III.4. Other Important Developments in Superconductivity Research

Prior to the discovery of high-temperature superconductivity, studies of heavy fermion superconductors enjoyed substantial attention because of their unconventional pairing symmetries as opposed to typical BCS superconductors. Similarly, organic superconductors known as the Bechgaard salts $(TMTSF)_2X$, where (TMTSF) refers to "tetramethyltetraselenefulvaline" and X includes molecules such as PF_6 and ClO_4, have attracted much research interest because the system $(TMTSF)_2X$ is a highly anisotropic (quasi-one-dimensional) organic material that exhibits complex phases in the pressure-temperature phase diagram, including spin density waves (SDW), non-Fermi liquid metallic state, (possibly triplet *p*-wave) superconductivity, and quantum Hall effect. Other actively researched areas include the newly discovered (in 2001) MgB_2 superconductor that is interesting for its two-gap superconductivity and promising for large scale applications (such as for high-field magnets, energy storage, and power transmission lines) at moderate temperatures (T_c ~ 39 K) because of its high and isotropic critical current density and high upper critical fields; the development of high-temperature superconducting devices for such applications as scanning magnetometers, microwave filters and receivers for cellular phone base stations, and SQUID sensors for magnetocardiography; and qubits for quantum computations based on the Josephson junction technology with conventional superconductors. Continuing search and synthesis of new superconductors is also an ongoing research effort worldwide. In the following we only focus on an overview of important physics issues associated with heavy-fermion superconductivity.

[Heavy-fermion superconductors]

In 1961, P. W. Anderson and P. Morel generalized the BCS theory for *s*-wave superconductors to include the consideration of non-zero angular momentum and the physical consequences of such BCS pairing ["Generalized Bardeen-Cooper-Schrieffer States and the proposed low-temperature phases of ^{3}He", Phys. Rev. **123**, 1911 (1961)]. Subsequently superfluidity was discovered in ^{3}He in 1972, and many experimental results shortly after this discovery have established ^{3}He as *p*-waves spin-triplet superfluids that exhibit three superfluid phases with broken time-reversal symmetry.

The metallic analogy of superfluidity in ^{3}He was discovered in heavy-fermion superconductors in 1979, [F. Steglich *et al.*, Phys. Rev. Lett. **43**, 1892 (1979)], particularly in the U-based compounds of UBe_{13}, UPt_3, URu_2Si_2, UNi_2Al_3, and UPd_2Al_3. In recent years, numerous heavy-fermion superconductors with exotic properties resembling competing orders in the cuprate superconductors, such as $CeCoIn_5$, $CeIrIn_5$, $CeCu_2Si_2$, $CePd_2Si_2$, $PuCoGa_5$, $PuRhGa_5$, and Sr_2RuO_4 have also been discovered. The unusual temperature dependence of the heat capacity, penetration depth, sound absorption, and critical fields have led to consensus that many, if not all, of the heavy-fermion superconductors exhibit unconventional pairing. However, the unconventional pairing in heavy-fermion superconductors is mostly triplet pairing and exhibit broken time-reversal symmetry, which differs from the $d_{x^2-y^2}$-wave singlet pairing in cuprate superconductors that preserves time-reversal symmetry.

In general, we note that the pairing amplitude takes the form

$$\Delta_{\alpha\beta}(\mathbf{k}_F) \sim \left\langle c_{\mathbf{k}_F\alpha}\, c_{-\mathbf{k}_F\beta} \right\rangle, \qquad \text{(III.261)}$$

where α and β refer to the spin labels of the quasiparticles, and equal time pairing has been assumed. In conventional superconductors with *s*-wave pairing symmetry, we find that

$$\Delta_{\alpha\beta}(\mathbf{k}_F) \sim \Delta_0(\mathbf{k}_F)\left(i\sigma_y\right)_{\alpha\beta}, \qquad \text{(III.262)}$$

where $\Delta_0(\mathbf{k}_F)$ is a complex amplitude that breaks the *U*(1) gauge symmetry. On the other hand, unconventional pairing occurs when the pairing amplitude spontaneously breaks one or more symmetries of the normal state besides the *U*(1) gauge symmetry. That is, for a symmetry operation $\mathcal{R} \in \mathcal{G}$ where the full symmetry group $\mathcal{G}$ of the system is given by

$$\mathcal{G} = \mathcal{G}_{\text{spin}} \otimes \mathcal{G}_{\text{space}} \otimes \mathcal{T} \otimes U(1), \qquad \text{(III.263)}$$

$\mathcal{T}$ represents the time-reversal symmetry, and $\mathcal{G}_{\text{spin}}$ and $\mathcal{G}_{\text{space}}$ respectively denote the symmetry groups of the spin and orbit wavefunctions of the system, we find that unconventional pairing occurs if $\mathcal{R} \otimes \Delta(\mathcal{R}^{-1} \otimes \mathbf{k}_F) \neq \Delta(\mathbf{k}_F)$. Moreover, the fermion statistics of quasiparticles in superconductors requires the pairing amplitude to obey the antisymmetric condition so that $\Delta_{\alpha\beta}(\mathbf{k}_F) = -\Delta_{\beta\alpha}(-\mathbf{k}_F)$.

In the case of negligible spin-orbit interaction, the normal state system is invariant under rotation so that $\mathcal{G}_{\text{spin}} = SO(3)$, and the α and β labels refer to the spin indices of the quasiparticle states near the Fermi level. On the other hand, for systems with strong spin-orbit interaction, such as in the case of heavy-fermion superconductors, the α and β labels in EQ. (III.250) are no longer the same as the eigenvalues of the spin operators for electrons. However, in this case the Kramers degeneracy in the absence of magnetic field still guarantees that each **k**-state is two-fold degenerate. Consequently, for strong spin-orbit interaction we may consider the α and β labels as pseudo-spin quantum numbers that take on two possible values.

To date, all superconductors exhibiting unconventional pairing have inversion symmetry in their crystalline structures. Consequently, the pairing interaction that drives the superconducting transition necessarily decomposes into even- and odd-parity sectors, and $\Delta_{\alpha\beta}(\mathbf{k}_F)$ for any of these superconductors with a single primary order parameter can therefore be expressed as follows:

$$\Delta_{\alpha\beta}(\mathbf{k}_F) = \Delta_0(\mathbf{k}_F)(i\sigma_y)_{\alpha\beta} + \Delta(\mathbf{k}_F) \bullet (i\boldsymbol{\sigma}\sigma_y)_{\alpha\beta}, \qquad \text{(III.264)}$$

where $\boldsymbol{\sigma}$ denote the Pauli matrices, and $\Delta_0(\mathbf{k}_F) = \Delta_0(-\mathbf{k}_F)$ for even parity and $\Delta(\mathbf{k}_F) = -\Delta(-\mathbf{k}_F)$ for odd parity. Furthermore, the general form of the order parameter can be expressed by

$$\Delta_0(\mathbf{k}_F) = \sum_{\Gamma}^{\text{even}} \sum_{i}^{d_\Gamma} \eta_i^{(\Gamma)} \mathrm{Y}_{\Gamma,i}(\mathbf{k}_F), \qquad \text{(singlet, } S = 0\text{)} \qquad \text{(III.265)}$$

$$\Delta(\mathbf{k}_F) = \sum_{\Gamma}^{\text{odd}} \sum_{i}^{d_\Gamma} \eta_i^{(\Gamma)} \mathbf{Y}_{\Gamma,i}(\mathbf{k}_F), \qquad \text{(triplet, } S = 1\text{)} \qquad \text{(III.266)}$$

where Γ denotes an irreducible representation of the point group, d_Γ is the dimension of a given irreducible representation Γ, $\eta_i^{(\Gamma)}$ is the coefficient associated with the basis function $\mathrm{Y}_{\Gamma,i}(\mathbf{k}_F)$ of the *i*th partner function belonging to an even-parity representation Γ, and $\mathbf{Y}_{\Gamma,i}(\mathbf{k}_F)$ denotes the basis function of the *i*th partner function belonging to an odd-parity representation Γ. Based on EQ. (III.346) and the experimental evidence of multi-component pairing amplitudes in various heavy-fermion superconductors, which is similar to the multiple superfluid phases in ^{3}He, possible scenarios for the unconventional pairing symmetry in heavy-fermion superconductors include 1) the representation of the order parameter has a dimension $d_\Gamma > 1$, and 2) multiple representations become accidentally degenerate at the Fermi level.

Among the best studied heavy-fermion superconductors, the UPt_3 compound has been one of the model systems that clearly reveal multiple superconducting phases and has been under intense theoretical investigation. In the following we describe the primary experimental phenomena and some theoretical development of the UPt_3 compound as an example of triplet heavy-fermion superconductors.

The crystalline structure of UPt_3 belongs to the point group D_{6h}, and its physical properties are generally consistent with the characteristics of a heavy-fermion system below a temperature $T^* \sim 10$ K. Specifically, the resistivity ρ, specific heat C_v, and magnetic susceptibility χ of UPt_3 satisfy the following temperature dependence:

$$\rho = \rho_0 + AT^2 , \qquad \rho_0 \approx 0.1\,\mu\Omega - \text{cm} ,$$
$$C_v = \gamma T , \qquad (\gamma/\gamma_0) = (m^*/m_e) \approx 500 \text{ ! (Thus the name "heavy-fermion")}$$
$$(\chi/\chi_0) \approx \text{constant} \sim (m^*/m_e) .$$

Superconductivity in UPt_3 was first discovered in 1984 when a resistive transition was observed at $T_c = 526$ mK with a very sharp transition width of $\delta T_c \sim 1$ mK. The existence of two superconducting phases at $H = 0$ was confirmed by specific heat measurements, which revealed two distinct transitions T_c and T_{c2} separated by ~ 50 mK, as illustrated in Fig. III.4.1 (a). Additionally, ultrasonic attenuation in magnetic fields at low temperatures also exhibited two distinct characteristic fields H^* and H_{c2}, as schematically shown in Fig. III.4.1 (b). By collecting information from experiments under varying H and T, an interesting H vs. T phase diagram emerges, revealing three different flux phases that meet at a tetra-critical point for all orientations of the magnetic field relative to the sample c-axis (which is normal to the hexagonal two-dimensional lattice), as shown in Fig. III.4.2.

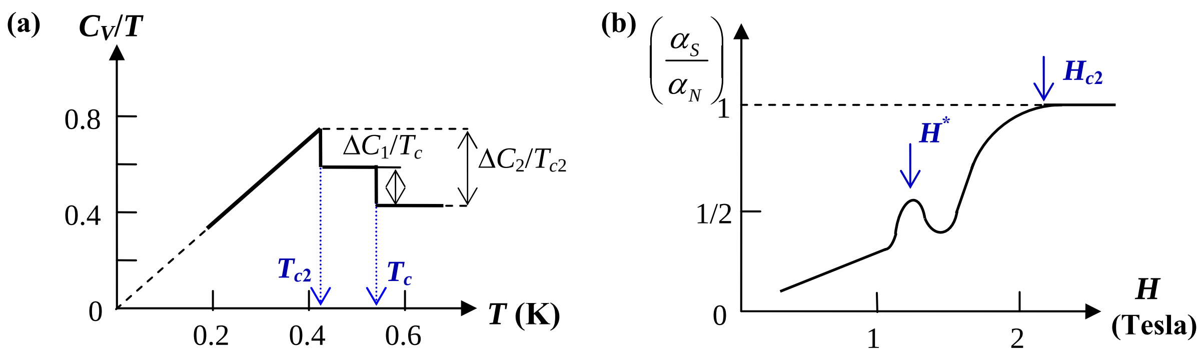
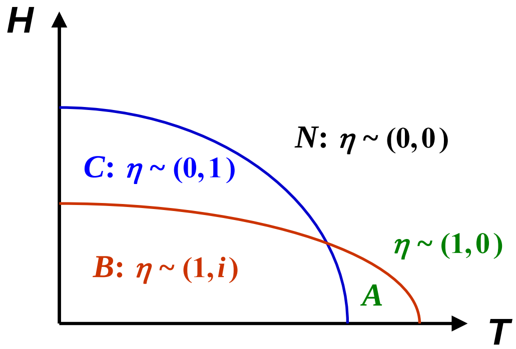


Fig. III.4.1 **(a)** Schematic of the zero-field specific heat (C_v) versus temperature (T) data in UPt_3. The pronounced steps with discontinuities ΔC_1 and ΔC_2 occur at two phase transition temperatures at T_c and T_{c2}, where $(T_c - T_{c2})/T_c \sim 10\%$ and $(\Delta C_2/T_{c2})/(\Delta C_1/T_c) \sim 1.25$. **(b)** Schematic of the relative ultrasound attenuation (α_S/α_N) versus magnetic field (H) data taken at $T = 50$ mK, where the subscripts S and N refer to superconductivity and normal state, respectively. The presence of two characteristic fields H^* and H_{c2} indicates three distinct flux phases in UPt_3.

To understand the physical origin of multiple superconducting phases in UPt_3, it is helpful to consider the irreducible representations and the basis functions of the symmetry group D_{6h}, which are given in Table V.5.1. Specifically, several theoretical models based on different symmetry-breaking scenarios have been proposed for the occurrence of multiple superconducting phases in UPt_3. These models consider the coupling of a multi-component superconducting order parameter, expressed in terms of the two-dimensional representations (E_{1g}, E_{2g}, E_{1u}, E_{2u}) of D_{6h}, to a symmetry-breaking field (SBF). This scenario is motivated by two empirical facts: the existence of two zero-field superconducting phases with relatively small ΔT_c; and the presence of antiferromagnetism and the apparent correlation of ΔT_c with the strength of antiferromagnetism.

As shown in Table III.4.1, there are four *E*-representations for strong spin-orbit coupling in UPt_3, including the even-parity representations (E_{1g}, E_{2g}) and the odd-parity representations (E_{1u}, E_{2u}). The *E*-representation models require a weak symmetry-breaking field that lowers the symmetry of the normal state so as to split the zero-field superconducting transition into two transitions and produce multiple superconducting phases in both zero and finite magnetic fields. A natural candidate for the symmetry-breaking field in UPt_3 is the in-plane antiferromagnetic order parameter that onsets at $T_N \sim 5$ K. This relatively low T_N is indicative of the weak strength of the antiferromagnetic coupling.



$$\Delta_0(\mathbf{k}_F) = \eta_1 Y_1(\mathbf{k}_F) + \eta_2 Y_2(\mathbf{k}_F)$$

$$\eta \equiv (\eta_1, \eta_2)$$

$Y_{1,2}(\mathbf{k}_F)$: the basis functions of the appropriate representations

(The *B*-phase breaks the time-reversal symmetry)

Fig. III.4.2 Schematic of the magnetic field (*H*) versus temperature (*T*) phase diagram of UPt_3.

Among the *E*-representation models, the most successful ones in explaining the superconducting properties of UPt_3 are based on E_{1g} and E_{2u}:

E_{1g} – spin-singlet & *d*-wave pairing with a uniaxial symmetry;
E_{2u} – spin-triplet & *f*-wave pairing with a uniaxial symmetry;

In the absence of spin-orbit coupling, the dimensionality of the odd-parity *E*-representation is three times larger than that of the corresponding spin-singlet *E*-representation. Therefore, the order parameter transforms according to the following:

$$\Delta_0(\mathbf{k}_F) \to \mathcal{R}_{\text{spin}}\,\Delta_0\left(\mathcal{R}_{\text{orbit}}^{-1}\,\mathbf{k}_F\right), \qquad \text{where } \mathcal{R}_{\text{orbit}} \in [D_{6h}]_{\text{orbit}} \text{ and } \mathcal{R}_{\text{spin}} \in SU(2)_{\text{spin}}\,. \tag{III.267}$$

Thus, the enlarged symmetry group for the normal state without spin-orbit coupling is:

$$\mathcal{G} = SU(2)_{\text{spin}} \otimes [D_{6h}]_{\text{orbit}} \otimes \mathcal{T} \otimes U(1)\ . \tag{III.268}$$

In reality, the presence of strong spin-orbit coupling in the uranium-based heavy-fermion metals reduces the symmetry group by allowing only joint rotations of the spin and orbit degrees of freedom. Therefore, the even and odd parity representations are described in terms of the pseudo spin-singlet and pseudo spin-triplet order parameters, and EQ. (III.267) is now modified into:

$$\Delta(\mathbf{k}_F) \to \mathcal{R}\,\Delta\left(\mathcal{R}^{-1}\,\mathbf{k}_F\right), \qquad \text{where } \mathcal{R} \in [D_{6h}]_{\text{spin-orbit}}\,. \tag{III.269}$$

Moreover, the full symmetry group in the normal state in the presence of strong spin-orbit coupling becomes

$$\mathcal{G} = [D_{6h}]_{\text{spin-orbit}} \otimes \mathcal{T} \otimes U(1)\,. \tag{III.270}$$

In the 1990's a series of empirical facts have led to strong support for E_{2u}-pairing symmetry in UPt_3. Hence, the system is a spin-triplet superconductor, which requires consideration of the relation between the orbital magnetic moment and the spin-degree of freedom of the pairs. There are two commonly discussed special cases. In the first case, we consider the (pseudo) spin-triplet order parameter being factorized into a single (pseudo) spin-vector and an orbital component, so that

$$\Delta\left(\mathbf{k}_F\right) = \mathbf{d}\,\Delta_0\left(\mathbf{k}_F\right), \tag{III.271}$$

where **d** is a real unit vector, and the notation $\Delta_0\left(\mathbf{k}_F\right)$ is an odd-parity orbital function. We note that generally the vector **d** defines the axis along which the pairs have zero spin projection. In other words, $\mathbf{d}\bullet\mathbf{S}\left(\mathbf{k}_F\right)=0$. Hence, for $\mathbf{d}\parallel\hat{z}$ (with $\hat{z}$ being a unit vector in spin space), from EQs. (III.264) and (III.271) we find that $\Delta_{\uparrow\uparrow}=\Delta_{\downarrow\downarrow}=0$ and $\Delta_{\uparrow\downarrow}=\Delta_{\downarrow\uparrow}=\Delta\left(\mathbf{k}_F\right)$. On the other hand, if $\mathbf{d}\perp\hat{z}$, then the same pairing state, now described in terms of a different choice of the quantization axis, is given by $\Delta_{\alpha\alpha}=\Delta_{\beta\beta}=\Delta\left(\mathbf{k}_F\right)$ where $\alpha\perp\hat{z}$ and $\beta\perp\hat{z}$ and $\Delta_{\alpha\beta}=\Delta_{\beta\alpha}=0$. That is, the pairing state is described as "equal-spin" pairing in an "easy-plane". In the second special case, one considers a complex unit vector **d**, so that the spin components of the order parameter spontaneously break the time-reversal symmetry from the condition $\mathbf{d}\bullet\mathbf{S}\left(\mathbf{k}_F\right)=0$.

More generally, the order parameter Δ is complex so that $\Delta\times\Delta^*\neq 0$, and it varies over the Fermi surface. These states are non-unitary, because the square of the spin-matrix representation of the order parameter is not proportional to the unit spin matrix. That is,

$$\left[\Delta^\dagger\Delta\right]_{\alpha\beta} = \left|\Delta\right|^2\delta_{\alpha\beta} + i\left[\Delta\times\Delta^*\bullet\sigma\right]_{\alpha\beta}. \tag{III.272}$$

Consequently, the spin degeneracy of the excitation spectrum is lifted, and the quasiparticle energy depends on the local pair spin at $\mathbf{k}_F$, where $\mathbf{S}_{\text{pair}}\left(\mathbf{k}_F\right)\sim i\,\Delta\left(\mathbf{k}_F\right)\times\Delta\left(\mathbf{k}_F\right)^*$.

In fact, paramagnetism can serve as an important probe for the spin structure of the superconducting order parameter, particularly as an experimental signature to differentiate even- and odd-parity superconductivity. In the case of odd-parity spin-triplet superconductors, the transition temperature, energy gap, and many other superconducting properties can depend strongly on the orientation of the magnetic field relative to the spin-quantization axis of the Cooper pairs. Let's consider the following two examples. First, if the Cooper pairs form spin-singlets, then the Zeeman energy that favors an unequal spin population has pair-breaking effect for all field directions. Second, if Cooper pairs form spin-triplets, we may consider the special case with a real **d**-vector. If $\mathbf{d}\parallel\hat{z}$, then the non-vanishing order parameter for the pair $\left(\mathbf{k}_F,-\mathbf{k}_F\right)$ can be a pure "opposite-spin state" with $\left|\uparrow\downarrow\right\rangle+\left|\downarrow\uparrow\right\rangle$. In this case, an external magnetic field $\mathbf{H}\parallel\hat{z}$ is pair-breaking for all $\mathbf{k}_F$, whereas $\mathbf{H}\perp\hat{z}$ is not pair-breaking because $\Delta_{\uparrow\uparrow}=\Delta_{\downarrow\downarrow}=0$ already. Similarly, if $\mathbf{d}\perp\hat{z}$, then a magnetic field along an "equal-spin-pairing" (ESP) direction can easily polarize the pairs and minimize the Zeeman energy by alternating the relative number of $\left|\uparrow\uparrow\right\rangle$ and $\left|\downarrow\downarrow\right\rangle$ pairs with essentially no loss in the condensation energy. Therefore, in general a magnetic field with $\mathbf{H}\perp\mathbf{d}\left(\mathbf{k}_F\right)$ for all $\mathbf{k}_F$ is not pair-breaking. However, if $\mathbf{H}\parallel\mathbf{d}\left(\mathbf{k}_F\right)$, then the field is pair-breaking for the pairs $\left(\mathbf{k}_F,-\mathbf{k}_F\right)$, similar to the case of spin-singlet pairing.

Table III.4.1 Basis functions for the irreducible representations of D_{6h}

Representation (Γ)	Basis Function	Dimension (d_Γ)
A_{1g}	1	1
A_{2g}	$\mathrm{Im}\left[\left(k_x + ik_y\right)^6\right]$	1
B_{1g}	$k_z\,\mathrm{Im}\left[\left(k_x + ik_y\right)^3\right]$	1
B_{2g}	$k_z\,\mathrm{Re}\left[\left(k_x + ik_y\right)^3\right]$	1
E_{1g}	$k_z\begin{pmatrix} k_x \\ k_y \end{pmatrix}$	2
E_{2g}	$\begin{pmatrix} k_x^2 - k_y^2 \\ 2k_xk_y \end{pmatrix}$	2
A_{1u}	$\hat{z}\,k_z$	1
A_{2u}	$\hat{z}\,k_z\,\mathrm{Im}\left[\left(k_x + ik_y\right)^6\right]$	1
B_{1u}	$\hat{z}\,\mathrm{Im}\left[\left(k_x + ik_y\right)^3\right]$	1
B_{2u}	$\hat{z}\,\mathrm{Re}\left[\left(k_x + ik_y\right)^3\right]$	1
E_{1u}	$\hat{z}\begin{pmatrix} k_x \\ k_y \end{pmatrix}$	2
E_{2u}	$\hat{z}\,k_z\begin{pmatrix} k_x^2 - k_y^2 \\ 2k_xk_y \end{pmatrix}$	2

In the case of UPt_3, the H_{c2} measurements reveal paramagnetic effect for $\mathbf{H} \parallel \hat{c}$ and no paramagnetic effect for $\mathbf{H} \perp \hat{c}$. Knowing that the order parameter is odd-parity spin-triplet state, we can conclude that $\mathbf{d} \parallel \hat{c}$ in UPt_3. That is, the orbital magnetic moment points along the crystalline *c*-axis, and the spins of the Cooper pairs lie in the basal plane. For the E_{2u} representation, we may express $\Delta\left(\mathbf{k}_F\right)$ as

$$\Delta\left(\mathbf{k}_F\right) = \hat{z}\left[\left(k_x^2 - k_y^2\right)\eta_1 + 2\left(k_xk_y\right)\eta_2\right]k_z\,. \qquad \text{(III.273)}$$

It is also worth noting that significant differences exist in odd-parity superconductors, depending on whether there is weak or strong spin-orbit coupling in the pairing channel. For an ESP state, in the absence of spin-orbit coupling, the **d**-vector will orient itself perpendicular to the magnetic field in order to minimize the Zeeman energy. Consequently, the measured spin susceptibility in the low-field limit will not show dependence on the field-orientation for $T < T_c$. On the other hand, if there is crystalline anisotropy and strong spin-orbit coupling, then a rotation of **d** under an applied field will cost energy on the order of T_c. Thus, spin-orbit coupling tends to select preferred directions for **d** in the crystal, so that the orientation of an applied magnetic field can be used to probe the spin structure of the order parameter directly.

Finally, we discuss the Landau-Ginzburg theory for triplet superconductors by considering the special case of the E_{2u}-pairing in UPt_3. It should be noted, however, that similar analysis described below can be done for E_{1g}, E_{1u} and E_{2g} representations.

We already know that the pairing state in UPt_3 is odd-parity triplet ESP state with strong spin-orbit coupling, so that $\mathbf{d} \parallel \hat{c}$ and

$$\Delta_{\alpha\beta}(\mathbf{k}_F) = \Delta_0(\mathbf{k}_F)\left[\mathbf{d}\bullet(i\sigma\sigma_y)\right]_{\alpha\beta}. \tag{III.275}$$

Choosing $\mathbf{d} \parallel \hat{z} \parallel \hat{c}$ so that $S_z = 0$ (because $\mathbf{d}\bullet\mathbf{S}(\mathbf{k}_F) = 0$), we have

$$\overleftrightarrow{\Delta}(\mathbf{k}_F) = \begin{pmatrix} 0 & \Delta_0(\mathbf{k}_F) \\ \Delta_0(\mathbf{k}_F) & 0 \end{pmatrix}, \tag{III.276}$$

where the two-dimensional orbital state is given by

$$\Delta_0(\mathbf{k}_F) = \left[\left(k_x^2 - k_y^2\right)\eta_1 + 2\left(k_x k_y\right)\eta_2\right]k_z. \tag{III.277}$$

In the case of strong spin-orbit coupling, the terms in the Landau-Ginzburg functional $\mathcal{F}$ must be invariant under the symmetry group $\mathcal{G} = [D_{6h}]_{\text{spin-orbit}} \otimes \mathcal{T} \otimes U(1)$. The form of $\mathcal{F}$ is governed by the linearly independent invariants that are constructed from fourth-order products of the form

$$\sum_{i,j,k,l} b_{ijkl}\,\eta_i \eta_j \eta_k^* \eta_l^* \tag{III.278}$$

and second-order gradient terms

$$\sum_{i,j,k,l} \kappa_{ijkl}\left(D_i\eta_j\right)\left(D_k\eta_l\right)^*, \qquad \text{where } D_i = \partial_i + 2ieA_i. \tag{III.279}$$

Thus, we may express $\mathcal{F}$ in the following general form with $(i,j) = (1,2)$:

$$\mathcal{F}[\boldsymbol{\eta},\mathbf{A}] = \int d^3\mathbf{r}\left\{\alpha(T)\,\boldsymbol{\eta}\bullet\boldsymbol{\eta}^* + \beta_1\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*\right)^2 + \beta_2\left|\boldsymbol{\eta}\bullet\boldsymbol{\eta}\right|^2 + \frac{1}{8\pi}|\mathbf{b}|^2 \right.$$
$$\left. + \kappa_1\left(D_i\eta_j\right)\left(D_i\eta_j\right)^* + \kappa_2\left(D_i\eta_i\right)\left(D_j\eta_j\right)^* + \kappa_3\left(D_i\eta_j\right)\left(D_j\eta_i\right)^* + \kappa_4\left(D_z\eta_j\right)\left(D_z\eta_j\right)^*\right\}, \tag{III.280}$$

and $\mathcal{F}$ at its minimum is the free energy difference between the superconducting state and the normal state; $\left[\alpha(T),\beta_1,\beta_2,\kappa_1,\kappa_2,\kappa_3,\kappa_4\right]$ are material parameters that can be calculated from microscopic theory or determined from experiments; and $\mathbf{b} = \partial \times \mathbf{A}$ is the magnetic field. Therefore, the equilibrium order parameter and the current distribution are determined by finding the stationary conditions of $\mathcal{F}$. That is,

$$\frac{\delta\mathcal{F}[\boldsymbol{\eta},\mathbf{A}]}{\delta\eta_i^*} = 0 \qquad \text{and} \qquad \frac{\delta\mathcal{F}[\boldsymbol{\eta},\mathbf{A}]}{\delta A_i} = 0. \tag{III.281}$$

Thus, from EQs. (III.360) and (III.361), we obtain the following two conditions:

$$\left(\kappa_1+\kappa_2+\kappa_3\right)D_x^2\eta_1 + \kappa_1 D_y^2\eta_1 + \kappa_4 D_z^2\eta_1 + \left(\kappa_2 D_x D_y + \kappa_3 D_y D_x\right)\eta_2$$

$$+2\beta_1\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*\right)\eta_1+2\beta_2\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}\right)\eta_1^*=-\alpha\eta_1, \tag{III.282}$$

$$\kappa_1 D_x^2\eta_2+\left(\kappa_1+\kappa_2+\kappa_3\right)D_y^2\eta_2+\kappa_4 D_z^2\eta_2+\left(\kappa_2 D_y D_x+\kappa_3 D_x D_y\right)\eta_1$$
$$+2\beta_1\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*\right)\eta_2+2\beta_2\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}\right)\eta_2^*=-\alpha\eta_2, \tag{III.283}$$

These two equations together with the following Maxwell' equation

$$\left(\nabla\times\mathbf{b}\right)_i=-16\pi e\,\mathrm{Im}\left[\kappa_1\eta_j\left(D_{\perp,i}\eta_j\right)^*+\kappa_2\eta_i\left(D_{\perp,j}\eta_j\right)^*+\kappa_3\eta_j\left(D_{\perp,j}\eta_i\right)^*+\kappa_4\delta_{iz}\eta_j\left(D_z\eta_j\right)^*\right] \tag{III.284}$$

form the basis for studying the *H*-vs.-*T* phase diagram, vortices and related magnetic properties of triplet superconductors. Specifically, there are two possible homogeneous states that are dependent on the sign of β_2. For $\left(-\beta_1\right)<\beta_2<0$, the equilibrium order parameter $\boldsymbol{\eta}=\eta_0\,\hat{x}$ (or any of the six degenerate states obtained by rotation) breaks rotational symmetry in the basal plane, but still preserves the time-reversal symmetry. For $\beta_2>0$, the order parameter retains full rotational symmetry (provided that each rotation is combined with a proper gauge transformation), but spontaneously breaks the time-reversal symmetry. This case corresponds to a doubly-degenerate equilibrium state, with an order parameter of the form

$$\boldsymbol{\eta}_+=\frac{1}{\sqrt{2}}\eta_0\left(x+iy\right)\equiv\frac{1}{2}\sqrt{\frac{|\alpha|}{\beta_1}}\left(x+iy\right)\qquad[\text{or } \boldsymbol{\eta}_-=\boldsymbol{\eta}_+^*]. \tag{III.285}$$

To derive better understanding of the phases associated with different parameters in the Landau-Ginzburg functional $\mathcal{F}$, we consider the simple case of zero-field solutions for a homogeneous system, so that all gradient terms can be neglected. In this case, we find that EQ. (III.360) becomes:

$$\mathcal{F}\left[\boldsymbol{\eta}\right]=\int d^3\mathbf{r}\left\{\alpha(T)\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*+\beta_1\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*\right)^2+\beta_2\left|\boldsymbol{\eta}\bullet\boldsymbol{\eta}\right|^2\right\}. \tag{III.286}$$

We may express $\boldsymbol{\eta}$ in terms of a global phase $\theta(\mathbf{r})$ and an internal, relative phase $\zeta(\mathbf{r})$, so that

$$\boldsymbol{\eta}=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=e^{i\theta}\begin{pmatrix}|\eta_1|e^{i\zeta/2}\\ |\eta_2|e^{-i\zeta/2}\end{pmatrix}. \tag{III.287}$$

The global phase $\theta(\mathbf{r})$ is associated with broken $U(1)$ symmetry and $\mathbf{p}_S\sim\nabla\theta$, whereas $\zeta(\mathbf{r})$ is associated with the broken time-reversal symmetry. Inserting these two phases into EQ. (III.286), we rewrite $\mathcal{F}$ into

$$\mathcal{F}\left[\boldsymbol{\eta}\right]=\int d^3\mathbf{r}\left\{\alpha(T)\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*+\beta_1\left(\boldsymbol{\eta}\bullet\boldsymbol{\eta}^*\right)^2+\beta_2\left|\boldsymbol{\eta}\bullet\boldsymbol{\eta}\right|^2\right\}. \tag{III.288}$$

Minimizing the free energy, we find two sets of stable solutions (see Fig.V.5.3), depending on the sign of β_2, and we note that β_1 is always > 0.

<u>Case I</u>: ($\boldsymbol{\beta_1>0}$ and $\boldsymbol{0>\beta_2>-\beta_1}$)

In this case, we have

$$\boldsymbol{\eta}=\eta_0 e^{i\theta}\begin{pmatrix}1\\0\end{pmatrix}, \tag{III.289}$$

which implies that there is broken rotational symmetry from $\mathcal{G} = [D_{6h}]_{\text{spin-orbit}} \otimes \mathcal{T} \otimes U(1)$ in the normal state to $\mathcal{G}' = [D_{2h}]_{\text{spin-orbit}} \otimes \mathcal{T}$ in the superconducting state.

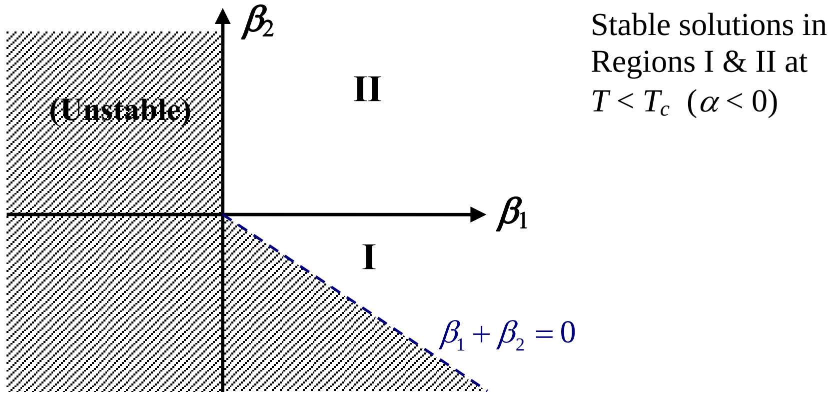


Fig. III.4.3 Schematic of the stable solutions for triplet-pairing superconductivity in the (β_1,β_2)-plane.

<u>Case II</u>: ($\beta_1 > 0$ and $\beta_2 > 0$)

In this case, we have

$$\boldsymbol{\eta}_+ = \eta_0 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta}_- = \eta_0 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}, \tag{III.290}$$

which implies that there is broken time-reversal symmetry from $\mathcal{G} = [D_{6h}]_{\text{spin-orbit}} \otimes \mathcal{T} \otimes U(1)$ in the normal state to $\mathcal{G}' = [D_{6h}]_{\text{spin-orbit}}$ in the superconducting state. Therefore, in this phase the *ab*-plane isotropy is preserved, and the order parameter exhibits two-fold degeneracy:

$$\Delta_+(\mathbf{k}_F) = \eta_0 \frac{1}{\sqrt{2}}\left[\left(k_x^2 - k_y^2\right) + i\,2\left(k_x k_y\right)\right]k_z\,, \tag{III.291}$$

$$\Delta_-(\mathbf{k}_F) = \eta_0 \frac{1}{\sqrt{2}}\left[\left(k_x^2 - k_y^2\right) - i\,2\left(k_x k_y\right)\right]k_z\,, \tag{III.292}$$

which can be visualized in Fig. III.4.4. Thus, there is an internal angular momentum:

$$M_{\text{orb}} = 2e\left(\kappa_2 - \kappa_3\right)\text{Im}\left[\boldsymbol{\eta} \times \boldsymbol{\eta}^*\right], \tag{III.293}$$

which can couple diamagnetically to an external field.

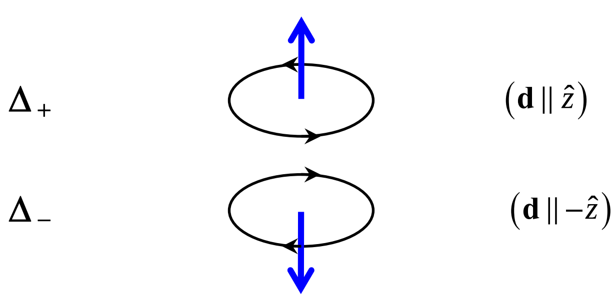


Fig. III.4.4 Degenerate order parameters Δ_+ and Δ_- for the stable solutions in Case II.

In principle, one can extend the above formalism to more complicated situations with finite magnetic fields and/or order parameter gradients. We shall not pursue further here. For more details on specific properties and phases associated with triplet superconductors, you may refer to some of the references listed below. However, before closing it is worth noting an interesting development associated with a heavy-fermion superconductor $SrRuO_4$. Although not yet rigorously comfirmed, this superconductor is believed to have a novel pairing symmetry of (p_x + ip_y) based on a number of theoretical and indirect experimental evidences. This specific pairing symmetry, if proven existent, can acquire novel "zero-mode" low-energy excitations in the vortex-core states. Specifically, if we consider 2n well separated vortices in a (p_x + ip_y)-pairing superconductor, there will be n zero modes for "Majorana fermions", so that there are a total of 2^n degenerate states for even fermion number and 2^{n-1} degenerate states for odd fermion number. Such degeneracy is crucial for the occurrence of non-abelian statistics in topological quantum computation.

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