N)$ is the potential energy, which includes the interaction energy among particles. We may take the continuum limit so that the discreteness of particles is replaced by a four-dimensional variable $x \equiv (t, \boldsymbol{x})$, and the discrete coordinates and momentum of particles by the field $\varphi(x)$ and its space-time derivatives $\partial_\mu\varphi(x)$, where $\partial_\mu \equiv (\partial/\partial t, \partial/\partial x^i)$ and the superscript index i runs through three spatial dimensions. The action S is now given by a 4-dimensional integral over the Lagrangian density $L(\varphi(x),\partial_\mu\varphi(x))$: $S = \int d^4x\, L(\varphi, \partial_\mu\varphi)$, so that $Z = \langle 0|\exp(-i\mathcal{H}t)\,|0\rangle$ becomes (after restoring the Planck constant)

$$Z = \int D\varphi\ \exp\left[\frac{i}{\hbar}\int d^4x\, L(\varphi,\partial_\mu\varphi)\right] = \int D\varphi\ \exp\left[iS(\varphi,\partial_\mu\varphi)\right]. \tag{II.37}$$

Using the Euler-Lagrangian variational procedure by minimizing the action and also integration by parts:

$$\delta S = \int d^4x\left[\frac{\delta L}{\delta\varphi}\delta\varphi + \frac{\delta L}{\delta\partial_\mu\varphi}\delta\partial_\mu\varphi\right] = \int d^4x\left[\left(\frac{\delta L}{\delta\varphi} - \partial_\mu\frac{\delta L}{\delta\partial_\mu\varphi}\right)\delta\varphi\right] = 0\,,$$

we obtain the equation of motion:

$$\partial_\mu\frac{\delta L}{\delta\partial_\mu\varphi} - \frac{\delta L}{\delta\varphi} = 0\,, \tag{II.38}$$

which is consistent with the classical field equation.

To make things more interesting than simply observing particles moving and interacting in vacuum, we introduce the source function $J(t,\boldsymbol{x}) = J(x)$ so that a term $J(x)\varphi(x)$ is added to the Lagrangian density. The generalized path integral becomes

$$Z = \int D\varphi \ \exp\left\{ i\int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + J(x)\varphi(x) \right] \right\}. \tag{II.39}$$

In general the functional integral in EQ. (II.39) cannot be done analytically unless the Lagrangian density has the following form:

$$L(\varphi, \partial_\mu \varphi) = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2 \right]. \tag{II.40}$$

This is known as the free or Gaussian theory. As mentioned previously, the equation of motion for the Gaussian theory (i.e. applying EQ. (II.38) to the Lagrangian density in EQ. (II.40)) yields the Klein-Gordon equation:

$$(\partial^2 + m^2)\varphi = 0. \tag{II.41}$$

Now let's evaluate the path integral in EQ. (II.39) within the Gaussian theory. If we assume that the fields that we are interested fall off rapidly at infinity, we can integrate EQ. (II.39) by parts under $\int d^4x$ and rewrite it into a more "friendly" expression

$$Z = \int D\varphi \ \exp\left\{ i\int d^4x \left[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J(x)\varphi(x) \right] \right\}. \tag{II.42}$$

Using the identity

$$\int_{-\infty}^{+\infty} dx \ e^{-\frac{1}{2}ax^2 + Jx} = \left(\frac{2\pi}{a} \right)^{1/2} e^{J^2/(2a)} \tag{II.43}$$

or its generalized form:

$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 ... \int_{-\infty}^{+\infty} dx_N \ \exp\left[-\frac{1}{2}(x \bullet A \bullet x) + J \bullet x \right] = \left(\frac{(2\pi)^N}{\det[A]} \right)^{1/2} \exp\left[\frac{1}{2}(J \bullet A^{-1} \bullet J) \right], \tag{II.44}$$

where A_{ij} is a real symmetric $N \times N$ matrix, we can treat the differential operator $-(\partial^2 + m^2)$ as the matrix A and rewrite EQ. (II.42) into

$$Z(J) = Z(J=0) \ \exp\left[-\frac{i}{2} \iint d^4x \, d^4y \, J(x) D(x-y) J(y) \right] \equiv Z(J=0) \, e^{iW(J)}, \tag{II.45}$$

$$W(J) \equiv -\frac{1}{2} \iint d^4x \, d^4y \, J(x) D(x-y) J(y), \tag{II.46}$$

where $D(x-y)$ is taken as the continuum limit of $(A^{-1})_{jk}$. In other words, the differential operator $-(\partial^2 + m^2)$ takes the place of the matrix A and we want to find the expression for the inverse of the differential operator. In the discrete case we have $A \bullet A^{-1} = I$ where I is an $N \times N$ unit matrix, or equivalently, $(A)_{ij}(A^{-1})_{jk} = \delta_{ik}$. In the continuum limit, the inverse of the differential operator should involve two variables x and y. Moreover, it is easy to argue that only the difference of x and y is relevant to the physics under consideration. Therefore we find that the inverse of $-(\partial^2 + m^2)$ can be represented by $D(x-y)$ and it satisfies the differential equation:

$$-(\partial^2 + m^2) D(x-y) = \delta^{(4)}(x-y), \tag{II.47}$$

and $\delta^{(4)}(x-y)$ is the four-dimensional Dirac delta function. Here the function $D(x)$ is known as the free-particle propagator, and its physical meaning will become clear in our later discussion. In fact, the propagator is directly related to the Green's function of differential equations, similar to what you have encountered in the course of electromagnetism. The solution for $D(x-y)$ in EQ. (II.47) can be obtained by expressing the Dirac delta function $\delta^{(4)}(x-y)$ in terms of its Fourier transformation

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)},$$

so that

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon}, \qquad (\varepsilon > 0). \tag{II.48}$$

Equation (II.48) can be evaluated by first integrating over k^0. Noting that $k^2 = (k^0)^2 - (\mathbf{k})^2$, there are two poles in the complex k^0-plane at $\pm\sqrt{\omega_k^{\ 2} - i\varepsilon}$, where $\omega_k \equiv \sqrt{\mathbf{k}^2 + m^2}$. For $\varepsilon \to 0$, the poles become $+[\omega_k - i\varepsilon/(2\omega_k)] \to (+\omega_k - i\varepsilon)$ and $-[\omega_k - i\varepsilon/(2\omega_k)] \to (-\omega_k + i\varepsilon)$ because a small number ε multiplied by any finite positive number is still ε. If $x^0 > 0$, the integration of k^0 from $-\infty$ to $+\infty$ is performed in the upper half-plane so that it picks up the pole at $(-\omega_k + i\varepsilon)$, yielding

$$\lim_{\varepsilon\to 0} D(x^0 > 0) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp\left[-i(\omega_k x^0 + \mathbf{k}\bullet\mathbf{x})\right]. \tag{II.49}$$

Similarly, for $x^0 < 0$, the integration of k^0 from $-\infty$ to $+\infty$ is performed in the lower half-plane so that it picks up the pole at $(+\omega_k - i\varepsilon)$, yielding

$$\lim_{\varepsilon\to 0} D(x^0 < 0) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp\left[i(\omega_k x^0 - \mathbf{k}\bullet\mathbf{x})\right]. \tag{II.50}$$

Noting that $x^0 = t$, we can consolidate the solution for $D(x)$ in EQs. (II.49) and (II.50) into a more concise form as follows:

$$D(t,\mathbf{x}) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ \exp\left[-i(\omega_k t + \mathbf{k}\bullet\mathbf{x})\right]\theta(t) + \exp\left[i(\omega_k t - \mathbf{k}\bullet\mathbf{x})\right]\theta(-t) \right\}. \tag{II.51}$$

For a point within the light cone $x = (t > 0, 0)$,

$$D(x) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[\exp\left(-i\omega_k t\right)\right], \tag{II.52}$$

which is a superposition of plane waves and is oscillatory in time. In contrast, for a point outside of the light cone, such as for $x = (t = 0, \mathbf{x} \neq 0)$,

$$D(x) = -i\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + m^2}} \left[\exp\left(-i\mathbf{k}\bullet\mathbf{x}\right)\right] \sim \exp\left(-m|\mathbf{x}|\right), \tag{II.53}$$

so that the propagator decays exponentially outside of the light cone. This result differs from classical physics where particles cannot propagate outside of the light cone, whereas quantum fields can leak out of the light cone over a length scale of m^{-1}.

[Free particle Green's function]

From EQs. (II.45) and (II.48), we may rewrite the $Z(J)$ into the following form:

$$Z(J) = Z(0)\left\{1 + \left(\frac{-i}{2}\right)\int dx\, dy\, J(x) D(x-y) J(y) + \frac{1}{2!}\left(\frac{-i}{2}\right)^2 \left[\int dx\, dy\, J(x) D(x-y) J(y)\right]^2 \right.$$
$$\left. + \frac{1}{3!}\left(\frac{-i}{2}\right)^3 \left[\int dx\, dy\, J(x) D(x-y) J(y)\right]^3 + \ldots \right\}.$$
$$= \sum_{n=0}^{\infty} \frac{i^n}{n!}\int dx_1 \ldots dx_n\, J(x_1)\ldots J(x_n)\, \tau(x_1,\ldots,x_n), \qquad \text{(II.54)}$$

where $\tau(x_1,\ldots,x_n)$ is an "n-point Green's function" given by

$$\tau(x_1,\ldots,x_n) = \frac{1}{i^n}\frac{\delta^n Z(J)}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0}. \qquad \text{(II.55)}$$

Therefore, $Z(J)$ is the generating functional for $\tau(x_1,\ldots,x_n)$, which has the physical meaning of the free particle Green's functions if n is even. Specifically, the two-point function may be considered as a single-particle Green's function, which is given by

$$\tau(x,y) = -\frac{\delta^2 Z(J)}{\delta J(x)\delta J(y)}\bigg|_{J=0} = \frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}\exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right]_{J=0}$$
$$= \frac{1}{i}\frac{\delta}{\delta J(x)}\left\{-\int d^4x_1\, D(y-x_1) J(x_1)\exp\left[-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1) D(x_1-x_2) J(x_2)\right]\right\}_{J=0}$$
$$= iD(x-y)\exp\left[-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1) D(x_1-x_2) J(x_2)\right]_{J=0}$$
$$+\left\{\int d^4x_1\, D(x-x_1) J(x_1)\int d^4x_1\, D(y-x_1) J(x_1)\exp\left[-\frac{i}{2}\int d^4x_1\, d^4x_2\, J(x_1) D(x_1-x_2) J(x_2)\right]\right\}_{J=0}$$
$$= iD(x-y). \qquad \text{(II.56)}$$

[Wick's theorem]

Using EQs. (II.48) and (II.55), it can be shown that the 3-point function, or more generally, any l-point function with l being an odd integer, always vanishes. That is,

$$\tau(x_1,\ldots,x_{2n+1}) = 0. \qquad \text{(II.57)}$$

On the other hand, any $2n$-point function can be expressed as the sum of products of 2-point functions given in EQ. (II.56):

$$\tau(x_1,\ldots,x_{2n}) = \frac{1}{2n!!}\sum_{\sigma\in S_{2n}}\tau\left(x_{\sigma(1)},x_{\sigma(2)}\right)\ldots\tau\left(x_{\sigma(2n-1)},x_{\sigma(2n)}\right), \qquad \text{(II.58)}$$

where S_{2n} represents the permutation group of $2n$ elements, and $2n!! = 2n\,(2n-2)\,(2n-4)\ldots 6\bullet 4\bullet 2$.

The results given in EQs. (II.57) and (II.58) are known as the *Wick's theorem*, which will be quite useful in our later studies of the quantum field theory. This theorem can be better understood if we consider the special case of EQ. (II.30): By operating $-2(d/da)$ repeatedly on EQ. (II.30) n times, we obtain

$$\left\langle x^{2n}\right\rangle \equiv \frac{\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}ax^2}} = \frac{1}{a^n}(2n-1)!! = \frac{1}{a^n}(2n-1)(2n-3)\ldots\cdot 5\cdot 3\cdot 1. \qquad \text{(II.59)}$$

This expression can be generalized to the cases with a replaced by a real symmetric $N \times N$ matrix A such that

$$\left\langle x_i x_j \ldots x_k x_l\right\rangle \equiv \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N\, e^{-\frac{1}{2}x\bullet A\bullet x} x_i x_j \ldots x_k x_l}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_N\, e^{-\frac{1}{2}x\bullet A\bullet x}}. \qquad \text{(II.60)}$$

Using Wick's theorem in EQ. (II.58), you can easily show that

$$\left\langle x_i x_j\right\rangle = \left(A^{-1}\right)_{ij} \quad \text{and} \quad \left\langle x_i x_j x_k x_l\right\rangle = \left(A^{-1}\right)_{ij}\left(A^{-1}\right)_{kl} + \left(A^{-1}\right)_{ik}\left(A^{-1}\right)_{jl} + \left(A^{-1}\right)_{il}\left(A^{-1}\right)_{jk}.$$

[Spin 0 and 1 bosons in strong and electromagnetic interactions]

The propagator that we have derived so far is associated with spin 0 mesons, so that the field only has one-degree of freedom. (Recall from your quantum mechanics that the degree of freedom is given by $2s+1$, where s denotes the spin of the particles.) In the following we shall investigate the forces incurred by particles of spin 0 and 1.

We have considered the path integral $Z(J) = Z(0)\, e^{iW(J)}$ in EQ. (II.45) for spin-0 particles in the free theory, where $W(J)$ is given by EQ. (II.46). For real $J(x)$, its Fourier transformation $J(k) \equiv \int d^4x\, e^{-ikx} J(x)$ satisfies the condition $J(k)^* = J(-k)$, so that $W(J)$ can be expressed in terms of the Fourier transformation

$$W(J) \equiv -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J(k). \qquad \text{(II.61)}$$

Suppose that we introduce two sources concentrated in two local regions 1 and 2 in space-time so that $J(x) = J_1(x) + J_2(x)$. There are four terms of the form $J_1^*J_1$, $J_2^*J_2$, $J_1^*J_2$, and $J_2^*J_1$ in EQ. (II.61), and we are interested in the behavior of the cross terms because we want to ask what quantum fluctuations in the field φ would do to the two localized sources J_1 and J_2. One of the cross terms is

$$W_{12}(J) \equiv -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J_2(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_1(k). \qquad \text{(II.62)}$$

Equation (II.62) indicates that $W_{12}(J)$ peaks at $k^2 = m^2$, which can be interpreted as a source initially in region 1 in space-time sends out a disturbance in the field, which is subsequently absorbed by a sink in region 2 in space-time, and the disturbance is a particle of mass m.

To obtain further insights, let's consider a simplified case of two static (*i.e.*, time-independent) and infinitely sharp disturbances in regions 1 and 2, such that $J_{1,2}(x) = \delta^{(3)}(\mathbf{x} - \mathbf{x}_{1,2})$. Noting that

$$J_1(k) = \int dx^0\left[\exp\left(-ik^0x^0\right)\right]\left[\exp\left(\mathbf{k}\bullet\mathbf{x}_1\right)\right] \quad \text{and} \quad J_2(k) = \int dy^0\left[\exp\left(-ik^0y^0\right)\right]\left[\exp\left(\mathbf{k}\bullet\mathbf{x}_2\right)\right],$$

the sum of the cross terms becomes:

$$W_{12}(J)+W_{21}(J)\equiv-\iint dx^0dy^0\int\frac{dk^0}{2\pi}\exp\left[ik^0(y^0-x^0)\right]\int\frac{d^3k}{(2\pi)^3}\frac{\exp\left[i\mathbf{k}\bullet(\mathbf{x}_1-\mathbf{x}_2)\right]}{k^2-m^2+i\varepsilon},$$
$$=\int dx^0\int\frac{d^3k}{(2\pi)^3}\frac{\exp\left[i\mathbf{k}\bullet(\mathbf{x}_1-\mathbf{x}_2)\right]}{\mathbf{k}^2+m^2-i\varepsilon}\equiv\tilde{W}(J)=(t)\int\frac{d^3k}{(2\pi)^3}\frac{\exp\left[i\mathbf{k}\bullet(\mathbf{x}_1-\mathbf{x}_2)\right]}{\mathbf{k}^2+m^2-i\varepsilon}. \qquad \text{(II.63)}$$

(*Hint:* To do the above integration, you may first integrate over dy^0, which picks up a delta function $\delta(k^0)$, and therefore you'll be left with integration over dx^0 and d^3k only.) Consequently, the relevant path integral is (taking the limit $\varepsilon\to 0^+$):

$$\tilde{Z}(J)\equiv\left\langle 0\,|\exp(-i\tilde{H}t)\,|\,0\right\rangle=\tilde{Z}(J=0)\exp[i\tilde{W}(J)]=\exp(-iEt),$$
$$E=-\int\frac{d^3k}{(2\pi)^3}\frac{\exp\left[i\mathbf{k}\bullet(\mathbf{x}_1-\mathbf{x}_2)\right]}{\mathbf{k}^2+m^2}=-\frac{1}{4\pi r}\exp(-mr). \qquad \text{(II.64)}$$

Equation (II.64) indicates that the energy associated with the two sources is negative, so that the force between the two sources is *attractive*. Moreover, the characteristic length for the attractive force generated by the field φ is of the order of m^{-1}. This is a very important result first derived by Yukawa, who proposed that the attraction between nucleons in the atomic nucleus is due to their coupling to a field like the φ field described here. The massive spin-0 particle associated with this field within a nucleus is now called the π meson or simply the pion. Yukawa was able to correctly predict the mass of the π meson through the known range of the nuclear force.

The result obtained in Eq. (II.64) suggests that the exchange of a particle (or a scalar field) of mass m between two sources can produce a force. Moreover, for $m\to 0$, we find that the energy decays with inverse distance, and the force decays with r^{-2}, which is the famous r^{-2} law for both Coulomb and gravitational forces! We shall examine the electromagnetic force in a moment.

There is another important piece of hidden information associated the functional integral $Z(J)$. If we perform the Taylor's expansion for $Z(J)$, we find that there are infinite terms associated with it:

$$Z(J)=Z(J=0)\exp[iW(J)]=Z(J=0)\sum_{n=0}^{\infty}\frac{[iW(J)]^n}{n!}. \qquad \text{(II.65)}$$

As we have seen earlier that the first order term represents the propagation from x_1 to x_2. Similarly, the second order term represents the propagation from x_1 to x_2 and from x_3 to x_4:

$$\frac{1}{2!}(\frac{-i}{2})^2\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4\,D(x_1-x_2)\,D(x_3-x_4)\,J(x_1)\,J(x_2)\,J(x_3)\,J(x_4).$$

This process is said to be "disconnected" – the propagation from x_1 to x_2 and from x_3 to x_4 are independent. So we have seen that within the Gaussian theory, the propagators (or equivalently the φ field) do not interact. As we shall see later in the discussion of Feynman diagrams, additional terms must be introduced into the Gaussian Lagrangian to produce "connected" terms that indicate interaction of the propagators.

Next, we proceed to consider quantum field theory for electromagnetism, known as quantum electrodynamics or simply QED. Although we all know that photons are massless, before we introduce the concepts of gauge invariance, it will be much more convenient if we first take the particles (i.e. photons in this case) as massive spin 1 mesons, or, vector mesons, and then set the mass to zero in the end of our calculations. (Remember that there are $2s$+1 degrees of freedom for the particle. Since photons have three polarizations, they are spin 1 vector mesons.)

Having decided on the particle degrees of freedom, the next task is to set up the Lagrangian density for QED. We first recall that the Maxwell's Lagrangian density for electromagnetism in the absence of sources can be given in a very concise form $L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A(x) = (\phi(x), \boldsymbol{A}(x))$ is the four-vector vector potential. (You are reminded that the magnetic field is given by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the electric field $\boldsymbol{E} = -\nabla\phi(x) + \partial\boldsymbol{A}/\partial t$, and the energy associated with electromagnetism contains terms associated with B^2, $\boldsymbol{E} \bullet \boldsymbol{B}$ and E^2.) Now if we introduce a source $J^\mu(x)$, where $J^0(x)$ represents an electrical charge and $J^i(x)$ an electrical current, and assume a small mass for photons, our Lagrangian density becomes $L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu$, where the term $\frac{1}{2} m^2 A_\mu A^\mu$ is analogous to the $m^2\varphi^2$ term in the scalar field Lagrangian that we have considered earlier. Hence, the field theory of the vector meson is now determined by the path integral $Z(J) = \int DA \exp[iS(A)] \equiv Z(J=0)\exp[iW(J)]$, with the action:

$$S(A) = \int d^4x\, L = \int d^4x \left\{ \frac{1}{2} A_\mu \left[\left(\partial^2 + m^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu + A_\mu J^\mu \right\}, \tag{II.66}$$

where $g^{\mu\nu}$ is the metric such that $A_\mu = g_{\mu\nu} A^\nu$, and in flat space we replace $g^{\mu\nu}$ by $\eta^{\mu\nu}$:

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The term $\frac{1}{2} g^{\mu\nu} A_\mu \partial^2 A_\nu$ in EQ. (II.66) is obtained through integration of the term $-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ by parts over space-time. Our objective is to express the action in a form that is quadratic in the vector potential plus a term linear in the vector potential that presents its coupling to the source. Given EQ. (II.66), we want to find the solution to a propagator in the vector meson field, $D_{\nu\lambda}(x)$:

$$\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] D_{\nu\lambda}(x) = \delta^\mu_\lambda \delta^{(4)}(x). \tag{II.67}$$

Solving EQ. (II.67) using the relations $[D_{\mu\nu}]^{-1} D_{\nu\lambda} = \delta_{\mu\lambda}$, $\delta^{(4)}(x) = \int [d^4k/(2\pi)^4] \exp(ikx)$, and $D_{\nu\lambda}(x) = \int [d^4k/(2\pi)^4]\, D_{\nu\lambda}(k) \exp(ikx)$, we arrive at $[-(k^2 - m^2) g^{\mu\nu} + k^\mu k^\nu] D_{\nu\lambda}(k) = \delta^\mu_\lambda$ and

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + \left(k_\nu k_\lambda / m^2\right)}{k^2 - m^2}. \tag{II.68}$$

Equation (II.68) may be obtained by directly inverting the (4 × 4) matrix $D^{-1}(k)$, or by noting that D and D^{-1} are both symmetric in their indices and verifying the following:

$$D_{\mu\nu}(k)\left[D^{-1}(k)\right]^{\nu\lambda} = \left[D^{-1}(k)\right]^{\lambda\nu} D_{\nu\mu}(k)$$
$$= -\left[-\left(k^2 - m^2\right) g^{\lambda\nu} + k^\lambda k^\nu \right] \frac{\left(g_{\nu\mu} - \dfrac{k_\nu k_\mu}{m^2} \right)}{\left(k^2 - m^2\right)} = \frac{\left(k^2 - m^2\right)\delta^\lambda_\mu - k^\lambda k_\mu - \dfrac{\left(k^2 - m^2\right)}{m^2} k^\lambda k_\mu + \dfrac{k^2}{m^2} k^\lambda k_\mu}{\left(k^2 - m^2\right)} = \delta^\lambda_\mu .$$

The physical significance of EQ. (II.68) is that it represents the massive vector meson propagator. Moreover, we find that

$$W(J) \equiv -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* D_{\mu\nu}(k) J^\nu(k) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + \left(k_\mu k_\nu / m^2\right)}{k^2 - m^2 + i\varepsilon} J^\nu(k). \tag{II.69}$$

Equation (II.69) can be further simplified by the condition of current conservation $\partial_\mu J^\mu(x) = 0$, which yields $k_\mu J^\mu(k) = 0$. Therefore we can drop the term $[J^\mu(k)^* (k_\mu k_\nu / m^2) J^\nu(k)]$ in EQ. (II.69), and arrive at

$$W(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_\mu(k) . \tag{II.70}$$

Comparing EQ. (II.70) for a vector meson with EQ. (II.62) for the scalar meson, we notice a difference sign, implying that the potential energy associated with two localized positive charges $J^0(x)$ is positive. We can now safely take the $m \to 0$ limit for photons, thanks to current conservation, and following similar steps as outlined before from EQ. (II.63) to EQ. (II.64), we obtain the potential energy for like charges as

$$\lim_{m\to 0} E = \frac{1}{4\pi r}\exp(-mr) \quad \to \quad E = \frac{1}{4\pi r} . \tag{II.71}$$

The force derived from EQ. (II.71) is consistent with the inverse squared distance law ($1/r^2$) of the Coulomb interaction between point charges. Thus, we have used relatively simple field theory arguments to derive the signs of interaction for the hardronic strong interaction of spin-0 mesons and the electromagnetic interaction of spin-1 photons. This derivation can be generalized from (3+1)-dimensions to (D+1)-dimensions, which will be left for you to work out.

Having introduced the basics of path integral formalisms and concepts of propagators, we are ready to discuss Feynman diagrams and Feynman's rules for interacting systems in Part II.3.

II.3. Feynman Diagrams and Feynman's Rules

What we have learnt in Part II.2 has been restricted to the Gaussian theory so that the path integrals are calculated under the harmonic approximation. While we have found interactions between two "lumps of sources" in our space-time through the exchange of particles such as the pions and photons, we have essentially assume that these particles (or modes) are independent and do not interact with each other, and therefore we have called such approximations as the free field theory, and the propagators thus derived in space-time do not cross. If we want to consider interactions among the modes, such as the modes scattering off each other, we must introduce into the Lagrangian anharmonic terms so that the equation of motion is no longer linear. For instance, we can add one anharmonic term like $(-\lambda\varphi^4/4!)$ to our free field theory, so that for the massive spin-0 field, we want to evaluate the following path integral:

$$Z(J,\lambda) = \int D\varphi \ \exp\left\{ i\int d^4x \left(\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \frac{\lambda}{4!}\varphi^4 + J\varphi \right) \right\} . \tag{II.72}$$

However, the addition of such terms will make the path integrals not solvable unless we perform perturbative calculations. A convenient way to perturbatively calculate the path integral of a system with interacting modes is to use the Feynman diagrams, as we shall study in this section. We shall begin with the consideration of Feynman diagrams and Feynman's rules for the simpler case of bosons. The related topics for fermions will be discussed later after we introduce the Dirac equations.

[Generating functionals for interacting fields of scalar bosons – general consideration]

We first consider the generating functionals for a general interaction Lagrangian density L_{int} of a scalar boson field φ, $L(\varphi) = L_0(\varphi) + L_{\text{int}}(\varphi)$, where we define the Lagrangian density and the action as

$$L_0(\varphi) = \frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2\right), \tag{II.73}$$

$$S = \int d^4x\, L(\varphi) = \int d^4x \left[\frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi - m^2\varphi^2\right) + L_{\text{int}}\right]$$
$$= -\int d^4x \left[\frac{1}{2}\varphi\left(\partial^2 + m^2\right)\varphi - L_{\text{int}}\right]. \tag{II.74}$$

The generating function $Z(J)$ is given by:

$$Z(J) = \frac{\int D\varphi \exp\left[iS + i\int d^4x\, J(x)\varphi(x)\right]}{\int D\varphi \exp(iS)}. \tag{II.75}$$

In the absence of interaction, *i.e.* $L_{\text{int}}(\varphi) = 0$, the generating function becomes $Z_0(J)$:

$$Z_0(J) = \int D\varphi \exp\left\{i\int d^4x\left[L_0(\varphi) + J(x)\varphi(x)\right]\right\} \Big/ \int D\varphi \exp\left[i\int d^4x\, L_0(\varphi)\right],$$
$$= \int D\varphi \exp\left\{i\int d^4x\left[-\frac{1}{2}\varphi\left(\partial^2 + m^2\right)\varphi + J(x)\varphi(x)\right]\right\} \Big/ \int D\varphi \exp\left[i\int d^4x\, L_0(\varphi)\right],$$
$$= \exp\left\{-\frac{i}{2}\int d^4x\, d^4y \left[J(x) D(x-y) J(y)\right]\right\}. \tag{II.76}$$

where $D(x-y)$ is the propagator defined in EQ. (II.48).

We shall prove in the following that for a given interaction L_{int}, a general relation exists between the interacting and non-interacting generating functionals $Z(J)$ and $Z_0(J)$, and the relation is given by

$$Z(J) = N \exp\left[i\int d^4x\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right] Z_0(J), \tag{II.77}$$

where N denotes the normalizing factor. We shall then apply EQ. (II.75) to the special case of the φ^4-theory.

To prove EQ. (II.77), we begin by considering the following derivative using EQ. (II.76):

$$\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z_0(J) = -\int d^4y\, D(x-y) J(y) \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right], \tag{II.78}$$

$$\Rightarrow \quad \left(\partial^2 + m^2\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z_0(J) = J(x) Z_0(J). \tag{II.79}$$

Equation (II.79) is the differential equation for the non-interacting generating functional. To find the corresponding differential equation for the interacting generating functional, we use EQ. (II.75) and consider the following derivative

$$\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z(J) = \frac{\int D\varphi \left\{\varphi(x) \exp\left[iS + i\int d^4x\, J(x)\varphi(x)\right]\right\}}{\int D\varphi \exp(iS)}. \tag{II.80}$$

For convenience, we define another functional:

$$\hat{Z}(\varphi) \equiv \frac{\exp(iS)}{\int D\varphi \exp(iS)}. \tag{II.81}$$

From EQs. (II.75) and (II.81), the derivative of $\hat{Z}(\varphi)$ relative to φ is given by

$$i\frac{\delta \hat{Z}(\varphi)}{\delta\varphi} = i\frac{\delta}{\delta\varphi}\frac{\exp\left\{-i\int d^4x\left[\frac{1}{2}\varphi\left(\partial^2+m^2\right)\varphi - L_{\text{int}}\right]\right\}}{\int D\varphi \exp(iS)} = \left(\partial^2+m^2\right)\varphi(x)\hat{Z}(\varphi) - \frac{\partial L_{\text{int}}(\varphi)}{\partial\varphi}\hat{Z}(\varphi). \tag{II.82}$$

If we multiply both sides of EQ. (II.82) by $\exp\left[i\int d^4x\, J(x)\,\varphi(x)\right]$ and integrate over the scalar field φ, the right-hand side becomes

$$\frac{\int D\varphi\left[\left(\partial^2+m^2\right)\varphi(x) - \frac{\delta L_{\text{int}}}{\delta\varphi}\right]\exp\left(iS + i\int d^4x\, J(x)\varphi(x)\right)}{\int D\varphi \exp(iS)} = \left(\partial^2+m^2\right)\frac{1}{i}\frac{\delta Z(J)}{\delta J} - \frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J\right)}Z(J), \tag{II.83}$$

where we have changed the argument of the derivative of L_{int} from φ to $(1/i)(\delta/\delta J)$ in EQ. (II.83). On the other hand, the left-hand side of EQ. (II.82) becomes

$$\begin{aligned} &i\int D\varphi \frac{\delta\hat{Z}(\varphi)}{\delta\varphi}\exp\left[i\int d^4x\, J(x)\varphi(x)\right] \\ &= i\exp\left[i\int d^4x\, J(x)\varphi(x)\right]\hat{Z}(\varphi)\Big|_{\varphi\to\infty} + \int D\varphi\, J(x)\hat{Z}(\varphi)\exp\left[i\int d^4x\, J(x)\varphi(x)\right] \\ &= J(x)Z(J). \end{aligned} \tag{II.84}$$

Equating EQs. (II.83) and (II.84), we obtain the differential equation for the interacting generating function $Z(J)$ as follows:

$$\left(\partial^2+m^2\right)\left(\frac{1}{i}\frac{\delta Z(J)}{\delta J(x)}\right) - \frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J\right)}Z(J) = J(x)Z(J). \tag{II.85}$$

In addition to the differential equations (II.79) and (II.85), we have the following commutation relation:

$$\left[J(x), \frac{1}{i}\frac{\delta}{\delta J(y)}\right] = i\delta(x-y), \tag{II.86}$$

by observing the functional analogue of

$$\left[x_i, \frac{1}{i}\frac{\partial}{\partial x_j}\right] = i\,\delta_{ij}.$$

Hence, we find from EQ. (II.86) that

$$\left[J(x), \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^n\right] = i\delta(x-y)\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1} + \frac{1}{i}\frac{\delta}{\delta J(y)}\left[J(x), \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1}\right]$$

$$= i\delta(x-y)\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1} + i\delta(x-y)\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1} + \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{2}\left[J(x),\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-2}\right]$$

$$= \cdots = i\delta(x-y)\, n\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1}, \qquad \text{(II.87)}$$

where we have repeatedly used the commutation relation $[A,BC]=[A,B]C+B[A,C]$.

Next, if we expand the interacting Lagrangian density L_{int} in its argument φ and then replace φ with $(1/i)(\delta/\delta J)$, we have

$$\left[J(x),\int d^4y\, L_{\text{int}}(\varphi)\right] = \left[J(x),\int d^4y \sum_{n=0}^{\infty}\frac{\varphi^n}{n!}\frac{\partial^n L_{\text{int}}(\varphi=0)}{\partial\varphi^n}\right]$$

$$\xrightarrow{\left(\varphi\to\frac{\delta}{i\delta J(y)}\right)} \left[J(x),\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] = i\frac{\delta L_{\text{int}}}{\delta\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}. \qquad \text{(II.88)}$$

Defining $A = -i\int d^4y\, L_{\text{int}}\left[\delta/\left(i\,\delta J(y)\right)\right]$ and $B = J(x)$, and using the fact that A commutes with $[A, B]$ from EQ. (II.88), we use the Hausdorff formula:

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots \qquad \text{(II.89)}$$

$$= B + [A,B] \qquad \text{if } A \text{ commutes with } [A, B].$$

Hence, we obtain the following identity:

$$\exp\left[-i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] J(x)\exp\left[i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] = J(x) - \frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J(x)\right)}. \qquad \text{(II.90)}$$

We may rearrange EQ. (II.90) by multiplying both sides by $N\exp\left\{i\int d^4y\, L_{\text{int}}\left[\delta/\left(i\,\delta J(y)\right)\right]\right\} Z_0(J)$, which yields:

$$NJ(x)\exp\left[i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] Z_0(J),$$

$$= N\exp\left[i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right]\left[J(x) - \frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J(x)\right)}\right] Z_0(J),$$

$$= N\exp\left[i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right]\left(\partial^2+m^2\right)\left(\frac{1}{i}\frac{\delta Z_0(J)}{\delta J(x)}\right) \qquad \text{(from EQ. (II.79))}$$

$$-N\frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J(x)\right)}\exp\left[i\int d^4y\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] Z_0(J),$$

$$= \left(\partial^2 + m^2\right)\left(\frac{1}{i}\frac{\delta Z(J)}{\delta J(x)}\right) - \frac{\delta L_{\text{int}}}{\delta\left(\delta/i\,\delta J(x)\right)} Z(J), \qquad \text{(from EQ. (II.85))}$$
$$= J(x) Z(J). \tag{II.91}$$

Canceling $J(x)$ from the first and the last lines of EQ. (II.91), we have thus proven EQ. (II.77).

[Application of Feynman diagrams and rules to the φ^4-theory]

Having derived the generating functionals for a general interaction Lagrangian density, we are ready to calculate the Green's functions in the interacting field case by perturbation theory. We begin by considering a simple case, the φ^4-theory, which corresponds to $L_{\text{int}} = (-\lambda\varphi^4/4!)$ where λ denotes the coupling constant. From EQ. (II.77), the normalized generating functional $\tilde{Z}(J)$ is given by

$$\tilde{Z}(J) \equiv Z(J)/Z(0) = [Z(0)]^{-1} \exp\left[i\int d^4z\, L_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right] \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right], \tag{II.92}$$

It is clear that the only way to treat EQ. (II.92) is as a power series in the coupling constant λ, which is equivalent to treating it via perturbation theory. Using $L_{\text{int}} = (-\lambda\varphi^4/4!)$ and replacing φ by $\left(-i\delta/\delta J\right)$, we may expand EQ. (II.92) to orders λ^0 and λ^1. The order λ^0 we simply recover the free particle generating functional $\tilde{Z}_0(J)$, whereas to order λ^1, we have the following terms:

$$\left(\frac{-i\lambda}{4!}\right)\int d^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right]$$
$$= \left(\frac{-i\lambda}{4!}\right)\int d^4z \left\{-3[D(0)]^2 + 6iD(0)\left[\int d^4x\, D(z-x) J(x)\right]^2 + \left[\int d^4x\, D(z-x) J(x)\right]^4\right\}$$
$$\times \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right]. \tag{II.93}$$

It is more convenient to express these terms diagrammatically. The free propagator $D(x-y)$ from x to y may be expressed as a line:

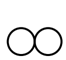
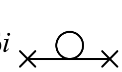
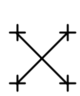
$$x \;\text{———}\; y \quad \Rightarrow \quad D(x-y). \tag{II.94}$$

Hence, $D(0)$ is expressed as a closed loop:

$$\bigcirc \quad \Rightarrow \quad D(0) \tag{II.95}$$

In addition, the external source (or sink) J is represented by a cross. Therefore, EQ. (II.93) becomes:

$$\left(\frac{-i\lambda}{4!}\right)\int d^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right]$$

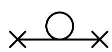
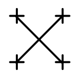
$= \left(\frac{-i\lambda}{4!}\right)\int d^4z \{-3$  $+ 6i$  $+$  $\}\exp\left[-\frac{i}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)\right]$. (II.96)

Similarly, it can be easily verified that the diagrammatic expression for $Z(0)$ is given by

$$Z(0)=\left\{\exp\left[i\int d^4z\left(\frac{-\lambda}{4!}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right]\exp\left[-\frac{i}{2}\int d^4x\,d^4y\,J(x)D(x-y)J(y)\right]\right\}_{J=0}$$

$$=1-\left(\frac{i\lambda}{4!}\right)\int d^4z\left[-3\right.$$  $$\left.\right] \qquad \text{(II.97)}$$

Combining EQs. (II.96) and (II.97), we can rewrite the normalized generating functional in EQ. (II.92) into the following form:

$$\tilde{Z}(J)=\left\{1-\left(\frac{i\lambda}{4!}\right)\int d^4z\left[6i\right.\right.$$  $$+$$  $$\left.\left.\right]\right\}\exp\left[-\frac{i}{2}\int d^4x\,d^4y\,J(x)D(x-y)J(y)\right]. \qquad \text{(II.98)}$$

To obtain a better sense for the generating functional and the diagrammatic expressions, we consider in the following the simplest case of (0+1)-dimension for the φ^4-theory. The generating functional becomes:

$$\begin{aligned}Z(J,\lambda)&=\int_{-\infty}^{+\infty}dq\;\exp\left(-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4+Jq\right)=\int_{-\infty}^{+\infty}dq\;\exp\left(-\frac{1}{2}m^2q^2+Jq\right)\exp\left(-\frac{\lambda}{4!}q^4\right)\\&=\int_{-\infty}^{+\infty}dq\;\left\{\exp\left(-\frac{1}{2}m^2q^2+Jq\right)\right\}\left[1+\left(\frac{-\lambda}{4!}\right)q^4+\frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2q^8+O(\lambda^3)\right]\\&=\left[1+\left(\frac{-\lambda}{4!}\right)\left(\frac{d}{dJ}\right)^4+\frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8+\ldots\right]\int_{-\infty}^{+\infty}dq\;\left\{\exp\left(-\frac{1}{2}m^2q^2+Jq\right)\right\}\\&=\exp\left[\frac{-\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right]\int_{-\infty}^{+\infty}dq\;\left\{\exp\left(-\frac{1}{2}m^2q^2+Jq\right)\right\}\\&=\left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}}\exp\left[\frac{-\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right]\exp\left(\frac{1}{2m^2}J^2\right).\end{aligned} \qquad \text{(II.99)}$$

We can further simplify the integral by suppressing the coefficient $(2\pi/m^2)^{1/2} = Z(J=0,\ \lambda=0) \equiv Z(0,0)$, so that we only consider $\tilde{Z}(J,\lambda)=Z(J,\lambda)/Z(0,0)$. The physical meaning of EQ. (II.99) can be understood in terms of its Taylor's expansion in powers of λ and J. As described earlier, each power of J represents an external source or sink, and each power of λ can be treated as representative of an interaction "vertex" of the propagators in the field due to the anharmonic contribution. Moreover, by comparing the simple (0+1)-dimensional problem in EQ. (II.99) with related (D+1)-dimensional problem that we have considered in Part II.2, we can associate each power of the coefficient $(1/2m^2)$ with a propagator. (Naturally for higher spatial dimensions the correct form of the propagator should be given by EQ. (II.48) so that the 4-momentum k is involved besides the mass.)

Let's consider a few examples. Suppose that we want to find the term of order λ and J^4 in EQ. (II.99). This term would involve

$$\left[\frac{-\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right]\left[\frac{1}{4!}\left(\frac{1}{2m^2}J^2\right)^4\right]=\frac{8!(-\lambda)}{(4!)^3(2m^2)^4}J^4 .$$

The physical significance of this term can be understood in terms of diagrams involving 4 external points (representing sources and sinks), one vertex of interaction, and 4 lines representing the propagators. The diagrams corresponding to such physical processes are illustrated in Figure II.3.1. Similarly, we can find out the coefficient associated with the term of the order of λ^2 and J^4 in EQ. (II.99) as:

$$\left[\frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8\right]\left[\frac{1}{6!}\left(\frac{1}{2m^2}J^2\right)^6\right]=\frac{12!(-\lambda)^2}{(2!)(4!)^3(6!)(2m^2)^6}J^4 ,$$

and the corresponding diagrams should involve 6 lines, 2 vertices, and 4 external points, as illustrated in Fig. II.3.2. As the third example, we find the coefficient associated with the term of the order of λ^2 and J^6 in EQ. (II.99) as:

$$\left[\frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8\right]\left[\frac{1}{7!}\left(\frac{1}{2m^2}J^2\right)^7\right]=\frac{14!(-\lambda)^2}{(2!)(4!)^2(6!)(7!)(2m^2)^7}J^6 .$$

The corresponding diagrams involve 7 lines, 2 vertices, and 6 external points, as shown in Fig. II.3.3.

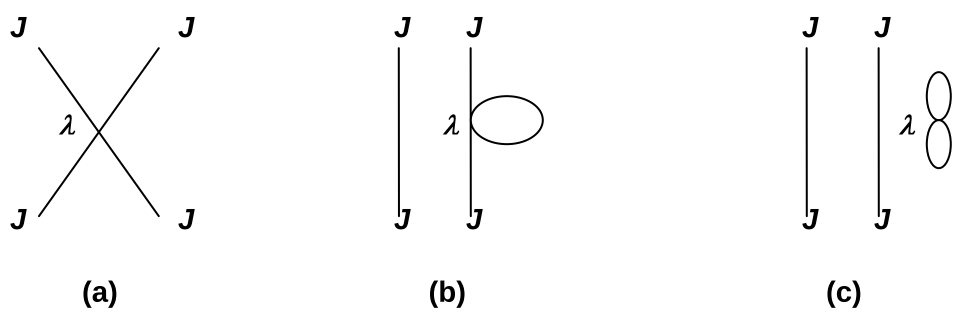


Figure II.3.1 Diagrams associated with the term $(1/m^2)^4\lambda J^4$.

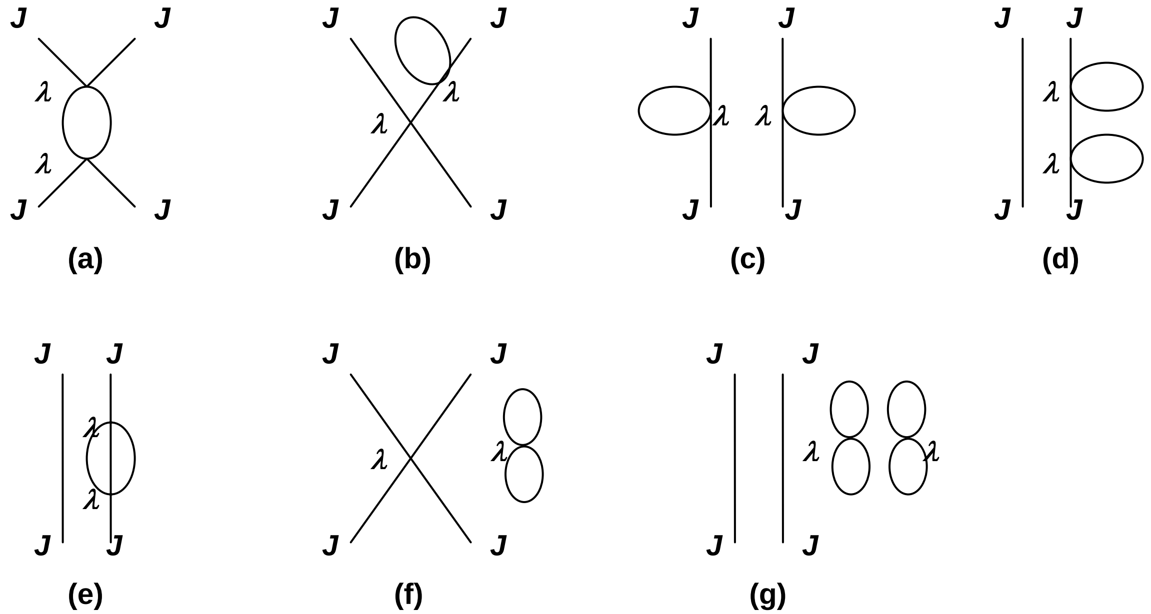


Figure II.3.2 Diagrams associated with the term $(1/m^2)^6\lambda^2J^4$.

There is in fact a well-defined relation between the powers of λ, J and $(1/m^2)$. If we denote the powers for λ, J and $(1/m^2)$ as N_λ, N_J and N_m, respectively, we find the relation $2N_m = 4\,N_\lambda + N_J$, because each line is associated with two ends, and the end points can be either vertices (with each vertex connecting to 4 end points) or external "legs". Moreover, it is clear from Figures II.3.1 – II.3.3 that there are connected graphs and disconnected graphs in the diagrams. The disconnected graphs do not have external legs, which imply that the corresponding terms do not contain any power in J. These graphs are associated with vacuum fluctuations and are not observables that involve interactions. If we rewrite the path integral into

$$Z(J,\lambda) = Z(J=0,\lambda)\exp\left[iW(J,\lambda)\right] = Z(J=0,\lambda)\sum_{N=0}^{\infty}\frac{1}{N!}\left[iW(J,\lambda)\right]^N, \qquad \text{(II.100)}$$

we can associate the term $Z(J = 0, \lambda)$ with those diagrams without external sources, whereas the exponential term in $W(J,\lambda)$ can be regarded as the sum of connected diagrams. What we are interested in calculating is the connected graphs because they are related to measurable quantities, whereas the disconnected graphs can be viewed as vacuum fluctuations that do not enter experimental observables.

Next, we want to generalize the result that we have obtained in EQ. (II.99) to multiple integrals for variables q_1, q_2, …, q_N:

$$Z(J,\lambda) = \int_{-\infty}^{+\infty}dq_1\int_{-\infty}^{+\infty}dq_2 ...\int_{-\infty}^{+\infty}dq_N\,\exp\left(-\frac{1}{2}q\bullet A\bullet q - \frac{\lambda}{4!}q^4 + J\bullet q\right), \qquad \text{(II.101)}$$

where $q^4 \equiv \sum_i (q_i)^4$, and A is a real symmetric $N \times N$ matrix. Equation (II.101) can be integrated out by considering the expansion in powers of λ:

$$Z(J,\lambda) = \left(\frac{(2\pi)^N}{\det[A]}\right)^{\frac{1}{2}}\exp\left[-\frac{\lambda}{4!}\sum_i\left(\frac{\partial}{\partial J_i}\right)^4\right]\exp\left(\frac{1}{2}J\bullet A^{-1}\bullet J\right). \qquad \text{(II.102)}$$

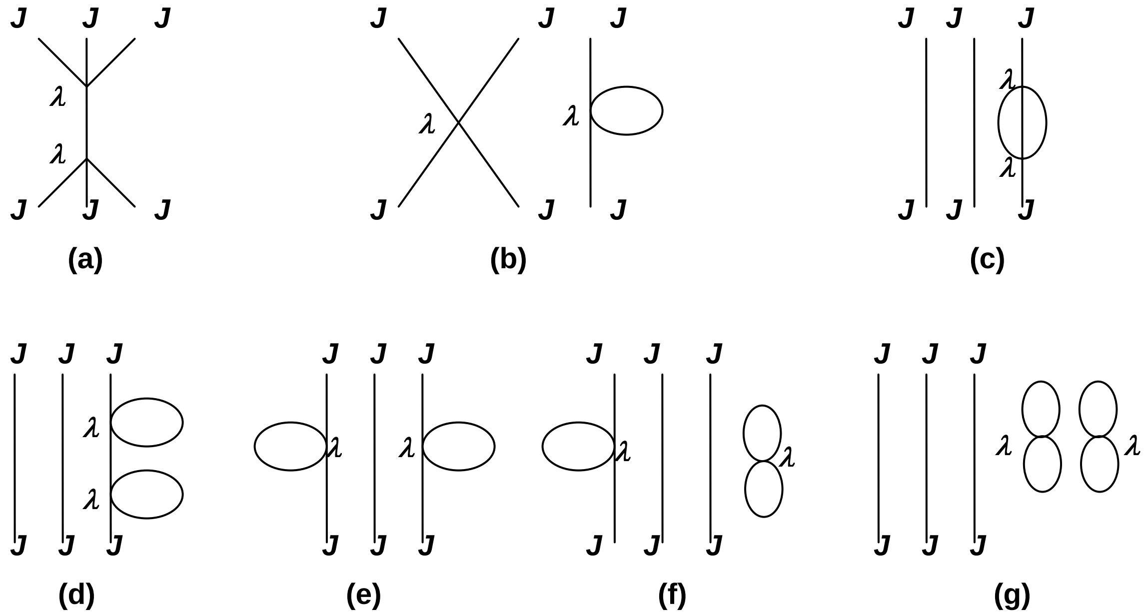


Figure II.3.3 Diagrams associated with the term $(1/m^2)^7\lambda^2 J^6$.

Alternatively, we can expand the path integral EQ. (II.101) in powers of J:

$$Z(J,\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}J_{i1}J_{i2}...J_{in}\prod_{l=1}^{N}\left(\int_{-\infty}^{+\infty}dq_l\right)q_{i1}q_{i2}...q_{in}\exp\left(-\frac{1}{2}q\bullet A\bullet q-\frac{\lambda}{4!}q^4\right)\equiv Z(0,0)\sum_{n=0}^{\infty}\frac{1}{n!}J_{i1}J_{i2}...J_{in}\;G^{(n)}_{i1...in} \tag{II.103}$$

followed by further expansion in λ and then evaluated by the Wick contraction. [You may also compare EQ. (II.103) with EQs. (II.54) and (II.55).] The function $G^{(n)}_{i1...in}$ is known as the "n-point Green's function" that has the meaning of a propagator. For instance, if we take $\lambda = 0$ as in the Gaussian theory, we find that the 2-point Green's function becomes:

$$G^{(2)}_{ij}(\lambda=0)=\frac{1}{Z(0,0)}\int_{-\infty}^{+\infty}\left(\prod_l dq_l\right)q_i q_j\exp\left(-\frac{1}{2}q\bullet A\bullet q\right)=\left(A^{-1}\right)_{ij}, \tag{II.104}$$

which represents propagation from i to j without interactions. On the other hand, the 2-point Green's function to order λ yields:

$$\begin{aligned}G^{(2)}_{ij}(\lambda)&=\frac{1}{Z(0,0)}\int_{-\infty}^{+\infty}\left(\prod_l dq_l\right)q_i q_j\left(1+\frac{-\lambda}{4!}\sum_n (q_n)^4+O(\lambda^2)\right)\exp\left(-\frac{1}{2}q\bullet A\bullet q\right)\\&=\left(A^{-1}\right)_{ij}+(-\lambda)\sum_n\left[\frac{1}{2}\left(A^{-1}\right)_{in}\left(A^{-1}\right)_{nn}\left(A^{-1}\right)_{jn}+\frac{1}{8}\left(A^{-1}\right)_{ij}\left(A^{-1}\right)_{nn}\left(A^{-1}\right)_{nn}\right]+O(\lambda^2)\,.\end{aligned} \tag{II.105}$$

The second term in EQ. (II.105) is associated with $(1/m^2)^3\lambda J^2$, and the corresponding diagrams for the first and the second terms are illustrated in Fig. II.3.4. Apparently the second term in the two-point Green's function $G^{(2)}_{ij}(\lambda\neq 0)$ is associated with vacuum fluctuations and because the interaction is not coupled directly to external observables.

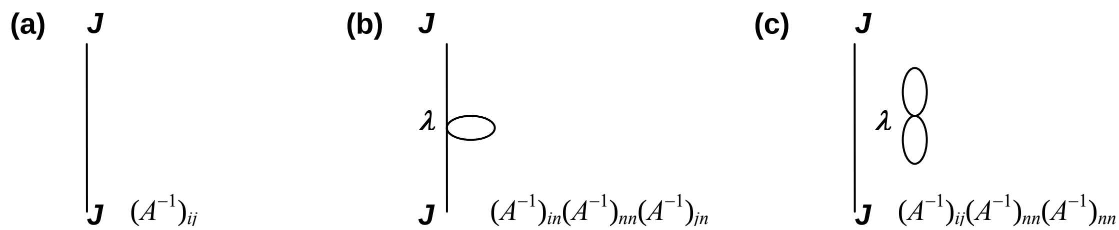


Figure II.3.4 Diagrams associated with terms up to $(1/m^2)^3\lambda J^2$.

Similarly, we can derive the 4-point Green's function $G^{(4)}_{ijkl}(\lambda)$ to first-order in λ:

$$G^{(4)}_{ijkl}(\lambda)=\frac{1}{Z(0,0)}\int_{-\infty}^{+\infty}\left(\prod_m dq_m\right)q_i q_j q_k q_l\left(1+\frac{-\lambda}{4!}\sum_n (q_n)^4+O(\lambda^2)\right)\exp\left(-\frac{1}{2}q\bullet A\bullet q\right)$$
$$=[\left(A^{-1}\right)_{ij}\left(A^{-1}\right)_{kl}+\left(A^{-1}\right)_{ik}\left(A^{-1}\right)_{jl}+\left(A^{-1}\right)_{il}\left(A^{-1}\right)_{jk}]-\lambda\sum_a[\left(A^{-1}\right)_{ia}\left(A^{-1}\right)_{ja}\left(A^{-1}\right)_{ka}\left(A^{-1}\right)_{la}]+O(\lambda^2)\,. \quad \text{(II.106)}$$

The first term in EQ. (II.106) corresponds to two parallel lines, whereas the second term corresponds to two lines crossing at a vertex, as shown in Fig. II.3.1(a).

Next, we take the continuum limit for quantum field theory, so that from EQ. (II.92), we find that the functional generator $Z(J,\lambda)$ becomes:

$$Z(J,\lambda)=\int D\varphi\ \exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4+J\varphi\right)\right\}$$
$$=Z(0,0)\exp\left(-\frac{i\lambda}{4!}\int d^4z\left[\frac{1}{i}\frac{\delta}{\delta J(z)}\right]^4\right)\int D\varphi\ \exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]+J\varphi\right)\right\}$$
$$=Z(0,0)\exp\left(-\frac{i\lambda}{4!}\int d^4z\left[\frac{1}{i}\frac{\delta}{\delta J(z)}\right]^4\right)\exp\left\{-\frac{i}{2}\int d^4x\int d^4y\,J(x)\,D(x-y)\,J(y)\right\}, \quad \text{(II.107)}$$

where the propagator $D(x–y)$ has the form given in EQ. (II.48) and reproduced below:

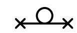
$$D(x-y)=\int\frac{d^4k}{(2\pi)^4}\ \frac{\exp[ik\bullet(x-y)]}{k^2-m^2+i\varepsilon}\,.$$

The expansion in EQ. (II.107) is known as the Schwinger way. It is worth noting that $D(x–y)$ describes the propagation from point x to point y in the absence of interaction. Alternatively, we can expand $Z(J,\lambda)$ in powers of J, similar to the discrete case in EQ. (II.103):

$$Z(J,\lambda)=Z(0,0)\sum_{s=0}^{\infty}\frac{1}{s!}\int d^4x_1d^4x_2\ldots d^4x_s\ J(x_1)J(x_2)...J(x_s)\ G^{(s)}(x_1,x_2,...,x_s)$$
$$=\sum_{s=0}^{\infty}\frac{1}{s!}\int d^4x_1d^4x_2\ldots d^4x_s\ J(x_1)J(x_2)...J(x_s)\int D\varphi\ \exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4\right)\right\}\varphi(x_1)\varphi(x_2)...\varphi(x_s)\,. \quad \text{(II.108)}$$

Here $G^{(s)}(x_1,x_2,...x_s)$ denotes the s-point Green's function in the presence of interaction, which has been given by EQ. (II.55):

$$G^{(s)}\left(x_1,\ldots,x_s\right)=\frac{1}{i^s}\frac{\delta^s Z(J)}{\delta J(x_1)\ldots\delta J(x_s)}\bigg|_{J=0}.$$

The expansion in EQ. (II.108) is known as the Wick way, in contrast to the Schwinger way in EQ. (II.107). In general, it is convenient to obtain the *s*-point Green's function using the diagrammatic expression of the generating functional given in EQ. (II.98). Specifically, we consider the 2-point Green's function $G(x_1,x_2)$ by inspecting the three diagrammatic terms in EQ. (II.98). We note that the first term simply gives the free propagator $iD(x-y)$, and the third term involves  four source terms and therefore does not contribute to the 2-point Green's function. Hence, we only need to consider the second term in  , which is:

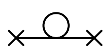
$$\frac{\lambda}{4}D(0)\int dx\,dy\,dz\; D(z-x)J(x)D(z-y)J(y)\exp\left[-\frac{i}{2}\int dx\,dy\; J(x)D(x-y)J(y)\right].$$

Using the definition of the two-point Green's function according to EQ. (II.108), we obtain

$$G^{(2)}(x_1,x_2)\equiv\frac{1}{Z(0,0)}\int D\varphi\;\exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4\right)\right\}\varphi(x_1)\varphi(x_2)\,. \qquad \text{(II.109)}$$

If we assume translation invariance, which is true if we consider interaction of particles in vacuum or in a crystal without defects, the Green's functions should be dependent only on the difference in the space-time coordinates of the source and sink. That is, $G(x_1, x_2) = G(x_1 - x_2)$. In this case, EQ. (II.109) can be more explicitly written out using the definition given in EQ. (II.55) and the expression of the generating functional in EQ. (II.98):

$$G^{(2)}(x_1-x_2)=iD(x_1-x_2)-\frac{\lambda}{2}D(0)\int dz\,D(z-x_1)D(z-x_2)+O(\lambda^2)$$

$= i$ $-\frac{\lambda}{2}$  $+\ O(\lambda^2)$

$$=\frac{i}{(2\pi)^4}\int d^4k\left[\frac{e^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\varepsilon}\right]-\frac{\lambda}{2}\frac{D(0)}{(2\pi)^8}\int d^4z\,d^4p\,d^4q\left[\frac{e^{-ip\cdot(x_1-z)}}{p^2-m^2+i\varepsilon}\right]\left[\frac{e^{-iq\cdot(x_2-z)}}{q^2-m^2+i\varepsilon}\right]+O(\lambda^2)$$

$$=\frac{i}{(2\pi)^4}\int d^4k\left[\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}\right]-\frac{\lambda}{2}\frac{D(0)}{(2\pi)^4}\int d^4p\,\frac{e^{-ip\cdot(x_1-x_2)}}{\left(p^2-m^2+i\varepsilon\right)^2}+O(\lambda^2)$$

$$=\frac{i}{(2\pi)^4}\int d^4p\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\varepsilon}\left[1+\frac{i\lambda}{2}\frac{D(0)}{p^2-m^2+i\varepsilon}\right]+O(\lambda^2)$$

$$\approx\frac{i}{(2\pi)^4}\int d^4p\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-m^2+i\varepsilon}\left[1-\frac{i\lambda}{2}\frac{D(0)}{p^2-m^2+i\varepsilon}\right]^{-1}$$

$$=\frac{i}{(2\pi)^4}\int d^4p\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-\left[m^2+\frac{1}{2}i\lambda D(0)\right]+i\varepsilon}$$

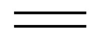
$$\equiv\frac{i}{(2\pi)^4}\int d^4p\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2-\left(m^2+\delta m^2\right)+i\varepsilon}\,. \qquad \text{(II.110)}$$

Hence, we find that the effect of interaction on the two-point Green's function is equivalent to renormalizing the mass according to the last two lines in EQ. (II.110).

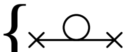
Similarly, the 4-point Green's function $G^{(4)}(x_1, x_2, x_3, x_4)$ is given by the following according to EQ. (II.108):

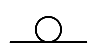
$$G^{(4)}(x_1,x_2,x_3,x_4) \equiv \frac{1}{Z(0,0)}\int D\varphi\ \exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \frac{\lambda}{4!}\varphi^4\right)\right\}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\,. \qquad \text{(II.111)}$$

Assuming translation invariance, $G^{(4)}(x_1, x_2, x_3, x_4)$ only depends on $(x_1 - x_4)$, $(x_2 - x_4)$, and $(x_3 - x_4)$. In the limit of $\lambda \to 0$, we recover the free-theory propagators $D(x_1, x_2,\ldots x_s)$, which correspond to $(s/2)$ parallel lines. In contrast, $G^{(s)}(x_1, x_2,\ldots x_s)$ involves interaction vertices. For instance, if we consider $G^{(4)}(x_1, x_2, x_3, x_4)$ in the first-order of λ, the Green's function corresponds to the scattering of particles through an interaction vertex (with the "internal point" denoted by w), which can be obtained by applying EQ. (II.55) to EQ. (II.98). The first term of order λ^0 in $G^{(4)}(x_1, x_2, x_3, x_4)$ corresponds to the free-particle 4-point function and is given by:

$$-\left[D(x_1-x_2)D(x_3-x_4)+D(x_1-x_3)D(x_2-x_4)+D(x_1-x_4)D(x_2-x_3)\right] = -3$$  . (II.112)

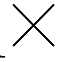
Clearly this term does not contribute to scattering. The second term of order λ in $G^{(4)}(x_1, x_2, x_3, x_4)$ is

$$\frac{\lambda}{4}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big\{$$  $$\exp\left[-\frac{i}{2}\int dx\,dy\,J(x)D(x-y)J(y)\right]\Big\}\Big|_{J=0}$$

$$= \frac{\lambda}{4}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big\{D(0)\int dxdy\,D(x-z)D(y-z)J(y)J(x)$$
$$\times\exp\left[-\frac{i}{2}\int dx\,dy\,J(x)D(x-y)J(y)\right]\Big\}\Big|_{J=0}$$
$$= -\frac{i\lambda}{2}D(0)\int dz\,\big[D(z-x_1)D(z-x_2)D(x_3-x_4)+D(z-x_1)D(z-x_3)D(x_2-x_4)$$
$$+D(z-x_1)D(z-x_4)D(x_2-x_3)+D(z-x_2)D(z-x_3)D(x_1-x_4)$$
$$+D(z-x_2)D(z-x_4)D(x_1-x_3)+D(z-x_3)D(z-x_4)D(x_1-x_2)\big]$$
$$= -3i\lambda$$  , (II.113)

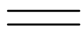
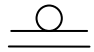
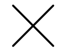
where we have simplified the diagrammatic expression by removing the crosses that explicitly illustrate the sources. The diagrammatic expression in EQ. (II.113) is the same as that illustrated in Fig. II.3.1(b).

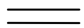
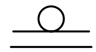
The third term in EQ. (II.98) that contributes to the first order in λ for the four-point Green's function is given by the following:

$$\frac{-i\lambda}{4!}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big\{$$  $$\exp\left[-\frac{i}{2}\int dx\,dy\,J(x)D(x-y)J(y)\right]\Big\}\Big|_{J=0}$$

$$= \frac{-i\lambda}{4!}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big\{\left[\int dx\,D(z-x)J(x)\right]^4\exp\left[-\frac{i}{2}\int dx\,dy\,J(x)D(x-y)J(y)\right]\Big\}\Big|_{J=0}$$
$$= -i\lambda\int dz\,D(x_1-z)D(x_2-z)D(x_3-z)D(x_4-z)$$
$$= -i\lambda$$  . (II.114)

This term is consistent with the diagram shown in Fig. II.3.1(a). Hence, the complete 4-point Green's function to order λ is therefore given by the sum of EQs. (II.112) – (II.114):

$G^{(4)}(x_1, x_2, x_3, x_4) = -3$  $-3i\lambda$  $-i\lambda$ 

$= -3$  $\frac{-i\lambda}{4!}[\,72$  $+ 24$  $]$. (II.115)

In practice, we can consider the first term in EQ. (II.115) as also involving a vertex that is entirely disconnected from the propagators, as illustrated in Fig. II.3.1(c). In this case, the coefficient associated with the term will become 9 rather than 3. This term apparently does not contribute to the scattering process. Similarly, the second term in EQ. (II.115) consists of two disconnected graphs, which does not contribute directly to scattering either. Therefore, the only connected graph for the 4-point Green's function to the first order in λ is given by the third term in EQ. (II.115).

Alternatively, we may write the 4-point Green's function into the following form:

$$G^{(4)}(x_1,x_2,x_3,x_4) \equiv \frac{1}{Z(0,0)}\left(-\frac{i\lambda}{4!}\right)\int d^4z \int D\varphi \ \exp\left\{i\int d^4x\left(\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right]\right)\right\}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\left[\varphi(w)\right]^4$$

$$= \frac{1}{Z(0,0)}\left(-i\lambda\right)\int d^4z \left[D_{1z}D_{2z}D_{z3}D_{z4} + \left(D_{12}D_{3z}D_{z4}D_{zz} + D_{13}D_{2z}D_{z4}D_{zz} + D_{14}D_{2z}D_{z3}D_{zz} + D_{23}D_{1z}D_{z4}D_{zz}\right)/2\right.$$
$$\left. + \left(D_{24}D_{1z}D_{z3}D_{zz} + D_{34}D_{1z}D_{z2}D_{zz}\right)/2 + \left(D_{12}D_{34}D_{zz}D_{zz} + D_{13}D_{24}D_{zz}D_{zz} + D_{14}D_{23}D_{zz}D_{zz}\right)/8\right], \qquad \text{(II.116)}$$

where $D_{1z} \equiv D(x_1 - z) = D(z - x_1) = D_{z1}$, etc. If we write out the propagators in EQ. (II.116) out explicitly:

$$D(x_a - z) = \int \frac{d^4k_a}{(2\pi)^4} \ \frac{\exp[\pm ik_a(x_a - z)]}{k_a{}^2 - m^2 + i\varepsilon},$$

the integration over z in EQ. (II.116) has a simple form:

$$\int d^4z \ \exp\left[-i\left(k_1 + k_2 - k_3 - k_4\right)z\right] = (2\pi)^4\delta^{(4)}\left(k_1 + k_2 - k_3 - k_4\right). \qquad \text{(II.117)}$$

Equation (II.117) implies that the momentum is conserved at the vertex.

To recast, we can in general view the functional $Z(J,\lambda)$ in the φ^4-theory as the result generated by applying the generator $\exp\left[-{i\lambda}/{4!}\int d^4z\left[\delta/\delta J(z)\right]^4\right]$ on the functional of the free theory $Z(J, \lambda = 0)$:

$$Z(J,\lambda = 0) = \exp\left\{-\frac{i}{2}\int d^4x\int d^4y\, J(x)\, D(x-y)\, J(y)\right\}.$$

Moreover, by writing $Z(J) = Z(0)\exp[iW(J)]$ and considering the 2-point and 4-point Green's functions, we find that the generating functional $W(J)$ generates only connected Feynman diagrams and connected Green's functions. Thus, similar to the definition of the n-point Green's function given in EQ. (II.55), we may define the irreducible n-point function $\phi\left(x_1,\ldots,x_n\right)$ by the following where $W\left(J\right) \equiv -i\ln Z\left(J\right)$:

$$\phi\left(x_1,\ldots,x_n\right) = \frac{1}{i^n}\left.\frac{\delta^n W\left(J\right)}{\delta J\left(x_1\right)\ldots\delta J\left(x_n\right)}\right|_{J=0}. \qquad \text{(II.118)}$$

Based on the discussions given above, we summarize the momentum space Feynman's rules for the bosonic scalar field theory as follows:

1. Draw a Feynman diagram of the process under consideration.
2. Label each line with a momentum k and associate it with the propagator $i/(k^2 - m^2 + i\varepsilon)$.
3. Associate with each interaction vertex the coupling constant $(-i\lambda)$ and the condition for momentum conservation, $(2\pi)^4\delta^{(4)}(\sum_i k_i - \sum_j k_j)$, where $\sum_i k_i$ denotes the sum of all incoming momenta and $\sum_j k_j$ denotes the sum of all outgoing momenta.
4. Momenta associated with the internal lines of a diagram are to be integrated with the measure $d^4k/(2\pi)^4$. This procedure corresponds to summing over all intermediate states in perturbation theory.
5. A numerical factor has to be inserted into diagrams that are symmetric, and the number depends on the specific symmetry of the diagrams.
6. For succinctness, we do not associate propagators with external lines.
7. The momentum conservation at each vertex is understood.

Let's go over a few examples to familiarize ourselves with the use of Feynman's rules. Consider the diagram shown in Fig. II.3.5 for two meson-meson scattering. There are two vertices, four external lines, and two internal lines in the process. The amplitude for the depicted process is given by

$$\frac{1}{2}(-i\lambda)^2\int\frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-m^2+i\varepsilon}\right)\left(\frac{i}{(k_1+k_2-k)^2-m^2+i\varepsilon}\right). \tag{II.119}$$

In the large k limit, the integrand goes as k^{-4} and the integral becomes logarithmically divergent. This "ultraviolet divergence" can be dealt with by introducing a physically justifiable "ultraviolet cutoff" Λ, because quantum field theory is an effective low-energy theory, and therefore cannot be expected to hold up to any arbitrarily high energy. The procedure of introducing a physically meaningful cutoff is known as "regularization" and "renormalization". We shall not go into further discussion about how to handle the divergence in quantum field theory because our primary focus will be on condensed matter physics, which only concerns with the low-energy spectrum. Topics on regularization and renormalization can be found in standard relativistic quantum field theory texts, such as the books by Peskin and Schroeder and by Ryder.

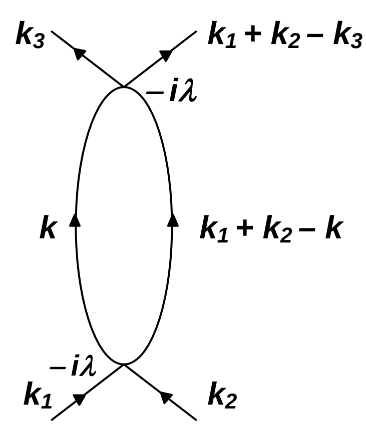


Figure II.3.5 A one-loop process of two meson-meson scattering.

As another example, consider the three-loop diagram for a two meson-meson scattering process in Fig. II.3.6. There are 4 vertices, 6 internal lines, and 4 external lines in the diagram. [Can you tell why only 6 internal lines are allowed from the given conditions?] We need to first decide on the total number of free variables in the problem. As in the previous example, we assume that the initial momenta (k_1, k_2) and the final momenta (k_3, $k_1+k_2-k_3$) are given, so that there are three free variables to be defined in the process, which we define as p, q, and r. Following Feynman's rules, we find the amplitude for this process is given by:

$$\frac{(-i\lambda)^4}{4}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\left\{\left(\frac{i}{p^2-m^2+i\varepsilon}\right)\left(\frac{i}{(k_1+k_2-p)^2-m^2+i\varepsilon}\right)\left(\frac{i}{q^2-m^2+i\varepsilon}\right)\right.$$
$$\left.\times\left(\frac{i}{(p-q-r)^2-m^2+i\varepsilon}\right)\left(\frac{i}{r^2-m^2+i\varepsilon}\right)\left(\frac{i}{(k_1+k_2-r)^2-m^2+i\varepsilon}\right)\right\}. \qquad \text{(II.120)}$$

Both Figs. II.3.5 and II.3.6 are connected graphs. As we have mentioned earlier, disconnected graphs are associated with vacuum fluctuations, during which particles can appear, interact, and vanish again into vacuum. These disconnected processes can be separated from the connected graphs, and they do not directly contribute to the experimental observables and are therefore of no interest to us at present.

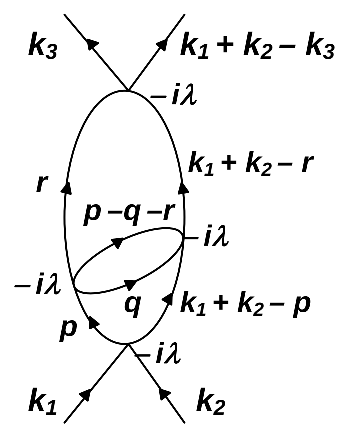


Figure II.3.6 A three-loop process of two meson-meson scattering.

A practical application of the path integral formalism and Feynman diagrams is to calculate the scattering amplitude of particles under interaction, from which the scattering cross section and the lifetime of decay of particles can be derived. In fact, the calculation of scattering amplitude using Feynman diagrams is also applicable to scattering processes in condensed matter physics systems if one takes the non-relativistic limit. We shall return to an example of calculating the scattering amplitude later in Part II.5, and we note that in general the scattering amplitude obtained from the scattering matrix (also known as the *S*-matrix) can be directly related to a corresponding Green's function, although we shall not go into more detailed discussion on the *S*-matrix here. You may refer to the textbooks by Peskin and Schroeder and by Ryder for in-depth consideration of the *S*-matrix.