3 Symmetry Protected Topological Phase

Having discussed the Majorana chain model, a similar picture can help us understand topological phases in a more general interacting setting.

3.1 Haldane phase in 1D

The spin 1 chain with anti-ferromagnetic Heisenberg interaction provides the first and simplest example of symmetry protected topological order. Consider the Hamiltonian

\[ H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \]  

where \( \vec{S} \) is the spin 1 operator

\[
S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]  

This Hamiltonian has a rich symmetry, including spin rotation and time reversal (\( \vec{S} \rightarrow -\vec{S} \)).

What is the ground state like? For classical spin, anti-ferromagnetic interaction leads to Neel order, with spins ordered as \( \uparrow \downarrow \uparrow \downarrow \ldots \). In 1D quantum chains, long range order cannot survive and usually gets reduced to power law decaying correlation. This is what happens in anti-ferromagnetic Heisenberg spin 1/2 chain, which is known to be gapless. For the spin 1 chain, it was thought to be gapless as well until Haldane pointed that it should actually be gapped. Moreover, as we are going to see below, it has symmetry protected topological order!

The Heisenberg model is very hard to solve and the existence of a gap remained a controversy for quite some time. On the other hand, exact ground state can be found for a slightly modified Hamiltonian proposed by Affleck, Kennedy, Lieb and Tasaki (AKLT).

\[ H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^2 \]  

By adding the quadratic interaction \( \frac{1}{3} \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^2 \), the Hamiltonian term becomes the projection to the total spin 2 sector for the two spin 1’s at site \( i \) and \( i + 1 \). That is to say, the total spin 2 sector has a higher energy, while the total spin 1 and total spin 0 sector has lower energy. Energy would be minimized if every pair of nearest neighbor spins satisfy this condition, i.e. if every pair of them has total spin 0 or 1 but not 2.

This condition can indeed be satisfied. As shown in Fig.1, consider each spin 1 (grey oval) as composed of two spin 1/2’s (yellow circle). Put the right spin 1/2 on site \( i \) and the left spin 1/2 on site \( i + 1 \) into a singlet state \( | \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle \). Now among the four spin 1/2’s on site \( i \) and \( i + 1 \), two
Figure 1: The valence bond ground state of the AKLT model. The spin 1 on each lattice site can be decomposed into two spin 1/2’s which form singlets between nearest neighbor pairs. At the two ends of an open chain, there are two isolated spin 1/2’s giving rise to a four fold ground state degeneracy.

of them has total spin 0, therefore the total spin of the four can only be either 0 or 1. In this way we have found the ground state of Eq. 3. The full wave function can be written as

$$\psi_{AKLT} = \prod_i P_i \prod_{i,i+1} \frac{1}{\sqrt{2}} (|\uparrow_i\downarrow_{i+1}\rangle - |\downarrow_i\uparrow_{i+1}\rangle)$$

where $P_i = |1\rangle\langle\uparrow_i| + |\downarrow_i\rangle\langle\downarrow_i| + \frac{1}{\sqrt{2}} (|\uparrow_i\downarrow_{i+1}\rangle - |\downarrow_i\uparrow_{i+1}\rangle)$ is the projection from two spin 1/2’s back to the spin 1 basis.

This wave function is called a ‘Valence Bond Solid’. In 1, it was shown that this wave function has a finite correlation length. That is

$$\langle \psi_{AKLT}|O_iO_j|\psi_{AKLT}\rangle - \langle \psi_{AKLT}|O_i|\psi_{AKLT}\rangle\langle \psi_{AKLT}|O_j|\psi_{AKLT}\rangle \sim e^{-|i-j|/\xi}$$

where $\xi$ is the correlation length. Moreover, the excitations on top of this ground state wave function consist of flipping the singlets into triplets and cost finite energy. Therefore, the ground state is gapped and is very different from the spin 1/2 case.

From this valence bond structure of the wave function, it is easy to see the nontrivial SPT order. On a closed ring with periodic boundary condition, all the spin 1/2’s are paired up and the ground state is unique. It is invariant under both spin rotation ($\prod_i e^{i\theta S_i^z}$) and time reversal ($\prod_i e^{i\pi S_i^y} K$, where $K$ is complex conjugation). There is no spontaneous symmetry breaking, yet the unique gapped ground state has a nontrivial order.

To see this, notice that on an open chain, the spin 1/2’s at the two ends of the chain are far away from each other and hence not coupled. They can point in any direction without affecting the energy (in the thermodynamic limit). Therefore, the ground state is 4 fold degenerate, with each spin 1/2 contributing 2 fold degeneracy. If spin rotation or time reversal symmetry is broken by adding, e.g., a magnetic field, the edge spin is polarized and the degeneracy is lifted. If either spin rotation or time reversal is preserved, the degeneracy is always robust and is a signature of the symmetry protected topological order in the AKLT model.

Such a statement holds not only for the AKLT model, but for all models in the same phase (supposedly including the Heisenberg model Eq. 1). If we vary the model within the phase, the edge state might change. For example, it might get wider and change from a spin 1/2 to a spin 3/2 by involving the neighboring spin 1. However, as long as it is a half integer spin, it always has a nontrivial degeneracy, indicating the nontrivial SPT order.

This is related to a special property of the half integer spins as compared to integer spins under spin rotation or time reversal symmetry. A half integer spin transforms under $2\pi$ spin rotation as
\( e^{i2\pi \sigma^z} = -1 \) while an integer spin transforms under \( 2\pi \) rotation as \( e^{i2\pi S^y} = 1 \). Therefore, these two cases cannot be smoothly connected. In particular, a half integer spin edge state cannot become a zero spin edge state which has no degeneracy. On the other hand, a integer spin can be connected to spin 0, hence removing the degeneracy.

Similarly under time reversal symmetry, a half integer spin transforms as \( T^2 = e^{i\pi \sigma^y} K e^{i\pi \sigma^y} K = -1 \) while an integer spin transforms as \( T^2 = e^{i\pi S^y} K e^{i\pi S^y} K = 1 \). Therefore integers spins and half integer spins are fundamentally different under time reversal symmetry. And the edge state cannot change from one to another without a phase transition.

Spin 1/2 under spin rotation and time reversal symmetry provides a prototypical example of a projective representation of symmetry. In general, for a symmetry group \( G \), a usual representation consists of matrices \( M(g), g \in G \), which satisfy

\[
M(g_1)M(g_2) = M(g_1g_2)
\]  

(6)

A projective representation consists of matrices \( M(g), g \in G \) which satisfy

\[
M(g_1)M(g_2) = \omega(g_1, g_2)M(g_1g_2)
\]

(7)

for some phase factor \( \omega(g_1, g_2) \neq 1 \). The \( \omega \)'s have to satisfy

\[
\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega^{s_1}(g_2, g_3)\omega(g_1, g_2g_3)
\]

(8)

where \( s_1 = 1 \) if \( g_1 \) is unitary symmetry, \( s_1 = -1 \) if \( g_1 \) is anti-unitary (involves time reversal). On the other hand, one can redefine the phase factor of \( M(g) \) by \( \alpha(g) \). Therefore, any two sets of \( \omega \)'s related by

\[
\tilde{\omega}(g_1, g_2) = \omega(g_1, g_2)\frac{\alpha(g_1)\alpha^{s_1}(g_2)}{\alpha(g_1g_2)}
\]

(9)

are considered equivalent. Eq. 8 and 9 lead to a classification of projective representations for symmetry group \( G \). In particular, for spin rotation \( G = SO(3) \), there are two classes and for time reversal there are two classes. Spin 1/2 corresponds to the nontrivial class in both cases.

This discussion of projective representation is important for the study of SPT phases because it has been shown that there is a one to one correspondence between one dimensional bosonic SPT phases with symmetry \( G \) and equivalence class of projective representations of group \( G \).