

## **PART I. Introduction & Overview of Modern Condensed Matter Physics**

Condensed matter physics is a branch of physics that investigates the physical phenomena associated with many-body interaction of materials in their “condensed” (i.e. liquid and solid) states. The objective of this summer course is to apply quantum field theory to several modern topics of condensed matter physics. Typically most condensed matter physics topics can be described in terms of non-relativistic quantum field theory. However, for topics involving gauge theory (*e.g.* spin liquid, high-temperature superconductivity) and topological field theory (*e.g.* fractional quantum-Hall states), knowledge of relativistic quantum field theory becomes necessary. In the interest of time, we shall focus on non-relativistic descriptions of the many-body interactions of fermions and bosons, although the basics of relativistic quantum field theory will be briefly reviewed before taking the relativistic quantum field theory to the non-relativistic limit, and concepts required for gauge theory, fractional statistics and topological field theory of anyons will also be covered when we discuss high-temperature superconductivity and fractional quantum Hall states.

Throughout this course we shall mostly use natural units in which the Dirac symbol  $\hbar$  (which is the Planck constant  $h$  divided by  $2\pi$ ) and the speed of light  $c$  are both set to 1, although at times we’ll restore them for comparison with experiments. Thus, in natural units both time and space are treated on the same footing, and mass is inversely proportional to length.

### **I.1. Overview of Modern Condensed Matter Physics**

The scope of condensed matter physics has evolved and expanded significantly in recently years, from the traditional “solid state physics” that is primarily representative of band structures, semiconductor physics and Landau symmetry-breaking theory of phase transitions, to a vast arena encompassing topics from highly interdisciplinary research such as nano- and biophysical sciences, optical lattices and Bose-Einstein condensation in laser-cooled atoms, quantum computation, to fundamental subjects such as gauge theory, quantum orders and phase transitions, fractional statistics, spin liquids, topological field theory, and string-net condensate for unification of fermions and bosons. In the development of modern condensed matter physics, quantum field theory (QFT) and group theory have played a very important role. In fact, QFT has found applications to a wide range of modern physics ranging from high energy physics, condensed matter physics, quantum gravity, to string theory. In this context, it appears natural to describe modern condensed matter physics in the language of QFT.

Generally speaking, quantum field theory is a confluence of quantum mechanics and special relativity, and it builds on important concepts of group theory and symmetries so that it can deal with any space-time dimensions, matter-energy interactions (including three out of four of the known fundamental forces in the universe: strong, electromagnetic and weak interactions) for both fermionic and bosonic fields, as well as topological order and fractional statistics. Specifically, most condensed matter physics topics can be described by non-relativistic quantum field theory. Interestingly, however, connections of condensed matter physics to relativistic quantum field theory can be found in various aspects of topological orders and defects, and also in certain aspects of strongly correlated electronic systems, such as in the fractional quantum Hall states and high-temperature superconductivity.

There are generally different approaches taken for dealing with QFT. The most common approaches include the canonical quantization formalism and the path integral formalism. Both approaches evoke the use of Feynman diagrams and are convenient for different purposes. In this course we shall adapt both approaches depending on the circumstances, with more emphasis on the path integral formalism because of

its elegance and convenience for introducing various important issues such as the Lorentz invariance, gauge invariance, topological orders, and renormalization theory.

The foundation of conventional condensed matter physics may be regarded as building on two conceptual cornerstones: the Fermi liquid theory, and the Landau symmetry-breaking theory of phase transitions. The Fermi liquid theory treats properties of electronic states in solids as perturbations of a ground state consisting of filling the single-particle energy levels. The Landau theory for phase transitions of matter classifies different phases of matter by their symmetries, so that phase transitions are associated with changes in the symmetry of the state of matter. However, these cornerstones can no longer hold grounds when facing the challenge of emerging phenomena in various modern topics of condensed matter physics. For instance, some of the strongly correlated electronic systems, such as high temperature superconductors, quantum Hall phenomena in two-dimensional electron gas, and “Luttinger liquid” in one-dimensional conducting systems, all involve properties beyond the perturbative descriptions of the conventional Fermi-liquid theory, and certain aspects of the strong correlation in these systems actually resemble phenomena encountered in high-energy physics. Similarly, conventional notions of broken symmetry associated with phase transitions are no longer applicable to the depiction of systems involving topological orders and their phase transitions. Well known examples include the fractional quantum Hall (FQH) systems and spin liquids, where transitions among different topological orders can occur without changing the corresponding symmetries. Hence, new foundations must be established to describe these emerging areas of condensed matter physics. This course is intended to first familiarize students with the conceptual foundation and basic language of conventional condensed matter physics, and then proceed to the introduction of representative new developments in modern condensed matter physics.

The course is structured as follows. Part I reviews the second quantization techniques, quantum dynamics, pictures of quantum dynamics and time-dependent perturbation theory, and the basic notions of low-energy excitations in solids. In Part II quantum field theory for many-body systems is introduced in terms of both canonical and path integral formalisms, followed by descriptions of the Feynman diagrams for relativistic quantum field theory, and then extended to non-relativistic quantum field theory for many-body systems using the Green’s function techniques. Superconductivity, a special state of matter involving a bosonic ground state and fermionic low-energy excitations, is studied in Part III for the case of conventional and heavy Fermion superconductors. Part IV introduces the concept of gauge theory that is beyond the global symmetry consideration, including the non-abelian gauge theory with local symmetries and the corresponding broken symmetry that leads to the Anderson-Higgs mechanism. Part V discusses recent developments in high-temperature superconductivity, including topics of and the phenomenology and various attempts in the microscopic theory. Finally, Part VI investigates topics related to topological field theory, including topological defects, fractional quantum Hall effects, fractional statistics, basic properties of abelian and non-abelian anyons, and possible applications of non-abelian anyons to quantum computation.

## **I.2. Review of the Second Quantization Techniques**

Second quantization is a useful tool for dealing with many-body phenomena in condensed matter physics. The method was developed by P. A. M. Dirac in 1927 for photons in radiation theory, and later on extended to fermions by E. Wigner and P. Jordon in 1928.

The key concept of 2<sup>nd</sup> quantization technique is to describe a system of particles in the “occupation number” space, in contrast to the description of particles in terms of ordinary coordinate wave functions  $\Psi$  as in the first quantization language. Specifically, the distinction between the languages of 1<sup>st</sup> and 2<sup>nd</sup> quantization is summarized as follows:

1<sup>st</sup> quantization description:  $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$  in terms of the coordinates  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$  of the particles;

2<sup>nd</sup> quantization description: Hilbert space  $(n_1, n_2, \dots, n_\infty)$  in terms of the occupation numbers  $n_1, n_2, \dots, n_\infty$  for the states of the particles;

Here a Hilbert space is referred to as an infinite-dimensional linear space consisting of linearly independent orthonormal functions  $\varphi_n(x)$ , such that any well-behaved function  $F(x)$  can be approximated by

$$F(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad \text{where } a_m = \int dx F(x) \varphi_m(x) \rho(x), \quad (\text{I.1})$$

and  $\rho(x)$  is the density function.

To convert 1<sup>st</sup> quantization expressions to 2<sup>nd</sup> quantization, let's consider the following Hamiltonian for  $N$ -interacting particles:

$$\mathcal{H} = \sum_{i=1}^N \mathcal{T}(\mathbf{r}_i, \dot{\mathbf{r}}_i) + \frac{1}{2} \sum_{i,j \neq i}^N \mathcal{V}(\mathbf{r}_i, \mathbf{r}_j), \quad (\text{I.2})$$

where  $\mathcal{T}$  and  $\mathcal{V}$  denote the kinetic and potential energies, respectively. The time-dependent many-body wave function  $\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)$  satisfies the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = \mathcal{H} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t), \quad (\text{I.3})$$

and

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = \sum_{\{\mathbf{k}_1, \dots, \mathbf{k}_N\}} c(\mathbf{k}_1, \dots, \mathbf{k}_N, t) \varphi_{\mathbf{k}_1}(\mathbf{r}_1) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N), \quad (\text{I.4})$$

where  $\varphi_{\mathbf{k}_i}(\mathbf{r}_i)$  ( $i = 1, 2, \dots, N$ ) denote the single-particle wave function,  $\mathbf{k}_i$  represent the quantum numbers (which need not be the wave vector unless specified), and  $c(\mathbf{k}_1, \dots, \mathbf{k}_N, t)$  are coefficients satisfying the following permutation conditions:

$$c(\mathbf{k}_1, \dots, \mathbf{k}_i, \dots, \mathbf{k}_j, \dots, \mathbf{k}_N, t) = -c(\mathbf{k}_1, \dots, \mathbf{k}_j, \dots, \mathbf{k}_i, \dots, \mathbf{k}_N, t) \quad \text{for fermions,} \quad (\text{I.5})$$

$$c(\mathbf{k}_1, \dots, \mathbf{k}_i, \dots, \mathbf{k}_j, \dots, \mathbf{k}_N, t) = c(\mathbf{k}_1, \dots, \mathbf{k}_j, \dots, \mathbf{k}_i, \dots, \mathbf{k}_N, t) \quad \text{for bosons.} \quad (\text{I.6})$$

In the case of fermions, we must keep track of the sign change. We may define occupations numbers and write a Slater determinant as follows:

$$\begin{aligned} & c(\mathbf{k}_1, \dots, \mathbf{k}_N, t) \varphi_{\mathbf{k}_1}(\mathbf{r}_1) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N) + [\text{all permutations with the set } (\mathbf{k}_1, \dots, \mathbf{k}_N)] \\ & \equiv f(n_1, \dots, n_\infty, t) \begin{vmatrix} \varphi_{\mathbf{k}_1}(\mathbf{r}_1) & \dots & \varphi_{\mathbf{k}_1}(\mathbf{r}_N) \\ \vdots & \ddots & \vdots \\ \varphi_{\mathbf{k}_N}(\mathbf{r}_1) & \dots & \varphi_{\mathbf{k}_N}(\mathbf{r}_N) \end{vmatrix}. \end{aligned} \quad (\text{I.7})$$

Hence, we find that

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = \sum_{\{n_1, \dots, n_\infty\}} f(n_1, \dots, n_\infty, t) \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_{\mathbf{k}_1}(\mathbf{r}_1) & \dots & \varphi_{\mathbf{k}_1}(\mathbf{r}_N) \\ \vdots & \ddots & \vdots \\ \varphi_{\mathbf{k}_N}(\mathbf{r}_1) & \dots & \varphi_{\mathbf{k}_N}(\mathbf{r}_N) \end{vmatrix}. \quad (\text{I.8})$$

For comparison, the wave function for bosons is given by

$$\begin{aligned} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) &= \sum_{\{\mathbf{k}_1, \dots, \mathbf{k}_N\}} c(\mathbf{k}_1, \dots, \mathbf{k}_N, t) \varphi_{\mathbf{k}_1}(\mathbf{r}_1) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N), \\ &\equiv \sum_{\{n_1, \dots, n_\infty\}} f(n_1, \dots, n_\infty, t) \Phi_{n_1, \dots, n_\infty}(\mathbf{r}_1, \dots, \mathbf{r}_N), \\ &\equiv \sum_{\{n_1, \dots, n_\infty\}} f(n_1, \dots, n_\infty, t) \left[ \left( \frac{N_1! N_2! \dots}{N!} \right)^{1/2} \sum_{\{\mathbf{k}_1, \dots, \mathbf{k}_N\}} \varphi_{\mathbf{k}_1}(\mathbf{r}_1) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N) \right] \end{aligned} \quad (\text{I.9})$$

where  $N = \sum_i N_i$  and  $N_i$  denotes the number of particles in state  $\mathbf{k}_i$ .

Next, we define a Hilbert space spanned by the basis vectors  $|n_1, n_2, \dots, n_\infty\rangle$  and also define the creation and annihilation operators  $a_i^\dagger$  and  $a_i$  that satisfy the anti-commutation relations for fermions:

$$\begin{aligned} \{a_i^\dagger, a_j\} &\equiv a_i^\dagger a_j + a_j a_i^\dagger = \delta_{ij}, \\ \{a_i, a_j\} &= \{a_i^\dagger, a_j^\dagger\} = 0. \end{aligned} \quad (\text{I.10})$$

For comparison, we note that the creation and annihilation operators  $b_i^\dagger$  and  $b_i$  for bosons satisfy the following commutation relations:

$$\begin{aligned} [b_i^\dagger, b_j] &\equiv b_i^\dagger b_j - b_j b_i^\dagger = \delta_{ij}, \\ [b_i, b_j] &= [b_i^\dagger, b_j^\dagger] = 0. \end{aligned} \quad (\text{I.11})$$

Returning to the case of fermions, a basis vector  $|n_1, n_2, \dots, n_\infty\rangle$  can now be written in terms of the creation operators:

$$|n_1, n_2, \dots, n_\infty\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_\infty^\dagger)^{n_\infty} |0\rangle, \quad (n_i = 0 \text{ or } 1), \quad (\text{I.12})$$

where  $|0\rangle$  denotes the vacuum state, so that

$$\begin{aligned} a_k |n_1, \dots, n_k, \dots, n_\infty\rangle &= 0, & \text{if } n_k = 0, \\ &= (-1)^{\sum_{i=1}^{k-1} n_i} |n_1, \dots, (n_k - 1), \dots, n_\infty\rangle, & \text{if } n_k = 1, \end{aligned} \quad (\text{I.13})$$

and

$$\begin{aligned} a_k^\dagger |n_1, \dots, n_k, \dots, n_\infty\rangle &= (-1)^{\sum_{i=1}^{k-1} n_i} |n_1, \dots, (n_k + 1), \dots, n_\infty\rangle, & \text{if } n_k = 0, \\ &= 0, & \text{if } n_k = 1, \end{aligned} \quad (\text{I.14})$$

with  $\sum_k = n_1 + n_2 + \dots + n_{k-1}$ . We note that creation operator  $a_k^\dagger$  is the Hermitian conjugate of the annihilation operator  $a_k$ . Equations (I.13) and (I.14) may be rewritten into the following equivalent expressions:

$$a_k |n_1, \dots, n_k, \dots, n_\infty\rangle = (-1)^{\sum_{j=1}^{k-1} n_j} \sqrt{n_k} |n_1, \dots, (n_k - 1), \dots, n_\infty\rangle, \quad (\text{I.15})$$

$$a_k^\dagger |n_1, \dots, n_k, \dots, n_\infty\rangle = (-1)^{\sum_{j=1}^{k-1} n_j} \sqrt{1 - n_k} |n_1, \dots, (n_k + 1), \dots, n_\infty\rangle. \quad (\text{I.16})$$

From EQ. (I.15) and EQ. (I.16), we obtain

$$a_k^\dagger a_k = n_k. \quad (\text{I.17})$$

For comparison, if we apply the bosonic operators to a state defined by  $|n_1, n_2, \dots, n_\infty\rangle$ , we obtain

$$b_k |n_1, \dots, n_k, \dots, n_\infty\rangle = \sqrt{n_k} |n_1, \dots, (n_k - 1), \dots, n_\infty\rangle, \quad (\text{I.18})$$

$$b_k^\dagger |n_1, \dots, n_k, \dots, n_\infty\rangle = \sqrt{n_k + 1} |n_1, \dots, (n_k + 1), \dots, n_\infty\rangle. \quad (\text{I.19})$$

In the case of bosons, there are no restrictions to the occupation number.

To express the Schrödinger equation of fermions in a second-quantization form, we first rewrite the Slater determinant in EQ. (I.8) into the following:

$$\frac{1}{\sqrt{N!}} \sum_{\mathbf{P} \in S_N} (-1)^p \mathbf{P} [\varphi_{\mathbf{k}_1}(\mathbf{r}_1) \varphi_{\mathbf{k}_2}(\mathbf{r}_2) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N)] \quad (\text{I.20})$$

where  $\mathbf{P}$  denotes the operator that permutes the order of electrons,  $S_N$  denotes the permutation group for  $N$  indistinguishable objects, and  $p$  is the number of permutations. Next, we define the wave function

$$\Psi(t) = \sum_{\{n'_1, \dots, n'_\infty\}} f(n'_1, \dots, n'_\infty, t) |n'_1, \dots, n'_\infty\rangle. \quad (\text{I.21})$$

Inserting the expressions of EQs. (I.8), (I.20) and (I.21) into EQ. (I.3), we obtain

$$\begin{aligned} i\hbar \frac{\partial \Psi(t)}{\partial t} &= i\hbar \sum_{\{n'_1, \dots, n'_\infty\}} \frac{\partial f(n'_1, \dots, n'_\infty, t)}{\partial t} \frac{1}{\sqrt{N!}} \sum_{\mathbf{P}} (-1)^p \mathbf{P} [\varphi_{\mathbf{k}_1}(\mathbf{r}_1) \varphi_{\mathbf{k}_2}(\mathbf{r}_2) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N)], \\ &= \mathcal{H} \sum_{\{n'_1, \dots, n'_\infty\}} f(n'_1, \dots, n'_\infty, t) \frac{1}{\sqrt{N!}} \sum_{\mathbf{P}} (-1)^p \mathbf{P} [\varphi_{\mathbf{k}_1}(\mathbf{r}_1) \varphi_{\mathbf{k}_2}(\mathbf{r}_2) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N)]. \end{aligned} \quad (\text{I.22})$$

We may multiply both sides of EQ. (I.22) by the conjugate of a particular Slater determinant that corresponds to a specific set of occupation numbers  $(n_1, \dots, n_\infty)$ , which yields

$$\begin{aligned} i\hbar \frac{\partial f(n_1, \dots, n_\infty, t)}{\partial t} &= \frac{1}{N!} \sum_{\mathbf{P}, \mathbf{P}'} (-1)^{p+p'} f(n_1, \dots, n_\infty, t) \int d\mathbf{r}_1 \dots d\mathbf{r}_N \\ &\quad \times \mathbf{P}' [\varphi_{\mathbf{k}_1}^*(\mathbf{r}_1) \varphi_{\mathbf{k}_2}^*(\mathbf{r}_2) \dots \varphi_{\mathbf{k}_N}^*(\mathbf{r}_N)] \mathcal{H} \mathbf{P} [\varphi_{\mathbf{k}_1}(\mathbf{r}_1) \varphi_{\mathbf{k}_2}(\mathbf{r}_2) \dots \varphi_{\mathbf{k}_N}(\mathbf{r}_N)], \end{aligned} \quad (\text{I.23})$$

For simplicity, we first consider a Hamiltonian with kinetic energy only. That is,  $\mathcal{V}(\mathbf{r}_i, \mathbf{r}_j) = 0$ , and

$$\mathcal{H} = \sum_i \mathcal{T}(\hat{\mathbf{r}}_i), \quad i = 1, 2, \dots, N. \quad (\text{I.24})$$

Since  $\mathcal{T}(\mathbf{r}_i)$  is a one-particle operator, the set of occupation numbers  $\{n'_1, \dots, n'_\infty\}$  cannot differ from a given set  $\{n_1, \dots, n_\infty\}$  in EQ. (I.23) by more than two numbers. Thus, the right side of EQ. (I.23) can be simplified into the following form if we assume that  $k < l$  and define  $|k\rangle$  ( $|l\rangle$ ) as the  $k^{\text{th}}$  ( $l^{\text{th}}$ ) state of which  $\{n_1, \dots, n_\infty\}$  has an occupation number  $n_k$  ( $n_l$ ):

$$\begin{aligned} & \sum_{k,l} (-1)^{\Sigma_k + \Sigma_l} f(n_1, \dots, n_k - 1, \dots, n_l + 1, \dots, n_\infty, t) \int d\mathbf{r} \varphi_k^*(\mathbf{r}) \mathcal{T}(\mathbf{r}) \varphi_l(\mathbf{r}) \\ & \equiv \sum_{k,l} (-1)^{\Sigma_k + \Sigma_l} f(n_1, \dots, n_k - 1, \dots, n_l + 1, \dots, n_\infty, t) \langle k | \mathcal{T} | l \rangle. \end{aligned} \quad (\text{I.25})$$

Therefore the Schrödinger equation using EQ. (I.23) and EQ. (I.25) becomes:

$$\begin{aligned} i\hbar \frac{\partial |\Psi(t)\rangle}{\partial t} &= \sum_{\{n_1, \dots, n_\infty\}} i\hbar \frac{\partial f(n_1, \dots, n_\infty, t)}{\partial t} |n_1, \dots, n_\infty\rangle, \\ &= \sum_{\{n_1, \dots, n_\infty\}} \sum_{k,l} f(n_1, \dots, n_k - 1, \dots, n_l + 1, \dots, n_\infty, t) \langle k | \mathcal{T} | l \rangle (-1)^{\Sigma_k + \Sigma_l} |n_1, \dots, n_\infty\rangle, \\ &= \sum_{\{n_1, \dots, n_\infty\}} \sum_{k,l} f(n_1, \dots, n_k - 1, \dots, n_l + 1, \dots, n_\infty, t) \langle k | \mathcal{T} | l \rangle a_k^\dagger a_l |n_1, \dots, n_k - 1, \dots, n_l + 1, \dots, n_\infty\rangle, \\ &= \sum_{k,l} \langle k | \mathcal{T} | l \rangle a_k^\dagger a_l \Psi(t). \end{aligned} \quad (\text{I.26})$$

We note that in EQ. (I.26) the sum over  $i$ ,  $\mathbf{P}$  and  $\mathbf{P}'$  give a factor  $N!$  that cancels the same factor in the denominator. Consequently, from EQ. (I.26) we find that for  $\mathcal{H} = \sum_{i=1}^N \mathcal{T}(\mathbf{r}_i)$ , its second-quantization form is:

$$\mathcal{H} = \sum_{k,l} \langle k | \mathcal{T} | l \rangle a_k^\dagger a_l. \quad (\text{I.27})$$

Following similar procedures that lead to EQ. (I.27), the interaction term of the Hamiltonian,  $\mathcal{V}(\mathbf{r}_i, \mathbf{r}_j)$ , can be expressed by the second-quantization form (see Problem Set 1):

$$\frac{1}{2} \sum_{k,l,s,t} \langle kl | \mathcal{V} | st \rangle a_k^\dagger a_l^\dagger a_t a_s \equiv \frac{1}{2} \sum_{k,l,s,t} \left[ \int d\mathbf{r}_1 d\mathbf{r}_2 \varphi_k^*(\mathbf{r}_1) \varphi_l^*(\mathbf{r}_2) \mathcal{V}(\mathbf{r}_1, \mathbf{r}_2) \varphi_s(\mathbf{r}_1) \varphi_t(\mathbf{r}_2) \right] a_k^\dagger a_l^\dagger a_t a_s. \quad (\text{I.28})$$

Hence, the second-quantization expression for a general Hamiltonian with both the kinetic and potential energy terms is given by:

$$\mathcal{H} = \sum_{k,l} \langle k | \mathcal{T} | l \rangle a_k^\dagger a_l + \frac{1}{2} \sum_{k,l,s,t} \langle kl | \mathcal{V} | st \rangle a_k^\dagger a_l^\dagger a_t a_s. \quad (\text{I.29})$$

A more general recipe to second-quantize an interaction Hamiltonian is by means of the field operator  $\psi(\mathbf{r})$  in the Hilbert space, where the field operator is defined in terms of a complete set of single particle states  $\varphi_k(\mathbf{r})$  characterized by the quantum number  $k$ :

$$\psi(\mathbf{r}) = \sum_k \varphi_k(\mathbf{r}) a_k, \quad (\text{I.30})$$

and  $a_k$  is a fermion operator. Using EQ. (I.30), we may obtain the second-quantization expression of an interaction Hamiltonian by writing the following:

$$\mathcal{H} = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \mathcal{T}(\mathbf{r}) \psi(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \mathcal{V}(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}). \quad (\text{I.31})$$

You can easily verify that EQ. (I.31) is entirely consistent with EQ. (I.29).

### [Coulomb interaction in a degenerate electron gas]

Now let's consider applying the above formalism to an explicit example, the Coulomb interaction in a degenerate electron gas. The Hamiltonian for  $N$  electrons of mass  $m_e$  and charge  $e$  is approximated by:

$$\mathcal{H}_{e-e} = \sum_{i=1}^N \frac{p_i^2}{2m_e} + \frac{e^2}{2} \sum_{i \neq j}^N \frac{\exp(-\mu|\mathbf{r}_i - \mathbf{r}_j|)}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (\text{I.32})$$

where  $\mu$  is the screening coefficient of the electron gas,  $\mathbf{r}_i$  and  $\mathbf{r}_j$  are the positions of the electrons, and  $p_i$  denotes the momentum of the  $i$ -th electron. The field operator is given by

$$\psi(\mathbf{r}) = \sum_{\mathbf{k}, \sigma} \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k} \cdot \mathbf{r}} \eta_\sigma a_{\mathbf{k}\sigma}, \quad (\text{I.33})$$

where  $\mathbf{k}$  denotes the wave vector,  $\sigma$  is the spin quantum number,  $\Omega$  is the volume, and

$$\eta_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{I.34})$$

are the spinors. We also note that the following condition is satisfied:

$$\int d\mathbf{r} \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) = \sum_{\mathbf{k}, \sigma} a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} = N. \quad (\text{I.35})$$

Using EQ. (I.31) and EQ. (I.33), the Coulomb interaction term of the Hamiltonian in EQ. (I.32) becomes:

$$\begin{aligned} & \frac{e^2}{2\Omega^2} \sum_{\mathbf{k}_1 \sigma_1} \sum_{\mathbf{k}_2 \sigma_2} \sum_{\mathbf{k}_3 \sigma_3} \sum_{\mathbf{k}_4 \sigma_4} \left\{ \int d\mathbf{r} \int d\mathbf{r}' e^{-i(\mathbf{k}_1 \cdot \mathbf{r} + \mathbf{k}_2 \cdot \mathbf{r}')} \left( \frac{e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) e^{i(\mathbf{k}_3 \cdot \mathbf{r} + \mathbf{k}_4 \cdot \mathbf{r}')} \right. \\ & \quad \left. \times \eta_{\sigma_1}^\dagger(\mathbf{r}) \eta_{\sigma_2}^\dagger(\mathbf{r}') \eta_{\sigma_4}(\mathbf{r}') \eta_{\sigma_3}(\mathbf{r}) a_{\mathbf{k}_1 \sigma_1}^\dagger a_{\mathbf{k}_2 \sigma_2}^\dagger a_{\mathbf{k}_4 \sigma_4} a_{\mathbf{k}_3 \sigma_3} \right\}, \\ & = \frac{e^2}{2\Omega^2} \sum_{\mathbf{k}_1 \sigma_1} \sum_{\mathbf{k}_2 \sigma_2} \sum_{\mathbf{k}_3 \sigma_3} \sum_{\mathbf{k}_4 \sigma_4} \left\{ \delta_{\sigma_1 \sigma_3} \delta_{\sigma_2 \sigma_4} \int d\mathbf{r} \int d\mathbf{r}' e^{-i(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \cdot \mathbf{r}} \left( \frac{e^{i(\mathbf{k}_2 - \mathbf{k}_4) \cdot (\mathbf{r} - \mathbf{r}')} e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) a_{\mathbf{k}_1 \sigma_1}^\dagger a_{\mathbf{k}_2 \sigma_2}^\dagger a_{\mathbf{k}_4 \sigma_4} a_{\mathbf{k}_3 \sigma_3} \right\}, \\ & = \frac{e^2}{2\Omega} \sum_{\mathbf{k}_1 \sigma_1} \sum_{\mathbf{k}_2 \sigma_2} \sum_{\mathbf{k}_3 \sigma_3} \sum_{\mathbf{k}_4 \sigma_4} \left\{ \delta_{\sigma_1 \sigma_3} \delta_{\sigma_2 \sigma_4} \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \left[ \frac{4\pi}{(\mathbf{k}_2 - \mathbf{k}_4)^2 + \mu^2} \right] a_{\mathbf{k}_1 \sigma_1}^\dagger a_{\mathbf{k}_2 \sigma_2}^\dagger a_{\mathbf{k}_4 \sigma_4} a_{\mathbf{k}_3 \sigma_3} \right\}, \quad (\text{I.36}) \end{aligned}$$

where  $\Delta(\mathbf{k})$  is the generalized delta function, and we have used the identity:

$$\int d^3x \left( \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{x} \right) e^{-\mu x} = \frac{4\pi}{q^2 + \mu^2}. \quad (\text{I.37})$$

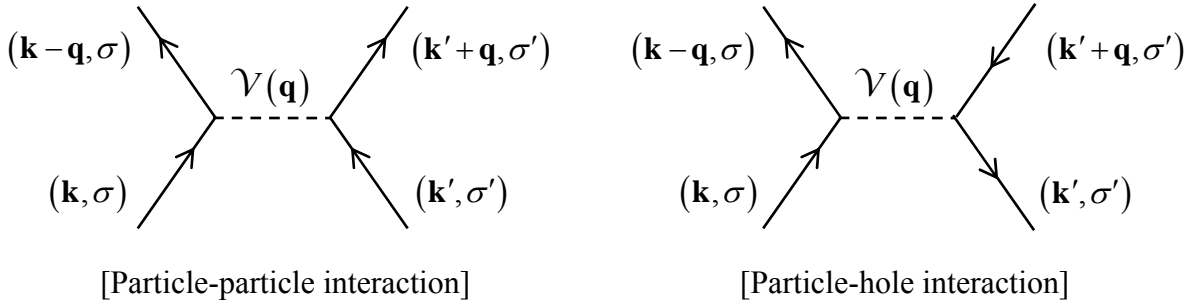
If we further define  $(\mathbf{k}_2 - \mathbf{k}_4) \equiv \mathbf{q}$ ,  $\sigma_1 \equiv \sigma$ ,  $\sigma_2 \equiv \sigma'$ , so that  $\mathbf{k}_1 = (\mathbf{k}_3 - \mathbf{q}) \equiv \mathbf{k} - \mathbf{q}$  and  $\mathbf{k}_2 = (\mathbf{k}_4 + \mathbf{q}) \equiv \mathbf{k}' + \mathbf{q}$ , EQ. (I.36) becomes

$$\frac{e^2}{2\Omega} \sum_{\mathbf{k}\sigma} \sum_{\mathbf{k}'\sigma'} \sum_{\mathbf{q}} \left( \frac{4\pi}{q^2 + \mu^2} \right) a_{\mathbf{k}-\mathbf{q},\sigma}^\dagger a_{\mathbf{k}'+\mathbf{q},\sigma'}^\dagger a_{\mathbf{k}',\sigma'} a_{\mathbf{k},\sigma}, \quad (\text{I.38})$$

The second-quantization expression for the total Hamiltonian of the degenerate electron gas is therefore:

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \left( \frac{\hbar^2 k^2}{2m_e} \right) a_{\mathbf{k},\sigma}^\dagger a_{\mathbf{k},\sigma} + \frac{e^2}{2\Omega} \sum_{\mathbf{k}\sigma} \sum_{\mathbf{k}'\sigma'} \sum_{\mathbf{q}} \left( \frac{4\pi}{q^2 + \mu^2} \right) a_{\mathbf{k}-\mathbf{q},\sigma}^\dagger a_{\mathbf{k}'+\mathbf{q},\sigma'}^\dagger a_{\mathbf{k}',\sigma'} a_{\mathbf{k},\sigma}. \quad (\text{I.39})$$

The physical significance of the interaction Hamiltonian can be readily seen from its second-quantization expression if we compare EQ. (I.39) with the corresponding diagrammatic depictions of the interaction in Figure I.2.1. Specifically, the Coulomb interaction leads to two types of processes, electron-electron scattering and electron-hole pair creation and annihilation. There is no spin flipping involved in the processes because the Hamiltonian does not include magnetic interaction.



**Fig.I.2.1** Diagrammatic depictions of the Coulomb interaction in EQ. (I.39). Here holes refer to the removal particles below the Fermi level.

Thus far we have only considered the second-quantization of fermionic systems. In the following we examine an example involving a bosonic system.

### [Quantization of the free electromagnetic field]

To obtain the second-quantization expression for the free electromagnetic field, we consider the energy  $U$  of an electromagnetic field in the absence of charges and currents, which is given by

$$U = \int d\mathbf{r} \left[ \frac{1}{2} \varepsilon_0 |\mathbf{E}(\mathbf{r})|^2 + \frac{1}{2\mu_0} |\mathbf{B}(\mathbf{r})|^2 \right] \equiv U_E + U_M. \quad (\text{I.40})$$

The electric and magnetic fields  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{B}(\mathbf{r})$  can be expressed in terms of the normal coordinate  $q_\alpha$  and the canonical momentum  $p_\alpha \equiv \dot{q}_\alpha$  as follows:

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}) &= -\frac{\partial \mathbf{A}(\mathbf{r})}{\partial t}, & \mathbf{B}(\mathbf{r}) &= \nabla \times \mathbf{A}(\mathbf{r}), \\
 \mathbf{A}(\mathbf{r}) &= c \left( \frac{\mu_0}{8\pi^3} \right)^{1/2} \sum_{\alpha} \hat{e}_{\alpha} \int d\mathbf{k} q_{\alpha}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}}, & (\alpha : \text{polarization}), \\
 \mathbf{B}(\mathbf{r}) &= ic \left( \frac{\mu_0}{8\pi^3} \right)^{1/2} \sum_{\alpha} \int d\mathbf{k} (\mathbf{k} \times \hat{e}_{\alpha}) q_{\alpha}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}}, \\
 \mathbf{E}(\mathbf{r}) &= -c \left( \frac{\mu_0}{8\pi^3} \right)^{1/2} \sum_{\alpha} \hat{e}_{\alpha} \int d\mathbf{k} \dot{q}_{\alpha}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}}.
 \end{aligned} \tag{I.41}$$

Assuming  $\nabla \cdot \mathbf{A} = 0$  so that  $\mathbf{k} \cdot \hat{e}_{\alpha} = 0$  and  $q_{\alpha}^*(\mathbf{k}, t) = q_{\alpha}(-\mathbf{k}, t)$  to ensure a real vector potential, we obtain

$$\begin{aligned}
 U_M &= \frac{c^2}{2(2\pi)^3} \sum_{\alpha, \alpha'} \int d\mathbf{k} (\mathbf{k} \times \hat{e}_{\alpha}) \cdot (\mathbf{k} \times \hat{e}_{\alpha'}) q_{\alpha}(\mathbf{k}, t) q_{\alpha'}^*(\mathbf{k}, t), \\
 &= \frac{c^2}{2(2\pi)^3} \sum_{\alpha\alpha'} \int d\mathbf{k} (k^2 \delta_{\alpha\alpha'}) q_{\alpha}(\mathbf{k}, t) q_{\alpha'}^*(\mathbf{k}, t) = \frac{1}{2(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} |q_{\alpha}(\mathbf{k}, t)|^2 \omega^2,
 \end{aligned} \tag{I.42}$$

$$U_E = \frac{1}{2(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} |\dot{q}_{\alpha}(\mathbf{k}, t)|^2 = \frac{1}{2(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} |p_{\alpha}(\mathbf{k}, t)|^2. \tag{I.43}$$

Note that in EQ. (I.42), we have used the dispersion relation  $\omega = ck$  for free electromagnetic field. Hence, we obtain the following expression for the Hamiltonian:

$$\mathcal{H} = \frac{1}{2(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} \left[ |p_{\alpha}(\mathbf{k}, t)|^2 + \omega^2 |q_{\alpha}(\mathbf{k}, t)|^2 \right]. \tag{I.44}$$

From EQ. (I.44) we note that  $q_{\alpha}$  and  $p_{\alpha}$  are in fact conjugates so that the corresponding Lagrangian  $\mathcal{L}$  of EQ. (I.44) yields  $p_{\alpha} = \partial \mathcal{L} / \partial \dot{q}_{\alpha} = \dot{q}_{\alpha}$ . Therefore  $q_{\alpha}$  and  $p_{\alpha}$  satisfy the following relations (by restoring  $\hbar$ ):

$$\begin{aligned}
 [q_{\alpha}^*(\mathbf{k}), p_{\alpha'}(\mathbf{k}')] &= i\hbar \delta_{\alpha\alpha'} \delta(\mathbf{k} - \mathbf{k}'), \\
 [q_{\alpha}(\mathbf{k}), q_{\alpha'}(\mathbf{k}')] &= [p_{\alpha}(\mathbf{k}), p_{\alpha'}(\mathbf{k}')] = 0.
 \end{aligned} \tag{I.45}$$

To second-quantize EQ. (I.45), we introduce the photon creation and annihilation operators  $b_{\alpha}^{\dagger}$  and  $b_{\alpha}$  so that

$$\begin{aligned}
 [b_{\alpha}(\mathbf{k}), b_{\alpha'}^{\dagger}(\mathbf{k}')] &= \delta_{\alpha\alpha'} \delta(\mathbf{k} - \mathbf{k}'), \\
 [b_{\alpha}(\mathbf{k}), b_{\alpha'}(\mathbf{k}')] &= [b_{\alpha}^{\dagger}(\mathbf{k}), b_{\alpha'}^{\dagger}(\mathbf{k}')] = 0.
 \end{aligned} \tag{I.46}$$

This leads to

$$b_{\alpha}(\mathbf{k}) = \left( \frac{\omega}{2\hbar} \right)^{1/2} q_{\alpha}(\mathbf{k}) + \frac{i}{(2\hbar\omega)^{1/2}} p_{\alpha}^*(\mathbf{k}),$$

$$b_{\alpha}^{\dagger}(\mathbf{k}) = \left(\frac{\omega}{2\hbar}\right)^{1/2} q_{\alpha}^*(\mathbf{k}) - \frac{i}{(2\hbar\omega)^{1/2}} p_{\alpha}(\mathbf{k}). \quad (\text{I.47})$$

Or equivalently,

$$\begin{aligned} q_{\alpha}(\mathbf{k}) &= \left(\frac{\hbar}{2\omega}\right)^{1/2} [b_{\alpha}(\mathbf{k}) + b_{\alpha}^{\dagger}(-\mathbf{k})], \\ p_{\alpha}(\mathbf{k}) &= \frac{1}{i} \left(\frac{\hbar\omega}{2}\right)^{1/2} [b_{\alpha}(\mathbf{k}) - b_{\alpha}^{\dagger}(-\mathbf{k})]. \end{aligned} \quad (\text{I.48})$$

Thus, we obtain

$$\mathcal{H} = \frac{1}{(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} \left[ b_{\alpha}^{\dagger}(\mathbf{k}) b_{\alpha}(\mathbf{k}) + \frac{1}{2} \right] \hbar\omega_{\alpha}(\mathbf{k}) \equiv \frac{1}{(2\pi)^3} \sum_{\alpha} \int d\mathbf{k} \left[ n_{\alpha}(\mathbf{k}) + \frac{1}{2} \right] \hbar\omega_{\alpha}(\mathbf{k}), \quad (\text{I.49})$$

where  $n_{\alpha}(\mathbf{k}) \equiv b_{\alpha}^{\dagger}(\mathbf{k}) b_{\alpha}(\mathbf{k})$  represents the number of photons with wave vector  $\mathbf{k}$ .

We note that similar techniques may be applied to phonons, the normal modes of lattice vibrations, except that the  $\mathbf{k}$ -vectors are associated with the lattice vibrations, and that  $\omega_{\alpha}(\mathbf{k})$  contains the dispersion relation of phonon branches, in contrast to the dispersion relation  $\omega = ck$  for the free electromagnetic field.

### I.3. Review of Pictures of Quantum Dynamics and Time-Dependent Perturbation Theory

In this section we review three pictures (the Schrödinger, interaction, and Heisenberg pictures) of quantum dynamics and their application to the time-dependent perturbation theory that will be essential for our later development of the Green's function and diagrammatic techniques.

We begin with general consideration for the solution to the following differential equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \mathcal{H}(t) |\Psi(t)\rangle, \quad (\text{I.50})$$

where  $\mathcal{H}(t)$  is a Hermitian operator, (which is usually the case when  $\mathcal{H}(t)$  corresponds to an energy operator of a physical system, although sometimes it is allowed to be non-Hermitian to represent dissipative systems). To find the general solution to EQ. (I.50) without restricting to any specific picture, we first introduce a time evolution operator  $U(t, t_0)$ , which relates the initial state  $|\Psi(t_0)\rangle$  to the final state  $|\Psi(t)\rangle$  by the expression:

$$|\Psi(t)\rangle \equiv U(t, t_0) |\Psi(t_0)\rangle. \quad (\text{I.51})$$

From EQs. (I.50) and (I.51), the time evolution operator satisfies the following differential equation and initial condition:

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = \mathcal{H}(t) U(t, t_0), \quad (\text{I.52})$$

$$U(t_0, t_0) = 1. \quad (\text{I.53})$$

Following EQs. (I.50) – (I.53) and assuming that  $\mathcal{H}(t)$  is Hermitian, the evolution for the adjoint space is given by:

$$-i\hbar \frac{\partial}{\partial t} \langle \Psi(t) | = \langle \Psi(t) | \mathcal{H}(t), \quad (\text{I.54})$$

$$-i\hbar \frac{\partial}{\partial t} U^\dagger(t, t_0) = U^\dagger(t, t_0) \mathcal{H}(t), \quad (\text{I.55})$$

$$U^\dagger(t_0, t_0) = 1. \quad (\text{I.56})$$

From EQs. (I.52) and (I.55), we find that

$$\frac{\partial}{\partial t} [U^\dagger(t, t_0) U(t, t_0)] = \frac{1}{i\hbar} [-U^\dagger(t, t_0) \mathcal{H}(t) U(t, t_0) + U^\dagger(t, t_0) \mathcal{H}(t) U(t, t_0)] = 0, \quad (\text{I.57})$$

which, together with the initial conditions, implies that:

$$U^\dagger(t, t_0) U(t, t_0) = 1 \Rightarrow U^\dagger(t, t_0) = U^{-1}(t, t_0) = U(t_0, t). \quad (\text{I.58})$$

In other words, the time evolution  $U$ -operator is unitary. In addition, we note that

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3).$$

In the special case of  $\mathcal{H}$  being a constant of time, the  $U$ -operator satisfies the group property.

Solving for quantum evolution is equivalent to finding the solution for the  $U$ -operator. From EQs. (I.52) and (I.53), the time evolution operator satisfies the following integral equation:

$$U(t, t_0) = U(t_0, t_0) - \frac{i}{\hbar} \int_{t_0}^t dt' \mathcal{H}(t') U(t', t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t dt' \mathcal{H}(t') U(t', t_0). \quad (\text{I.59})$$

Equation (I.59) may be solved by means of iteration:

$$\begin{aligned} U(t, t_0) &= 1 + \left(\frac{-i}{\hbar}\right) \int_{t_0}^t dt' \mathcal{H}(t') U(t', t_0) = 1 + \left(\frac{-i}{\hbar}\right) \int_{t_0}^t dt' \mathcal{H}(t') \left[1 + \left(\frac{-i}{\hbar}\right) \int_{t_0}^{t'} dt'' \mathcal{H}(t'') U(t'', t_0)\right], \\ &= 1 + \left(\frac{-i}{\hbar}\right) \int_{t_0}^t dt' \mathcal{H}(t') + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^t dt' \mathcal{H}(t') \int_{t_0}^{t'} dt'' \mathcal{H}(t'') + \dots \end{aligned} \quad (\text{I.60})$$

In EQ. (I.60) the second term can be rewritten as:

$$\begin{aligned} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \mathcal{H}(t') \mathcal{H}(t'') &= \frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^t dt'' [\mathcal{H}(t') \mathcal{H}(t'') \theta(t' - t'') + \mathcal{H}(t'') \mathcal{H}(t') \theta(t'' - t')], \\ &\equiv \frac{1}{2} \int_{t_0}^t dt' \int_{t_0}^t dt'' \hat{T} [\mathcal{H}(t') \mathcal{H}(t'')], \end{aligned} \quad (\text{I.61})$$

where we have introduced a time-ordering operator  $\hat{T}$  that orders the operators in the bracket from left to right with descending times, and the step function  $\theta(t_i - t_j)$  is defined by:

$$\begin{aligned}\theta(t_i - t_j) &= 1 && \text{for } t_i > t_j, \\ &= 0 && \text{for } t_i < t_j.\end{aligned}\tag{I.62}$$

The expression in EQ. (I.61) can be generalized to all terms in EQ. (I.60) by considering the following:

$$\begin{aligned}\hat{T}\left[\prod_{i=1}^n \mathcal{H}(t_i)\right] &= \hat{T}\left[\mathcal{H}(t_1) \dots \mathcal{H}(t_n)\right] \\ &= \sum_{\sigma \in S_n} \theta(t_{\sigma(1)} - t_{\sigma(2)}) \dots \theta(t_{\sigma(n-1)} - t_{\sigma(n)}) \mathcal{H}(t_{\sigma(1)}) \dots \mathcal{H}(t_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \left( \prod_{i=1}^{n-1} \theta(t_{\sigma(i)} - t_{\sigma(i+1)}) \prod_{i=1}^n \mathcal{H}(t_{\sigma(i)}) \right),\end{aligned}\tag{I.63}$$

where the summation is over  $S_n$ , the permutation group of  $n$  objects. Hence, we have

$$\begin{aligned}\int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \hat{T}\left[\mathcal{H}(t_1) \dots \mathcal{H}(t_n)\right] \\ &= \sum_{\sigma \in S_n} \left[ \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \left( \prod_{i=1}^{n-1} \theta(t_{\sigma(i)} - t_{\sigma(i+1)}) \prod_{i=1}^n \mathcal{H}(t_{\sigma(i)}) \right) \right] \\ &= \sum_{\sigma \in S_n} \left[ \int_{t_0}^t d(t_{\sigma^{-1}(1)}) \dots \int_{t_0}^t d(t_{\sigma^{-1}(n)}) \left( \prod_{i=1}^{n-1} \theta(t_i - t_{i+1}) \prod_{i=1}^n \mathcal{H}(t_i) \right) \right] \\ &= \sum_{\sigma \in S_n} \left[ \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \left( \prod_{i=1}^{n-1} \theta(t_i - t_{i+1}) \prod_{i=1}^n \mathcal{H}(t_i) \right) \right] \\ &= n! \left[ \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \left( \prod_{i=1}^{n-1} \theta(t_i - t_{i+1}) \prod_{i=1}^n \mathcal{H}(t_i) \right) \right] \\ &= n! \left[ \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n \mathcal{H}(t_1) \dots \mathcal{H}(t_n) \right].\end{aligned}\tag{I.64}$$

From EQs. (I.60) and (I.64), we find that the  $U$ -operator takes the form:

$$\begin{aligned}U(t, t_0) &= \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \hat{T}\left[\mathcal{H}(t_1) \dots \mathcal{H}(t_n)\right], \\ &= \hat{T} \left\{ \exp \left[ \left( \frac{-i}{\hbar} \right) \int_{t_0}^t dt' \mathcal{H}(t') \right] \right\}.\end{aligned}\tag{I.65}$$

In EQ. (I.65) we note that the term associated with  $n = 0$  is simply 1. In the event that  $\mathcal{H}$  is time independent, the time-evolution operator takes a simple form  $U(t, t_0) = \exp[-i\mathcal{H}(t - t_0)/\hbar]$ .

Having derived the above general solution for the differential equation EQ. (I.50), we now apply it to special cases in the following.

### [The Schrödinger picture]

The first case is the Schrödinger picture that you are probably most familiar with. The Schrödinger picture assumes that the state vectors  $|\Psi_s(t)\rangle$  are time dependent and the Hermitian operator  $\mathcal{H}(t)$  is the Hamiltonian of a physical system. Therefore, the Schrödinger equation takes the form:

$$i\hbar \frac{\partial}{\partial t} |\Psi_s(t)\rangle = \mathcal{H}(t) |\Psi_s(t)\rangle. \quad (\text{I.66})$$

For an initial value of the state vector  $|\Psi_s(t_0)\rangle$  at  $t = t_0$ ,  $|\Psi_s(t)\rangle$  satisfies

$$\begin{aligned} |\Psi_s(t)\rangle &= U(t, t_0) |\Psi_s(t_0)\rangle, \\ &\rightarrow e^{-i\mathcal{H}(t-t_0)/\hbar} |\Psi_s(t_0)\rangle \text{ if } \mathcal{H} \text{ is time independent.} \end{aligned} \quad (\text{I.67})$$

Thus, for a given solution  $|\Psi_s(t_0)\rangle$  at time  $t_0$ , the unitary transformation in EQ. (I.67) generates the solution to the Schrödinger equation at time  $t$  if  $\mathcal{H}$  is time independent.

### [The interaction picture]

In dealing with many realistic physical problems, we are often interested in a Hamiltonian that consists of two terms, one is a model Hamiltonian  $\mathcal{H}_0$  typically chosen as a soluble term, and the other is the “interaction” term  $\mathcal{H}'$ :

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'. \quad (\text{I.68})$$

The state vectors of the interaction picture are defined according to the following form:

$$|\Psi_I(t)\rangle = U_0(0, t) |\Psi_s(t)\rangle, \quad (\text{I.69})$$

where  $U_0$  refers to the time-evolution operator of the Hamiltonian  $\mathcal{H}_0$ . In the event that  $\mathcal{H}_0$  is a constant of time, we have  $U_0(0, t) = \exp(i\mathcal{H}_0 t / \hbar)$  and the state vectors of the interaction picture are given by:

$$|\Psi_I(t)\rangle = e^{i\mathcal{H}_0 t / \hbar} |\Psi_s(t)\rangle. \quad (\text{I.70})$$

Using EQs. (I.52) and (I.67) – (I.69), we obtain the state of motion of the interaction state vector:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} |\Psi_I(t)\rangle &= i\hbar \frac{\partial U_0(0, t)}{\partial t} |\Psi_s(t)\rangle + U_0(0, t) i\hbar \frac{\partial}{\partial t} |\Psi_s(t)\rangle, \\ &= -U_0(0, t) \mathcal{H}_0 |\Psi_s(t)\rangle + U_0(0, t) \mathcal{H} |\Psi_s(t)\rangle, \\ &= -U_0(0, t) \mathcal{H}_0 U_0(t, 0) |\Psi_I(t)\rangle + U_0(0, t) (\mathcal{H}_0 + \mathcal{H}') U_0(t, 0) |\Psi_I(t)\rangle, \\ &= U_0(0, t) \mathcal{H}' U_0(t, 0) |\Psi_I(t)\rangle, \\ &\equiv \mathcal{H}'_I(t) |\Psi_I(t)\rangle. \end{aligned} \quad (\text{I.71})$$

We note that the order of  $\mathcal{H}_0$  and  $\mathcal{H}'$  must be tracked carefully because in general they do not commute. Equation (I.71) has essentially the same form as EQ. (I.50) except that it is associated with  $\mathcal{H}'_I(t)$  and  $|\Psi_I(t)\rangle$  rather than  $\mathcal{H}$  and  $|\Psi(t)\rangle$ .

The definition of  $\mathcal{H}'_I(t)$  in EQ. (I.71) may be compared with the matrix element of any arbitrary operator  $O_s(t)$  in the Schrödinger picture:

$$\langle \Psi'_s(t) | O_s(t) | \Psi_s(t) \rangle = \langle \Psi'_I(t) | U_0(0,t) O_s(t) U_0(t,0) | \Psi_I(t) \rangle \equiv \langle \Psi'_I(t) | O_I(t) | \Psi_I(t) \rangle. \quad (I.72)$$

Therefore,  $\mathcal{H}'_I(t)$  is an operator in the interaction picture, and both the operators and the state vectors in the interaction picture are dependent on time. Moreover, if  $O_s$  is time independent, we find that

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} O_I(t) &= i\hbar \frac{\partial}{\partial t} [U_0(0,t) O_s U_0(t,0)] = U_0(0,t) [O_s \mathcal{H}_0 - \mathcal{H}_0 O_s] U_0(t,0), \\ &= [U_0(0,t) O_s U_0(t,0)] \mathcal{H}_{0I} - \mathcal{H}_{0I} [U_0(0,t) O_s U_0(t,0)] = [O_I(t), \mathcal{H}_{0I}]. \end{aligned} \quad (I.73)$$

More generally, for a time-dependent operator  $O_s(t)$ , we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} O_I(t) &= i\hbar \frac{\partial}{\partial t} [U_0(0,t) O_s(t) U_0(t,0)] = U_0(0,t) \left[ O_s(t) \mathcal{H}_0 - \mathcal{H}_0 O_s(t) + i\hbar \frac{\partial O_s(t)}{\partial t} \right] U_0(t,0), \\ &= [U_0(0,t) O_s(t) U_0(t,0)] \mathcal{H}_{0I} - \mathcal{H}_{0I} [U_0(0,t) O_s(t) U_0(t,0)] + i\hbar U_0(0,t) \frac{\partial O_s(t)}{\partial t} U_0(t,0), \\ &\equiv [O_I(t), \mathcal{H}_{0I}] + i\hbar \left( \frac{\partial O}{\partial t} \right)_I. \end{aligned} \quad (I.74)$$

Equation (I.73) can be readily applied to the creation and annihilation operators. For a representation in which  $\mathcal{H}_0$  is diagonal, we can express  $\mathcal{H}_0$  in terms of the creation and annihilation operators  $c_{\mathbf{k}}^\dagger$  and  $c_{\mathbf{k}}$  as follows:

$$\mathcal{H}_0 = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}. \quad (I.75)$$

Hence, the time dependence of the creation and annihilation operators  $c_{\mathbf{k}I}^\dagger(t)$  and  $c_{\mathbf{k}I}(t)$  in the interaction picture can be determined by using EQs. (I.70) and (I.73):

$$i\hbar \frac{\partial}{\partial t} c_{\mathbf{k}I}(t) = [c_{\mathbf{k}I}(t), \mathcal{H}_{0I}] = e^{i\mathcal{H}_0 t/\hbar} [c_{\mathbf{k}}, \mathcal{H}_0] e^{-i\mathcal{H}_0 t/\hbar} = \hbar \omega_{\mathbf{k}} e^{i\mathcal{H}_0 t/\hbar} c_{\mathbf{k}} e^{-i\mathcal{H}_0 t/\hbar} = \hbar \omega_{\mathbf{k}} c_{\mathbf{k}I}(t), \quad (I.76)$$

which leads to a solution:

$$c_{\mathbf{k}I}(t) = c_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} \quad \text{and} \quad c_{\mathbf{k}I}^\dagger(t) = c_{\mathbf{k}}^\dagger e^{i\omega_{\mathbf{k}} t}. \quad (I.77)$$

Since any operators in the Schrödinger picture can be expressed in terms of a complete set of operators  $c_{\mathbf{k}}^\dagger$  and  $c_{\mathbf{k}}$ , the operators in the interaction picture can then be obtained by making the following substitutions:

$$c_{\mathbf{k}} \rightarrow c_{\mathbf{k}l}(t) \quad \text{and} \quad c_{\mathbf{k}}^\dagger \rightarrow c_{\mathbf{k}l}^\dagger(t). \quad (\text{I.78})$$

Now we want to find an explicit solution to the state vector in EQ. (I.71). Evidently, EQ. (I.71) is essentially the same as the general differential equation in EQ. (I.50) if we make the substitutions  $\mathcal{H}(t) \rightarrow \mathcal{H}'_l(t)$  and  $|\Psi(t)\rangle \rightarrow |\Psi_l(t)\rangle$ . Therefore, we simply follow the same prescription as before by introducing a unitary operator  $U_l(t, t_0)$  that describes the time evolution of the state vector  $|\Psi_l(t)\rangle$  from time  $t_0$  to time  $t$ :

$$|\Psi_l(t)\rangle \equiv U_l(t, t_0) |\Psi_l(t_0)\rangle. \quad (\text{I.79})$$

Clearly  $U_l(t_0, t_0) = 1$ . Moreover, from EQs. (I.67) and (I.69), we can rewrite EQ. (I.79) into the following:

$$|\Psi_l(t)\rangle = U_0(0, t) |\Psi_s(t)\rangle = U_0(0, t) U(t, t_0) |\Psi_s(t_0)\rangle = U_0(0, t) U(t, t_0) U_0(t_0, 0) |\Psi_l(t_0)\rangle. \quad (\text{I.80})$$

Hence, we obtain an explicit expression for  $U_l(t, t_0)$  by comparing EQs. (I.79) and (I.80):

$$U_l(t, t_0) = U_0(0, t) U(t, t_0) U_0(t_0, 0). \quad (\text{I.81})$$

The operator  $U_l(t, t_0)$  is unitary and satisfies the same group property as  $U(t, t_0)$ . In addition, from EQs. (I.71) and (I.79), we find that

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} |\Psi_l(t)\rangle &= i\hbar \frac{\partial}{\partial t} U_l(t, t_0) |\Psi_l(t_0)\rangle = \mathcal{H}'_l(t) |\Psi_l(t)\rangle = \mathcal{H}'_l(t) U_l(t, t_0) |\Psi_l(t_0)\rangle, \\ \Rightarrow i\hbar \frac{\partial}{\partial t} U_l(t, t_0) &= \mathcal{H}'_l(t) U_l(t, t_0). \end{aligned} \quad (\text{I.82})$$

Consequently, similar to our previous derivation for the general time-evolution  $U$ -operator, in the interaction picture  $U_l(t, t_0)$  is given by:

$$\begin{aligned} U_l(t, t_0) &= 1 - \frac{i}{\hbar} \int_{t_0}^t dt' \mathcal{H}'_l(t') U_l(t', t_0) \\ &= \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n \hat{T} [\mathcal{H}'_l(t_1) \dots \mathcal{H}'_l(t_n)], \\ &= \hat{T} \left\{ \exp \left[ \left( \frac{-i}{\hbar} \right) \int_{t_0}^t dt' \mathcal{H}'_l(t') \right] \right\}. \end{aligned} \quad (\text{I.83})$$

The expressions in EQ. (I.83) will be important for our diagrammatic consideration later.

An immediate application of EQ. (I.83) is found in time-dependent perturbation theory. To the first-order of time-dependent perturbation, the  $U$ -operator is approximated by:

$$U_l(t, t_0) \approx 1 - \left( \frac{i}{\hbar} \right) \int_{t_0}^t dt_1 \mathcal{H}'_l(t_1), \quad (\text{I.84})$$

which is used to find the transition probabilities between the eigenstates of the unperturbed Hamiltonian  $\mathcal{H}_0$ . Specifically, the transition probability  $\mathcal{P}_{i \rightarrow f}(t)$  from an initial eigenstate  $|\varphi_i\rangle$  to a final eigenstate  $|\varphi_f\rangle$  of the

unperturbed Hamiltonian  $\mathcal{H}_0$  due to a time-dependent perturbation Hamiltonian  $\mathcal{H}'_I(t)$  is given by the formula:

$$\begin{aligned}
 \mathcal{P}_{i \rightarrow f}(t) &= \left| \langle \varphi_f | U(t, t_0) | \varphi_i \rangle \right|^2 = \left| \langle \varphi_f | U_0(t, 0) U_I(t, t_0) U_0(0, t_0) | \varphi_i \rangle \right|^2, \\
 &= \left| \langle \varphi_f | e^{-i\mathcal{H}_0 t/\hbar} U_I(t, t_0) e^{i\mathcal{H}_0 t_0/\hbar} | \varphi_i \rangle \right|^2 = \left| \langle \varphi_f | U_I(t, t_0) | \varphi_i \rangle \right|^2, \\
 &\approx \left| \langle \varphi_f | 1 - \left( \frac{i}{\hbar} \right) \int_{t_0}^t dt_1 \mathcal{H}'_I(t_1) | \varphi_i \rangle \right|^2 = \left| \langle \varphi_f | 1 - \left( \frac{i}{\hbar} \right) \int_{t_0}^t dt_1 e^{i\mathcal{H}_0 t_1/\hbar} \mathcal{H}' e^{-i\mathcal{H}_0 t_1/\hbar} | \varphi_i \rangle \right|^2, \\
 &= \left| \delta_{if} - \left( \frac{i}{\hbar} \right) \int_{t_0}^t dt_1 e^{i(E_f - E_i)t_1/\hbar} \langle \varphi_f | \mathcal{H}' | \varphi_i \rangle \right|^2. \\
 &\xrightarrow{(i \neq f)} \frac{1}{\hbar^2} \left| \int_{t_0}^t dt_1 e^{i(E_f - E_i)t_1/\hbar} \langle \varphi_f | \mathcal{H}' | \varphi_i \rangle \right|^2. \tag{I.85}
 \end{aligned}$$

In EQ. (I.85) we have taken the unperturbed Hamiltonian  $\mathcal{H}_0$  as time-independent. The fifth line in EQ. (I.85) is the familiar expression for the transition probability between two different eigenstates in the first-order time-dependent perturbation theory.

### [The Heisenberg picture]

In the Heisenberg picture the state vector is defined as

$$| \Psi_H(t) \rangle \equiv U(0, t) | \Psi_S(t) \rangle = | \Psi_S(0) \rangle, \tag{I.86}$$

which immediately gives the equation of motion of  $| \Psi_H(t) \rangle$ :

$$i\hbar \frac{\partial}{\partial t} | \Psi_H(t) \rangle = i\hbar \frac{\partial}{\partial t} | \Psi_S(0) \rangle = 0, \tag{I.87}$$

implying that the state vector in the Heisenberg picture is time independent. The operator in the Heisenberg picture  $O_H(t)$  can be related to the operator in the Schrödinger picture  $O_S$  by considering the matrix element of  $O_S$ :

$$\langle \Psi'_S(t) | O_S(t) | \Psi_S(t) \rangle = \langle \Psi'_H | U(0, t) O_S(t) U(t, 0) | \Psi_H \rangle \equiv \langle \Psi'_H | O_H(t) | \Psi_H \rangle. \tag{I.88}$$

From EQs. (I.87) and (I.88) we find that all the time dependence in the Heisenberg picture is ascribed to the operator  $O_H(t)$  whereas the corresponding state vector is time independent, in contrast to the situation in the Schrödinger picture. We may also rewrite the operator  $O_H(t)$  in terms of the operator in the interaction picture by using EQs. (I.69) and (I.81):

$$\begin{aligned}
 O_H(t) &= U(0, t) O_S(t) U(t, 0) = U_I(0, t) O_S(t) U_0(t, 0) U_I(t, 0) U_0(0, t), \\
 &= U_I(0, t) [U_0(0, t) O_S(t) U_0(t, 0)] U_I(t, 0) = U_I(0, t) O_I(t) U_I(t, 0). \tag{I.89}
 \end{aligned}$$

In addition, for all three pictures at  $t = 0$ , we find that

$$|\Psi_H\rangle = |\Psi_S(0)\rangle = |\Psi_I(0)\rangle, \quad (I.90)$$

and

$$O_S = O_I(0) = O_H(0). \quad (I.91)$$

In other words, all three pictures coincide at  $t = 0$ . Moreover, from EQs. (I.79) and (I.90), we obtain the relation:

$$|\Psi_H\rangle = |\Psi_I(0)\rangle = U_I(0, t_0) |\Psi_I(t_0)\rangle, \quad (I.92)$$

which implies that the eigenstate in the Heisenberg picture can be derived from the eigenstate at a given time  $t_0$  in the interaction picture through the  $U_I$ -operator.

### [Adiabatically turning on the interaction]

Having introduced all three pictures and the time-evolution operators, we are now well equipped to discuss time-dependent perturbation theory. A representative case is to consider adiabatically turning on an interaction within a non-interacting system, assuming that the eigenstates and eigen-energies are known for the non-interacting system described by  $\mathcal{H}_0$ . The Hamiltonian  $\mathcal{H}$  for such a process is given by:

$$\mathcal{H} = \mathcal{H}_0 + e^{-\varepsilon|t|} \mathcal{H}', \quad (\varepsilon > 0) \quad (I.93)$$

where  $\varepsilon$  is a small positive quantity. In general we take  $\varepsilon \rightarrow 0^+$  in the end of our calculations. Clearly  $\mathcal{H} \rightarrow \mathcal{H}_0$  for  $t \rightarrow \pm \infty$ , and  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$  at  $t = 0$ . Moreover, any physically significant result should not be dependent on the exact choice of  $\varepsilon$ . Following EQ. (I.83), we introduce a new  $\varepsilon$ -dependent  $U_I$ -operator,  $U_{I\varepsilon}$ , so that we can derive a solution for the Schrödinger equation in the interaction picture. We have

$$|\Psi_I(t)\rangle \equiv U_{I\varepsilon}(t, t_0) |\Psi_I(t_0)\rangle, \quad (I.94)$$

and

$$U_{I\varepsilon}(t, t_0) = \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n e^{-\varepsilon(|t_1| + \dots + |t_n|)} \hat{T}[\mathcal{H}'_I(t_1) \dots \mathcal{H}'_I(t_n)]. \quad (I.95)$$

In the limit of  $t_0 \rightarrow -\infty$  where  $\mathcal{H} \rightarrow \mathcal{H}_0$ , we define a time-independent stationary eigenstate for  $\mathcal{H}_0$  in the Schrödinger picture as  $|\Phi_0\rangle$  and the corresponding eigen-energy as  $E_0$ , so that

$$\mathcal{H}_0 |\Phi_0\rangle = E_0 |\Phi_0\rangle. \quad (I.96)$$

For the state vector in the interaction picture to be time independent and coincide with the solution for the unperturbed Schrödinger equation in the  $t_0 \rightarrow -\infty$  limit, we have:

$$|\Psi_I(t_0)\rangle = |\Phi_0\rangle = U_0(0, t_0) |\Psi_S(t_0)\rangle. \quad (I.97)$$

$$|\Psi_S(t_0)\rangle = U_0(t_0, 0) |\Phi_0\rangle. \quad (I.98)$$

As time increases from  $-\infty$ , the interaction is turned on adiabatically for small  $\varepsilon$  and the state vector in the interaction picture develops in time. At  $t = 0$ , the interaction reaches its full strength, and we have the following relation

$$|\Psi_H\rangle = |\Psi_I(0)\rangle = U_{I\varepsilon}(0, -\infty)|\Psi_I(-\infty)\rangle = U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle. \quad (\text{I.99})$$

In other words, we can obtain an exact eigenstate of  $\mathcal{H}$  from the eigenstate of the unperturbed Hamiltonian  $\mathcal{H}_0$  by using the  $\varepsilon$ -dependent  $U_I$ -operator.

Next we want to find out whether we can still obtain physically meaningful results if we have  $\varepsilon \rightarrow 0^+$ . The answer to this question is proven by the following theorem of Gell-Mann and Low [M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951)].

### [Gell-Mann & Low Theorem]

If the following quantity exists to all orders in perturbation theory:

$$\frac{U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle}{\langle\Phi_0|\Psi_{I\varepsilon}(0)\rangle} \equiv \frac{|\Psi_{I\varepsilon}(0)\rangle}{\langle\Phi_0|\Psi_{I\varepsilon}(0)\rangle} \equiv \frac{|\Psi_\varepsilon\rangle}{\langle\Phi_0|\Psi_\varepsilon\rangle},$$

then an exact eigenstate  $|\Psi\rangle$  of the Hamiltonian  $\mathcal{H}$  can be derived from the unperturbed ground state  $|\Phi_0\rangle$  via the relation

$$|\Psi\rangle \equiv \frac{|\Psi_I(0)\rangle}{\langle\Phi_0|\Psi_I(0)\rangle} = \lim_{\varepsilon \rightarrow 0} \frac{U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle}{\langle\Phi_0|\Psi_{I\varepsilon}(0)\rangle} = \lim_{\varepsilon \rightarrow 0} \frac{U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle}{\langle\Phi_0|U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle}, \quad (\text{I.100})$$

where the Hamiltonian  $\mathcal{H}$  in the time-evolution operator satisfies:

$$\mathcal{H}|\Psi\rangle = E|\Psi\rangle, \quad (\text{I.101})$$

and  $E$  denotes the exact eigen-energy of  $\mathcal{H}$ .

**Proof:** Given the definitions  $|\Psi_\varepsilon\rangle \equiv |\Psi_{I\varepsilon}(0)\rangle \equiv U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle$ , we consider the quantity

$$(\mathcal{H}_0 - E_0)|\Psi_\varepsilon\rangle = (\mathcal{H}_0 - E_0)U_{I\varepsilon}(0, -\infty)|\Phi_0\rangle = [\mathcal{H}_0, U_{I\varepsilon}(0, -\infty)]|\Phi_0\rangle. \quad (\text{I.102})$$

To evaluate the commutator in EQ. (I.102), we recall the expression for  $U_{I\varepsilon}$  in EQ. (I.95) and so we first examine the following commutator:

$$\begin{aligned} [\mathcal{H}_0, \mathcal{H}'_I(t_i)\mathcal{H}'_I(t_j)\cdots\mathcal{H}'_I(t_k)] &= [\mathcal{H}_0, \mathcal{H}'_I(t_i)]\mathcal{H}'_I(t_j)\cdots\mathcal{H}'_I(t_k) + \mathcal{H}'_I(t_i)[\mathcal{H}_0, \mathcal{H}'_I(t_j)]\cdots\mathcal{H}'_I(t_k) \\ &\quad + \mathcal{H}'_I(t_i)\mathcal{H}'_I(t_j)\cdots[\mathcal{H}_0, \mathcal{H}'_I(t_k)]. \end{aligned} \quad (\text{I.103})$$

Here  $\mathcal{H}'_I(t_i)\mathcal{H}'_I(t_j)\cdots\mathcal{H}'_I(t_k)$  represents an arbitrary time ordering of the  $n$  indices. From EQ. (I.73), we have

$$-i\hbar \frac{\partial}{\partial t} \mathcal{H}'_I(t) = [\mathcal{H}_0, \mathcal{H}'_I(t)], \quad (\text{I.104})$$

so that each commutator in EQ. (I.103) yields a time derivative of  $\mathcal{H}'_I(t)$  and EQ. (I.103) for all possible time orderings becomes:

$$\left[ \mathcal{H}_0, \mathcal{H}'_i(t_i) \mathcal{H}'_i(t_j) \dots \mathcal{H}'_i(t_k) \right] = \frac{\hbar}{i} \left( \sum_{\nu=1}^n \frac{\partial}{\partial t_\nu} \right) \left[ \mathcal{H}'_i(t_i) \mathcal{H}'_i(t_j) \dots \mathcal{H}'_i(t_k) \right]. \quad (\text{I.105})$$

Consequently, EQ. (I.102) can be rewritten as

$$(\mathcal{H}_0 - E_0) |\Psi_\varepsilon\rangle = - \sum_{n=1}^{\infty} \left( \frac{-i}{\hbar} \right)^{n-1} \frac{1}{n!} \int_{-\infty}^0 dt_1 \dots \int_{-\infty}^0 dt_n e^{\varepsilon(t_1 + \dots + t_n)} \left( \sum_{\nu=1}^n \frac{\partial}{\partial t_\nu} \right) \hat{T} \left[ \mathcal{H}'_i(t_1) \dots \mathcal{H}'_i(t_n) \right] |\Phi_0\rangle \quad (\text{I.106})$$

We note that each time-derivate term in EQ. (I.106) makes the same contribution. If we further assume that  $\mathcal{H}'_i(t)$  is proportional to a coupling constant  $g$ , we may integrate EQ. (I.106) by parts with respect to one of the time variables, and also use the following identity:

$$\left( \frac{-i}{\hbar} \right)^{n-1} \frac{1}{(n-1)!} g^n = i\hbar g \frac{\partial}{\partial g} \left( \frac{-i}{\hbar} \right)^n \frac{1}{n!} g^n, \quad ,$$

we find that EQ. (I.106) becomes

$$(\mathcal{H}_0 - E_0) |\Psi_\varepsilon\rangle = -\mathcal{H}' |\Psi_\varepsilon\rangle + i\hbar\varepsilon g \frac{\partial}{\partial g} |\Psi_\varepsilon\rangle, \quad (\text{I.107})$$

so that

$$(\mathcal{H} - E_0) |\Psi_\varepsilon\rangle = i\hbar\varepsilon g \frac{\partial}{\partial g} |\Psi_\varepsilon\rangle. \quad (\text{I.108})$$

Multiplying EQ. (I.108) by  $[\langle \Phi_0 | \Psi_\varepsilon \rangle]^{-1} \langle \Phi_0 |$ , we obtain

$$\begin{aligned} \frac{\langle \Phi_0 | (\mathcal{H} - E_0) | \Psi_\varepsilon \rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} &= \frac{\langle \Phi_0 | (E_0 + \mathcal{H}' - E_0) | \Psi_\varepsilon \rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} = \frac{\langle \Phi_0 | \mathcal{H}' | \Psi_\varepsilon \rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} \equiv (E - E_0) \\ &= i\hbar\varepsilon g \frac{\langle \Phi_0 | \partial / \partial g | \Psi_\varepsilon \rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} = i\hbar\varepsilon g \frac{\partial}{\partial g} \ln \langle \Phi_0 | \Psi_\varepsilon \rangle, \end{aligned} \quad (\text{I.109})$$

where we have used the fact that  $(\partial / \partial g) \langle \Phi_0 | = 0$ . Moreover, from EQs. (I.108) and (I.109), we find

$$\left( \mathcal{H} - E_0 - i\hbar\varepsilon g \frac{\partial}{\partial g} \right) \frac{|\Psi_\varepsilon\rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} = \frac{|\Psi_\varepsilon\rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} \left[ i\hbar\varepsilon g \frac{\partial}{\partial g} \ln \langle \Phi_0 | \Psi_\varepsilon \rangle \right] = (E - E_0) \frac{|\Psi_\varepsilon\rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle}, \quad (\text{I.110})$$

which yields

$$(\mathcal{H} - E) \frac{|\Psi_\varepsilon\rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} = i\hbar\varepsilon g \frac{\partial}{\partial g} \left[ \frac{|\Psi_\varepsilon\rangle}{\langle \Phi_0 | \Psi_\varepsilon \rangle} \right] \xrightarrow{(\varepsilon \rightarrow 0)} (\mathcal{H} - E) \frac{|\Psi_I(0)\rangle}{\langle \Phi_0 | \Psi_I(0)\rangle} = 0. \quad (\text{I.111})$$

Thus, we have proven EQ. (I.101) by taking  $\varepsilon \rightarrow 0$  in EQ. (I.111). We remark that while it is justified to take the limit of  $\varepsilon \rightarrow 0$  in EQ. (I.111), the same cannot be trivially applied to EQ. (I.109), because the term

$(\partial/\partial g)\ln\langle\Phi_0|\Psi_\varepsilon\rangle$  in fact diverges in the  $\varepsilon \rightarrow 0$  limit, whereas the term  $(\partial/\partial g)[|\Psi_\varepsilon\rangle/\langle\Phi_0|\Psi_\varepsilon\rangle]$  in EQ. (I.109) is finite for  $\varepsilon \rightarrow 0$ .

### [Comparison with the time-independent perturbation theory]

Before closing our discussion of the time-dependent perturbation theory, we demonstrate in the following that the time-independent perturbation theory is equivalent to a special case of the time-dependent perturbation theory if we take a time-evolution operator  $U_I(0, -\infty)$ .

From EQ. (I.95), the operator  $U_I(0, -\infty)$  can be explicitly given by the following:

$$U_{I\varepsilon}(0, -\infty) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^0 dt_1 \dots \int_{-\infty}^0 dt_n e^{-\varepsilon(|t_1| + \dots + |t_n|)} \hat{T}[\mathcal{H}'_I(t_1) \dots \mathcal{H}'_I(t_n)]. \quad (\text{I.112})$$

Thus, the lowest order term in the construction of the matrix element  $\langle\Phi_f|U_I(0, -\infty)|\Phi_0\rangle$  between two unperturbed eigenstates  $|\Phi_0\rangle$  and  $|\Phi_f\rangle$  is:

$$\begin{aligned} \langle\Phi_f|U_{I1}(0, -\infty)|\Phi_0\rangle &= \left(\frac{-i}{\hbar}\right) \int_{-\infty}^0 dt_1 e^{-\varepsilon|t_1|} \langle\Phi_f|\mathcal{H}'_I(t_1)|\Phi_0\rangle = \left(\frac{-i}{\hbar}\right) \int_{-\infty}^0 dt_1 e^{-\varepsilon|t_1|} \langle\Phi_f|e^{i\mathcal{H}_0 t_1/\hbar} \mathcal{H}' e^{-i\mathcal{H}_0 t_1/\hbar} |\Phi_0\rangle \\ &= \left(\frac{-i}{\hbar}\right) \int_{-\infty}^0 dt_1 e^{i(E_f - E_0 - i\hbar\varepsilon)t_1/\hbar} \langle\Phi_f|\mathcal{H}'|\Phi_0\rangle = -\frac{\langle\Phi_f|\mathcal{H}'|\Phi_0\rangle}{(E_f - E_0 - i\hbar\varepsilon)}, \end{aligned} \quad (\text{I.113})$$

where  $E_f$  and  $E_0$  are the eigenvalues of the unperturbed states. Equation (I.113) is consistent with the familiar expression for the perturbation amplitude in the first-order time-independent perturbation theory.

Similarly, if we denote the intermediate state vectors by  $|\Phi_p\rangle$  and use the identity  $\sum_p |\Phi_p\rangle\langle\Phi_p| = 1$ , the second-order term in the construction of the matrix element  $\langle\Phi_f|U_{I2}(0, -\infty)|\Phi_0\rangle$  becomes:

$$\begin{aligned} \langle\Phi_f|U_{I2}(0, -\infty)|\Phi_0\rangle &= \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 e^{-\varepsilon(|t_1| + |t_2|)} \\ &\quad \times \left[ \sum_p \langle\Phi_f|e^{i\mathcal{H}_0 t_1/\hbar} \mathcal{H}' e^{-i\mathcal{H}_0 t_1/\hbar} |\Phi_p\rangle\langle\Phi_p|e^{i\mathcal{H}_0 t_2/\hbar} \mathcal{H}' e^{-i\mathcal{H}_0 t_2/\hbar} |\Phi_0\rangle \right], \\ &= \left(\frac{+i}{\hbar}\right) \sum_p \int_{-\infty}^0 dt_1 e^{i(E_f - E_p - i\hbar\varepsilon)t_1/\hbar} e^{i(E_p - E_0 - i\hbar\varepsilon)t_1/\hbar} \frac{\langle\Phi_f|\mathcal{H}'|\Phi_p\rangle\langle\Phi_p|\mathcal{H}'|\Phi_0\rangle}{(E_p - E_0 - i\hbar\varepsilon)}, \\ &= \sum_p \frac{\langle\Phi_f|\mathcal{H}'|\Phi_p\rangle\langle\Phi_p|\mathcal{H}'|\Phi_0\rangle}{(E_f - E_0 - i2\hbar\varepsilon)(E_p - E_0 - i\hbar\varepsilon)}. \end{aligned} \quad (\text{I.114})$$

Evidently EQ. (I.114) is consistent with the second-order time-independent perturbation theory.

In comparison with the ordinary perturbation theory, the time-dependent formalism introduced in this section is more convenient in various ways. For instance, the time-dependent formalism can better handle the poles encountered in the perturbation theory because we may use the following relation to deal with the results in EQs. (I.113) and (I.114):

$$\lim_{\alpha \rightarrow 0^+} \frac{1}{x \mp i\alpha} = \mathcal{P}\left(\frac{1}{x}\right) \pm i\pi\delta(x), \quad (\text{I.115})$$

where  $\mathcal{P}(\dots)$  represents the principal value of the function inside the parenthesis. Another advantage is that one can easily separate parts of the problem that are associated with the disconnected parts of the system. This point will be verified in general terms when we introduce Feynman diagrams in Chapter II. For now, we use a simple example in the following to illustrate this concept.

Consider the Hamiltonian  $\mathcal{H} = \mathcal{H}_0^a + \mathcal{H}_0^b + \mathcal{V}_a + \mathcal{V}_b$ , where  $a$  and  $b$  denote different regions in space that are not connected physically so that operators of the respected regions necessarily commute. The  $U_I$ -operator for the Hamiltonian in the interaction picture is:

$$\begin{aligned} U_I(0, -\infty) &= \hat{T} \left\{ \exp \left[ -\frac{i}{\hbar} \int_{-\infty}^0 dt \mathcal{V}(t) \right] \right\} = \hat{T} \left\{ \exp \left[ -\frac{i}{\hbar} \int_{-\infty}^0 dt e^{i(\mathcal{H}_0^a + \mathcal{H}_0^b)t/\hbar} (\mathcal{V}_a + \mathcal{V}_b) e^{-i(\mathcal{H}_0^a + \mathcal{H}_0^b)t/\hbar} \right] \right\} \\ &= \hat{T} \left\{ \exp \left[ -\frac{i}{\hbar} \left( \int_{-\infty}^0 dt e^{i\mathcal{H}_0^a t/\hbar} \mathcal{V}_a e^{-i\mathcal{H}_0^a t/\hbar} + \int_{-\infty}^0 dt e^{i\mathcal{H}_0^b t/\hbar} \mathcal{V}_b e^{-i\mathcal{H}_0^b t/\hbar} \right) \right] \right\} \\ &= U_{Ia}(0, -\infty) U_{Ib}(0, -\infty). \end{aligned} \quad (\text{I.116})$$

Therefore the  $U_I$ -operator can be written as the product of independent operators if the corresponding Hamiltonian involves disconnected parts.

#### I.4. Various Forms of Low-Energy Excitations in Condensed Matter

Generally speaking, the low energy excitations in solids may be described in terms of quantized units of energy in various fields. These excitations often combine both aspects of waves and particles, and are typically divided into bosonic and fermionic excitations. Interestingly, for two-dimensional electronic systems, the low-energy excitations can be characteristic of neither bosons nor fermions. Rather, they obey fractional or even non-abelian statistics. Such excitations are known as anyons, which will be covered in the Part VI of this summer course later. In the following we list definitions of various representative low-energy excitations in *three-dimensional* solids.

##### [Some representative low-energy excitation fields in solids]

**Quasiparticles:** electrons dressed by the interactions with the electron gas in a metal -- (fermions)

**Polarons:** charged particles associated with the polarization field in ionic crystals -- (fermions)

**Phonons:** elastic excitations of the crystalline lattice -- (bosons)

**Magnons:** elementary excitations of electron spins coupled together by the exchange interactions in a magnetic system -- (bosons)

**Plasmons:** collective Coulomb excitations of the electron gas in a metal -- (bosons)

**Excitons:** coupled electron-hole pairs associated with the dielectric polarization field -- (bosons)

As an example, we consider the collective elastic motion of an electron gas relative to the rigid background of positive charge, known as the plasmons. The Hamiltonian density of the electron gas can be expressed in terms of the displacement fields  $u_\alpha$  and the conjugate fields  $\pi_\alpha$  ( $\alpha$ : polarization) as follows:

$$H_{el} = \frac{1}{2n_e m_e} \sum_{\alpha} \pi_{\alpha} \pi_{\alpha} + \frac{1}{2} \lambda \sum_{\alpha, \beta} \frac{\partial u_{\alpha}}{\partial r_{\alpha}} \frac{\partial u_{\beta}}{\partial r_{\beta}} + \frac{1}{2} (\rho_e - \rho_0) \phi(\mathbf{r}), \quad (\text{I.117})$$

where  $n_e$  and  $\rho_e$  denote the volume density and the charge density of electrons, respectively;  $\rho_0$  is the uniform positive charge background,  $\lambda$  represents the bulk modulus, and  $\phi(\mathbf{r})$  is the electrostatic potential.

The plasmons are associated with the longitudinal waves of electron density fluctuations that destroy the local charge neutrality, thereby inducing a Coulomb restoring force. To understand the nature of plasmons, we begin with the charge density fluctuations  $\delta\rho$  due to local dilation of the electron gas:

$$\delta\rho(\mathbf{r}) = \rho_e(\mathbf{r}) - \rho_0(\mathbf{r}) = -n_e e \frac{\partial u_l(\mathbf{r})}{\partial r_l}, \quad (l: \text{longitudinal polarization}) \quad (\text{I.118})$$

and 
$$\nabla^2 \phi(\mathbf{r}) = -4\pi \delta\rho(\mathbf{r}). \quad (\text{I.119})$$

The longitudinal displacement field  $u_l$  may be written as  $u_l(\mathbf{r}) = \sum_{\mathbf{k}} Q_{\mathbf{k}}^l e^{i\mathbf{k}\cdot\mathbf{r}}$  so that

$$\delta\rho(\mathbf{r}) = -in_e e \sum_{\mathbf{k}} k Q_{\mathbf{k}}^l e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (\text{if } \mathbf{k} \parallel \hat{e}_l \text{ and } \mathbf{k} = k \hat{e}_l). \quad (\text{I.120})$$

Similarly, the electrostatic potential can be expressed by  $\phi(\mathbf{r}) = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$  so that

$$\nabla^2 \phi(\mathbf{r}) = -\sum_{\mathbf{k}} k^2 \varphi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = i4\pi n_e e \sum_{\mathbf{k}} k Q_{\mathbf{k}}^l e^{i\mathbf{k}\cdot\mathbf{r}} \rightarrow \varphi_{\mathbf{k}} = -i4\pi n_e e Q_{\mathbf{k}}^l / k. \quad (\text{I.121})$$

Consequently, the spatial integration of the third term in EQ. (I.117) becomes

$$\begin{aligned} \int d\mathbf{r} \frac{1}{2} \delta\rho(\mathbf{r}) \phi(\mathbf{r}) &= \sum_{\mathbf{k}, \mathbf{k}'} \int d\mathbf{r} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \left( -i \frac{n_e e}{2} k Q_{\mathbf{k}}^l \right) \left( \frac{-i4\pi n_e e Q_{\mathbf{k}'}^l}{k'} \right) \\ &= 2\pi n_e^2 e^2 \sum_{\mathbf{k}} Q_{\mathbf{k}}^l Q_{-\mathbf{k}}^l. \end{aligned} \quad (\text{I.122})$$

The conjugate momentum density field is given by  $\pi_{\alpha}(\mathbf{r}) = \sum_{\mathbf{k}} P_{\mathbf{k}}^{\alpha} e^{-i\mathbf{k}\cdot\mathbf{r}}$  so that

$$\int d\mathbf{r} \pi_{\alpha}(\mathbf{r}) \pi_{\alpha}(\mathbf{r}) = \sum_{\mathbf{k}} P_{\mathbf{k}}^{\alpha} P_{-\mathbf{k}}^{\alpha}. \quad (\text{I.123})$$

In addition, the elastic energy in the second term of EQ. (I.117) can be rewritten into the form

$$\frac{\lambda}{2} \sum_{\mathbf{k}} \sum_{\alpha, \beta} k_{\alpha} k_{\beta} Q_{\mathbf{k}}^{\alpha} Q_{-\mathbf{k}}^{\beta}. \quad (\text{I.124})$$

Consequently, for  $\mathbf{k} \parallel \hat{e}_l$  and from EQs. (I.122) – (I.124), we obtain the Hamiltonian of the electron gas:

$$\mathcal{H}_{el} = \int dr H_{el} = \sum_{\mathbf{k}} \left[ \frac{1}{2n_e m_e} P_{\mathbf{k}} P_{-\mathbf{k}} + \frac{1}{2} (\lambda k^2 + 4\pi n_e^2 e^2) Q_{\mathbf{k}} Q_{-\mathbf{k}} \right]. \quad (\text{I.125})$$

From EQ. (I.125), we may define the plasmon frequency  $\omega_{\mathbf{k}}$  and the plasma frequency  $\omega_p$  as follows:

$$\omega_{\mathbf{k}}^2 = \left( \frac{\lambda}{n_e m_e} \right) k^2 + \frac{4\pi n_e^2 e^2}{n_e m_e} \equiv \left( \frac{\lambda}{n_e m_e} \right) k^2 + \omega_p^2, \quad (\text{I.126})$$

$$\omega_p \equiv \left( \frac{4\pi n_e^2 e^2}{n_e m_e} \right)^{1/2}. \quad (\text{I.127})$$

In general the plasmon frequencies for collective electron excitations are much higher than the phonon frequencies associated with the lattice vibration because of the much smaller mass density in the former. More quantitatively, if we take a typical value of  $n_e \sim 10^{23} \text{ cm}^{-3}$  and  $m_e \sim 10^{-27} \text{ g}$ , we obtain  $\omega_p \sim 10^{16} \text{ s}^{-1}$ . On the other hand, even if we take the maximum  $k$ -value in the crystal, *i.e.*,  $k_{\text{max}} \sim 10^8 \text{ cm}^{-1}$ , the first quantity in EQ. (I.126),  $[\lambda k_{\text{max}}^2 / (n_e m_e)]$ , is still much smaller than  $\omega_p^2$  if we take a typical value  $\lambda \sim 10^9 \text{ dynes/cm}^2$ . Consequently, the plasmon excitations of the electron gas in a solid can be regarded as nearly uniform.

Finally, we conclude this example of low-energy excitations in solids by second-quantizing plasmons. Introducing the boson operators:

$$\begin{aligned} b_{\mathbf{k}} &= \left( \frac{n_e m_e \omega_{\mathbf{k}}}{2\hbar} \right)^{1/2} Q_{\mathbf{k}} + i \left( \frac{1}{2n_e m_e \hbar \omega_{\mathbf{k}}} \right)^{1/2} P_{\mathbf{k}}^*, \\ b_{\mathbf{k}}^\dagger &= \left( \frac{n_e m_e \omega_{\mathbf{k}}}{2\hbar} \right)^{1/2} Q_{\mathbf{k}}^* - i \left( \frac{1}{2n_e m_e \hbar \omega_{\mathbf{k}}} \right)^{1/2} P_{\mathbf{k}}, \end{aligned} \quad (\text{I.128})$$

we obtain the second-quantized Hamiltonian for the electron gas:

$$\mathcal{H}_{el} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left( b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \frac{1}{2} \right). \quad (\text{I.129})$$

Equation (I.129) is analogous to EQ. (I.49), except that the dispersion relations associated with the bosonic excitations differ in the two cases.

### **Further Readings**

1. Fetter and Walecka, "Quantum Theory of Many-Particle Systems": Sections 1 – 3 and 6.
2. Kittel, "Quantum Theory of Solids": Chapters 1 and 2.
3. Schweber, "An Introduction to Relativistic Quantum Field Theory", Chapter 6.
4. Merzbacher, "Quantum Mechanics", Chapters 14, 21 and 22.