1 Superfluidity and Bose Einstein Condensate

1.6 Superfluid phase: topological defect

Besides such smooth gapless excitations, superfluid can also support a very special type of excitation – a topological defect called vortex. Consider a configuration of the order parameter field as

$$\psi(\vec{r}) = \rho(r)e^{i\theta(\vec{r})}, \quad \theta(\vec{r}) = \arctan\left(\frac{r_y}{r_x}\right)$$ (1)

That is, the magnitude of the order parameter only depends on the distance from the origin while the phase component points in the same direction as the position vector. At the origin, the phase of the order parameter is not well defined and we need to require that $$\lim_{r \to 0} \rho(r) = 0$$.

Figure 1: Vortices in BEC.

This type of configuration cannot be obtained from smooth deformation of the order parameter field starting from a uniform one. In particular, there is a quantitative measure which can show that a vortex configuration is fundamentally different from a uniform configuration, or any smooth deformation of it. To define the measure, choose a smooth closed path encircling the center of the vortex. Along the path, calculate the integral

$$n = \frac{1}{2\pi} \oint_c \frac{\partial\theta}{\partial l} dl$$ (2)

If we calculate it along a circular path, we can easily see it is 1. In fact, it stays one even if we change the path as long as the path encloses center of the vortex, because it basically measures how much the phase changes along the path. Similarly, it is 1 for the configuration on the right. On the other hand, we can see that this integral is equal to zero if we choose a path which does
not enclose a vortex, e.g., in a uniform order parameter field or a smooth deformation of it. $n$ is called the winding number of the order parameter field around certain point. It can take only integer values 0, 1, 2 and also $-1$, $-2$, etc... The configuration with positive $n$ are called vortices and the ones with negative $n$ are called anti-vortices. $n$ is a called a topological quantum number with the most important feature of being discrete.

Because $n$ is discrete, there is no way to for it to change from one value to another in a smooth way. It can change, for example, by moving one vortex from the outside to the inside of a path. But in the process, when the vortex center passes the path, at this particular point, the differentiation of $\theta$ is not well defined and there is some singularity. Moreover, in a closed system, if we integrate along the 'boundary', which is either actually one point (in a sphere) or going back and forth along the same path (torus), the topological quantum number calculated is always 0. There can still be vortices and anti-vortices in the system but they must appear in pairs. In a system with open boundary condition (a disc), vortices can be created by rotating the condensate, as each vortex carries certain amount of angular momentum with them.

While the goldstone excitations can have very small energy, the vortex excitations can cost a lot of energy. In the homework, we are going to see that a vortex and anti-vortex pair has a potential energy of $\ln(R/R_0)$, corresponding to an effective attractive interaction between the pair that decays as $1/R$ (similar to Coulomb force in 2D).

The winding of the order parameter field around a vortex results in the so-called 'superflow' in a superfluid. That is, there is non-dissipative current flowing around a vortex. To see this, we can calculate the current in the superfluid as

$$j = -\frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

where we have used the single particle formulation of the current operator for the superfluid out of their similarity. We are going to argue below why this formulation is still valid in the many-body setting. Suppose $\psi = \rho e^{i\theta}$. Then

$$j = -\frac{i\hbar}{2m} \left( \rho e^{-i\theta} (\nabla \rho e^{i\theta} + i \nabla \theta \rho e^{i\theta}) - \rho e^{i\theta} (\nabla \rho e^{-i\theta} - i \nabla \theta \rho e^{-i\theta}) \right) = \frac{\hbar}{m} \rho^2 \nabla \theta$$

Therefore, the current is proportional to the gradient in the phase factor part of the order parameter. It does not depend on how the magnitude part changes. In the vortex configurations given above, there are hence no current flowing in the radial direction. All current is flowing in the angular direction around the vortex and in one direction. On the other hand, in the smooth gapless configuration we talked about above, the current oscillate over time.

Such a superflow is easy to understand if we think of $\psi$ as a single particle wave function. Imagine a single particle moving on a circle or radius $r$, if the wave function has uniform phase factor, then it means the wave length is infinity and momentum is zero, so there is no current. If the wave function has a phase factor that is changing periodically along the circle, that is, there is a factor of the form $e^{ikr \theta}$, then the wave length is finite and the momentum is finite, there is a finite current.

Of course, to make the connection, we need to show why $j$ as defined above really presents the current in the many-body second quantized setting. In the second quantized description of many-body systems, current operator $j$ is one that satisfies the continuity condition for density operator

$$\frac{\partial}{\partial t} n_r + \nabla \cdot j_r = 0$$

(5)
and from the Heisenberg formulation we know that
\[
\frac{\partial}{\partial t} n_{\vec{r}} \propto [n_{\vec{r}}, H] = \sum_{\vec{r}'} b_{\vec{r}}^\dagger b_{\vec{r}'} - b_{\vec{r}'}^\dagger b_{\vec{r}} = \left( b_{\vec{r}+x}^\dagger b_{\vec{r}+x} - b_{\vec{r}+x} b_{\vec{r}+x}^\dagger \right) - \left( b_{\vec{r}-x}^\dagger b_{\vec{r}} - b_{\vec{r}} b_{\vec{r}-x}^\dagger \right) + \text{terms in } y \text{ direction}
\]  

(6)

Therefore, we can choose the current operator to be
\[
j_x = b_{\vec{r}}^\dagger b_{\vec{r}+x} - b_{\vec{r}+x}^\dagger b_{\vec{r}}
\]

(7)

In the superfluid phase, we replace the \( b \) and \( b^\dagger \) operators by their expectation value \( \psi \) and \( \psi^* \) and we get
\[
j_x = \psi^*_{\vec{r}}\psi_{\vec{r}+x} - \psi^*_{\vec{r}+x}\psi_{\vec{r}} = \psi^*_{\vec{r}} \left( \psi_{\vec{r}} + a \frac{\partial}{\partial x}\psi_{\vec{r}} \right) - \left( \psi^*_{\vec{r}} + a \frac{\partial}{\partial x}\psi^*_{\vec{r}} \right) \psi_{\vec{r}} = a \left( \psi^*_{\vec{r}} \frac{\partial}{\partial x}\psi_{\vec{r}} - \psi_{\vec{r}} \frac{\partial}{\partial x}\psi^*_{\vec{r}} \right)
\]

(8)

Combined with the \( y \) direction component, we get the full current operator.

The quantization of vortex on the one hand can be understood as a result of symmetry breaking and the existence of an order parameter: around a certain path, the phase factor of the order parameter must wind around an integer multiple of \( 2\pi \), so that the order parameter field can be smooth and well defined. On the other hand, the quantization can be understood as a result of condensation. We are going to argue why in the superfluid phase, where the bosons condense, we can only have integer vortices, not fractional ones.

First, notice that as a boson moves around in an order parameter background, it accumulates phase shifts along the way. That is, as the boson hops from \( \vec{r} \) to \( \vec{r}' \) with
\[
b_{\vec{r}}^\dagger b_{\vec{r}'}
\]
the phase factor of its wave function changes by \( \theta_{\vec{r}'} - \theta_{\vec{r}} \) where \( \theta \) is the phase factor of complex number \( \psi \).

Secondly, recall that the ground state wave function can be written as a superposition of components with any number of bosons and for a fixed number of bosons, the component contains all possible configurations of the bosons. This corresponds to the ‘condensate’ picture where the particles numbers can fluctuate and the particles can move around freely. Such a superposition is only possible if integer vortices are allow because otherwise, as the bosons move around a fractional vortex, it accumulates a phase factor other than \( 2\pi \). In that case, we can no longer write the wave function as a single-valued superposition of all possible configurations and the condensate must break down. In a superconductor, we are going to see how similar argument shows that we can have \( \pi \) vortices as well as all the integer ones.

1.7 Landau’s description of symmetry breaking

Landau has a simple way to explain why there are symmetry breaking and symmetric phases and how phase transition happens between the two. Landau’s approach involves the following elements:

1. There exists an order parameter.

2. The energy (free energy) is a (simple) functional of the order parameter field, usually expanded in low order polynomials of the order parameter field and its derivatives.
3. The energy (free energy) functional has the required symmetry.

4. Minimize the energy / free energy functional over the order parameter field.

For the Boson gas / liquid model, Landau first chose a complex order parameter $\psi$, then wrote the energy / free energy functional as

$$F = F_0 + a|\psi|^2 + b|\psi|^4 + c|\nabla \psi|^2 + \ldots$$

which contains the simplest (lowest order) terms of $\psi$ which are $U(1)$ symmetric.

To minimize this functional, first notice that it is minimized when the derivative term is zero, so the equilibrium configuration has a uniform $\psi$. This uniform equilibrium value of $\psi$ is then determined by $a$ and $b$. In particular, when $a > 0$, $b > 0$, the minimum is achieved at $\psi = 0$. There is a unique equilibrium configuration, which does not break the $U(1)$ symmetry of the system. On the other hand, when $a < 0$, $b > 0$, the functional is minimized when $|\psi|^2 = -\frac{a}{2b}$. There are an infinite number of $\psi$ that satisfy this condition and they differ in the phase factor. The $U(1)$ symmetry is spontaneous broken in this case. Therefore, as $a$ is tuned from positive to zero to negative, a phase transition happens from symmetric to symmetry breaking phase. $a$, $b$ are in general functions of temperature, chemical potential and various parameters of the system. We are going to see that a very similar and more interesting theory can be formulated to explain superconductivity.

## 2 Superconductivity

Now let’s talk about superconductivity. Superconductivity is in some sense the analog of a superfluid in a fermion system. Theoretically, it is more complicated than a superfluid, but superconductivity was actually discovered earlier, in 1911, before superfluid helium was discovered in 1927. The first superconductor was Hg, and it became superconducting at $4.1K$.

The key characteristic of a superconductor is of course its zero resistance to current. That is, there can be persistent current flowing along a superconducting ring without the need to supply external energy. On the other hand, superconductor is also characterized by the more exotic and essential property of diamagnetism. That is, small magnetic field cannot penetrate through a superconductor and is always perfectly screened by currents along the edge of the sample. This is called the ‘Meissner’ effect, which is described by

$$\mathbf{B} = \mu_{sc} \mathbf{H} = 0$$

therefore, the permeability $\mu_{sc} = 0$. The magnetic susceptibility $\chi$ is

$$\chi = \frac{M}{H} = \left( \frac{\mathbf{B} - \mathbf{H}}{\mu_0} \right) / \mathbf{H} = -1$$

The magnetization is generated by a persistent current in the superconductor which is sustainable only if the external magnetic field to be screened is small enough. When the external magnetic field is large enough, above a critical field strength $H_c$, the superconductor (at least part of it) becomes a normal conductor.

Nowadays, the Meissner effect is often considered a more handy definition of superconductivity. We are going to see how these properties can be properly understood in terms of symmetry breaking and condensation.
2.1 London equation

The London equation provides a macroscopic phenomenological explanation of the relation between zero resistivity and Meissner effect.

In the first step, it was assumed that the electrons move without friction in a superconductor (which is of course a highly classical assumption without a good explanation). In this case, the electron motion is described by

\[ m \dot{v} = -eE \]  

where \( m \) is the mass of the electron, \( v \) their velocity and \( E \) the applied electric field. Current is related to electron motion by

\[ j = -env \]  

where \( n \) is the density of conduction electrons. Combining these two equations we get the first London equation

\[ \frac{\partial}{\partial t}j = \frac{e^2n}{m}E \]  

That is, in a superconductor, electric field is not proportional to current, but rather the time derivative of the current!

Now using the Maxwell equation

\[ \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]  

we get

\[ \frac{m}{ne^2} \nabla \times \frac{\partial j}{\partial t} + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]  

Integrating on both sides, we arrive at the second London equation

\[ \nabla \times j = -\frac{ne^2}{mc}B \]  

That is, the magnetic field is proportional to the spatial derivative of the current.

When there is no time dependent electric field, we can combine the second London equation with Ampere’s law

\[ \nabla \times B = \mu_0 j \]  

and obtain

\[ \nabla \times \nabla \times B = -\mu_0 \frac{ne^2}{mc}B \]  

Define \( \lambda = \sqrt{\frac{mc}{\mu_0 ne^2}} \) and using the fact that \( \nabla \cdot B = 0 \), we get

\[ \nabla^2 B = \frac{1}{\lambda^2}B \]  

A generic solution reads

\[ B = B_0 e^{-|x|/\lambda} \]  

If \( x \) measures the distance from the surface of the superconductor into the bulk, then we can see that the magnetic field is exponentially decaying into the bulk. The length scale \( \lambda \) over which the decay happens is called the penetration depth of the superconductor.
Compare Eq. 18 with $\mathbf{B} = \nabla \times \mathbf{A}$, we find that

$$j = -\frac{1}{\mu_0 \lambda^2} \mathbf{A}$$

(23)

which is usually called “London equation.”