Superfluidity and Bose Einstein Condensate

A few comments about coherent state first. Recall that for a boson mode, a coherent state is the eigenstate of the annihilation operator $b$

$$|\gamma\rangle = e^{-|\gamma|^2/2} \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}} |n\rangle$$  \hspace{1cm} (1)

such that

$$b|\gamma\rangle = \gamma|\gamma\rangle$$  \hspace{1cm} (2)

We can check that the above complicated form can actually be simplified into

$$|\gamma\rangle = N e^{\gamma b^\dagger} |0\rangle$$  \hspace{1cm} (3)

where $N = e^{-|\gamma|^2/2}$ is simply the normalization factor. Using this, we can write the superfluid mean field ground state of the Bose-Hubbard model as

$$\prod_i |\gamma\rangle \propto \prod_i e^{\gamma b_i^\dagger} |0\rangle = e^{\gamma \sum_i b_i^\dagger |0\rangle} = e^{\gamma \sqrt{N_s} b_{k=0}^\dagger |0\rangle}$$  \hspace{1cm} (4)

where $N_s$ is the total number of lattice sites. That is, the ground state is a big coherent state of the $k = 0$ boson mode, with average boson number $|\gamma|^2 N_s$. And we see again $|\gamma|^2$ is the average boson density.

Because $b$ is not a Hermitian operator, $\gamma$ is in general a complex number. Moreover, coherent states with different eigenvalue are not orthogonal to each other

$$\langle \gamma | \gamma' \rangle = e^{-\left(|\gamma|^2 + |\gamma'|^2 - 2\gamma^* \gamma'\right)}$$  \hspace{1cm} (5)

In particular, if $\gamma$ and $\gamma'$ have the same amplitude $\rho$ but different phase factor $\theta$ and $\theta'$, then their overlap is

$$\langle \gamma | \gamma' \rangle = e^{-2\rho^2 (1 - \cos \delta \theta)} e^{-i2\rho^2 \sin \delta \theta}$$  \hspace{1cm} (6)

On the other hand, coherent states are complete in that

$$\mathbb{I} = \frac{1}{\pi} \int d^2 \gamma |\gamma\rangle \langle \gamma|$$  \hspace{1cm} (7)

In the last lecture we talked about the ground state of the superfluid phase being a tensor product of the same coherent states on each lattice site $\prod |\gamma\rangle$. When we apply a global symmetry transformation, the state changes to $\prod |\gamma e^{i\alpha}\rangle$. On each lattice site, the states before and after the transformation have finite but < 1 overlap. However, when we take the inner product over the total wave function, the overlap decays to 0.
1.5 Superfluid phase: excitations

On top of a coherent superfluid ground state background with a uniform order parameter, as shown in Fig.1.5 (a), we can generate excitations by varying the order parameter by different amount at different spatial locations. For example we can vary the order parameter in a smooth way as a wave. The variation of the order parameter from one point to another costs energy and the faster the order parameter changes the higher the energy. If we make an infinitely smooth variation of the order parameter field, the energy cost can be infinitely small. Therefore, this type of wave excitation is gapless. The existence of gapless excitation is again a consequence of spontaneous symmetry breaking. Whenever (a continuous) symmetry is spontaneously broken, the ground state has a uniform order parameter field and global rotation of this order parameter field does not require energy. However, smooth variations on top of this uniform field does cost energy. Such variations form the Goldstone mode whose energy goes to zero when the variation becomes infinitely smooth.

How does energy go to zero as the variation becomes more and more smooth? The smoothness of the variation is described by the wave number \( k \). There are usually two different types of behavior for the excitation energy \( E_k \) as the wave number goes to zero.

Linear dispersion: \( E_k \propto |k| \), or Quadratic dispersion: \( E_k \propto |k|^2 \) \( (8) \)

Let’s see which case the weakly interacting boson gas belongs to.

To study the dynamics of the order parameter field, we need to find the equation that governs its dynamics. To do this, we take the Heisenberg formulation of quantum mechanics which says that the time evolution of an the order parameter operator \( \hat{b}_\vec{r} \) is given by

\[
 i\hbar \frac{\partial}{\partial t} b_{\vec{r}}(t) = [b_{\vec{r}}, H]
\]

where we have used \( \vec{r} \) to label the location of the lattice sites in the 2D plane. To calculate the term on the right hand side, we use the fact that \( [b_{\vec{r}}, b_{\vec{r}'}] = \delta_{\vec{r}-\vec{r}'} \) and \( [b_{\vec{r}}, n_{\vec{r}'}] = \delta_{\vec{r}-\vec{r}'} b_{\vec{r}'} \) and find

\[
 i\hbar \frac{\partial}{\partial t} b_{\vec{r}}(t) = -t(b_{\vec{r}+\hat{x}} + b_{\vec{r}-\hat{x}} + b_{\vec{r}+\hat{y}} + b_{\vec{r}-\hat{y}}) - (\mu + U/2)b_{\vec{r}} + Un_{\vec{r}}b_{\vec{r}}
\]

If we take the continuum limit, the equation becomes

\[
 i\hbar \frac{\partial}{\partial t} b_{\vec{r}}(t) = \left( -a^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - (\mu + U/2 + 4t) + Un_{\vec{r}} \right) b_{\vec{r}}
\]

\( (11) \)
Of course, in its exact form, this is an equation of operators with \( b_r \) at different \( \vec{r} \) all coupled to each other and is very hard to solve. However, in the superfluid phase near ground state, we can treat \( b_r(t) \) as numbers \( \psi_r(t) = \langle b_r(t) \rangle \). Then the equation for the evolution of \( \psi_r(t) \) formally looks like a Schrödinger equation in first quantization

\[
\frac{i\hbar}{\partial t} \psi_r(t) = \left( -a^2 t \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - (\mu + U/2 + 4t) + U|\psi_r(t)|^2 \right) \psi_r(t) \tag{12}
\]

where the operator \( n_r \) has been replaced by \( |\psi_r(t)|^2 \) in the last term. This term, which includes the variable \( \psi_r \), makes the equation nonlinear. This is the so-called Gross-Pitaevskii equation for superfluid.

Solving the Gross-Pitaevskii equation would allow us to understand the dynamics of the order parameter field. However, this is hard because the equation is nonlinear. If we are only interested in the low energy excitation, which corresponds to smooth small variations on top of a uniform background, we can assume that the solution takes the form

\[
\psi_r(t) = \psi_0 + u(\vec{r})e^{-i\omega t} + v(\vec{r})e^{i\omega t} \tag{13}
\]

That is, we assume that variation on top of the uniform background takes a plane wave form and the variation is much smaller in magnitude than the background \(|u|, |v| \ll |\psi_0|\). Note that if the equation is linear, each plane wave component would be independent of each other. However, for the Gross-Pitaevskii equation we will see that components with opposite frequencies \( \omega \) and \(-\omega\) mix with each other due to the nonlinear interaction term, so we need to combine these two components when solving for eigen-modes.

Now we can solve for this equation order by order. Plugging this form of the solution into the Gross-Pitaevskii equation and retaining terms to zeroth order in \( u \) and \( v \), we find

\[
|\psi_0|^2 = \frac{\mu + U/2 + 4t}{U} \tag{14}
\]

which is exactly the mean field solution we found before for the ground state. Now retain the first order terms in \( u \) and \( v \), we find the left hand side to be

\[
\hbar \omega (-u(\vec{r})e^{-i\omega t} + v(\vec{r})e^{i\omega t}) \tag{15}
\]

and the right hand side to be

\[
(-a^2 t \nabla^2 + U|\psi_0|^2) (u(\vec{r})e^{-i\omega t} + v(\vec{r})e^{i\omega t}) + U\psi_0^2 (u^*(\vec{r})e^{i\omega t} + v^*(\vec{r})e^{-i\omega t}) \tag{16}
\]

Combining the left and right hand side we get a set of coupled equation of \( u \) and \( v \)

\[
\begin{align*}
\hbar \omega u &= (-a^2 t \nabla^2 + U|\psi_0|^2)u + U\psi_0^2 v^* \\
\hbar \omega v^* &= (-a^2 t \nabla^2 + U|\psi_0|^2)v^* + U\psi_0^2 u 
\end{align*} \tag{17}
\]

Suppose that \( u \) and \( v \) contains only momentum \( k \) and \(-k\) components, i.e. we are looking for plane wave solutions. We find that

\[
\begin{pmatrix} u \\ -v^* \end{pmatrix} = \begin{pmatrix} a^2 tk^2 + U|\psi_0|^2 & -U\psi_0^2 \\ U\psi_0^2 & -a^2 tk^2 - U|\psi_0|^2 \end{pmatrix} \begin{pmatrix} u \\ -v^* \end{pmatrix} \tag{18}
\]

and the eigenvalues are

\[
\hbar \omega = \pm \sqrt{a^2 tk^2(a^2 tk^2 + U|\psi_0|^2)} \tag{19}
\]
This is the conclusion we are looking for: the relation between the energy and the momentum of the smooth variations on top of the uniform background. In particular, as long as $U$ is nonzero, at small $k$, $\omega_k \propto k$, giving rise to a linear dispersion.

And a linear dispersion is the key reason why a superfluid can flow without viscosity.

To understand this, consider a particle moving relative to a superfluid. If the particle has mass $m$, moves with velocity $v$, then it has momentum $p = mv$ and energy $E = \frac{p^2}{2m}$. In a normal fluid, the particle can be slowed down by scattering with particles in the fluid. In the scattering process, momentum and energy is conserved, so the change in the momentum and energy in the particle is transferred to particles in the fluid. However, such a transfer may not

Suppose that the momentum of the particle changes from $p_1$ to $p_2$, then correspondingly energy changes from $E_1 = \frac{p_1^2}{2m}$ to $E_2 = \frac{p_2^2}{2m}$. The ratio between energy difference and momentum difference is

$$\frac{\Delta E}{\Delta p} = \frac{1}{2m} \frac{p_1^2 - p_2^2}{p_1 - p_2} = \frac{p_1 + p_2}{2m}$$

which can be arbitrarily small for small $p_1$ and $p_2$.

On the other hand, in a superfluid, $\epsilon$ is proportional to $k$. Therefore, $\Delta \epsilon$ is proportional to $\Delta p = \hbar \Delta k$ and their ratio is a constant. Because of this, when $p_1$ and $p_2$ are small, their energy difference may be too small to excite anything in the superfluid. Therefore, the particle would move forward without scattering and without viscosity. The same argument holds if the fluid is moving in a container.

In the homework, we are going to derive the dispersion relation in a different but closely related way.

A few comments about the order parameter field: as we see, the dynamics of the order parameter field obeys an equation very much like single particle Schrodinger equation, except for the nonlinear term. Therefore, $\psi_\vec{r}$ is often called the ‘wave function’ of the system. It has the right property that $|\psi_\vec{r}|^2$ represents the average number density at $\vec{r}$, but we should also keep in mind that it is only the expectation value of a dynamical operator. In superfluid Helium, even though interaction is so strong that we cannot think of the system as individual bosons occupying certain single particle states any more, we can still talk about the macroscopic wave function $\psi_\vec{r}$, which is the expectation value of the order parameter.