Physics 223b Homework 4 Due 06/06/18 by 5pm

1. Majorana chain with time reversal symmetry

Consider the Majorana chain in the topological phase

\[ H = \sum_j i\gamma_{2j}\gamma_{2j+1} \]  \hspace{1cm} (1)

where time reversal symmetry acts as

\[ i \rightarrow -i, \ \gamma_{2j-1} \rightarrow \gamma_{2j-1}, \ \gamma_{2j} \rightarrow -\gamma_{2j} \] \hspace{1cm} (2)

Take \( M \) copies of the Majorana chain, so that on an open chain there are \( M \) Majorana modes at each end of the chain. In particular, at the left end, the Majorana modes transform under time reversal as

\[ \gamma_1^{(m)} \rightarrow \gamma_1^{(m)}, \ m = 1, 2, ..., M \] \hspace{1cm} (3)

(1) Consider an increasing number of Majorana chains. What is the smallest number \( M = M_0 \) where the edge degeneracy can be completely removed by adding four fermion interaction terms (of the form \( \gamma \gamma \gamma \gamma \)) to the edge Majorana modes without breaking time reversal symmetry?

\( M_0 = 8 \). To see this, first note that it is impossible to have no degeneracy if there are an odd number of Majoranas; they must come in pairs if we wish to gap them out. Two Majoranas combine into one complex fermion, so \( M_0 \) chains will have a Hilbert space of dimension \( 2^{M_0/2} \). Let’s examine the case with 2 chains. The only possible perturbation is

\[ i\gamma_1^1\gamma_2^2 \] \hspace{1cm} (4)

But this is forbidden by time reversal symmetry, as \( i \rightarrow -i \). So 2 is still degenerate.

4 is next, and there is an allowed perturbation:

\[ \gamma_1^1\gamma_2^2\gamma_3^3\gamma_4^4 \] \hspace{1cm} (5)

Perturbations of this type partially lift the degeneracy; if there are \( N \) degenerate modes then \( N/2 \) will be lifted above the rest. But this perturbation is not enough; let’s work in the basis where

\[ n_1 = i\gamma_1^1\gamma_2^2, \ n_2 = i\gamma_3^3\gamma_4^4 \] \hspace{1cm} (6)

If we want to use the fermion language, we can combine Majoranas into fermions like

\[ c_1 = \frac{\gamma_1^1 + i\gamma_2^2}{2}, \ c_2 = \frac{\gamma_3^3 + i\gamma_4^4}{2} \] \hspace{1cm} (7)
and the operators become

\[ n_1 = 2c_1^\dagger c_1 - 1, \quad n_2 = 2c_2^\dagger c_2 - 1 \]  

(8)

With eigenvalues \( \pm 1 \) depending on whether the fermionic mode is occupied or unoccupied.

Our perturbation is \(-n_1n_2\), and the two states \(|n_1 = 1, n_2 = 1\rangle\) and \(|n_1 = -1, n_2 = -1\rangle\) are still degenerate. So \(4\) will not work either.

6 Majorana chains offer a lot more possible terms: 15, to be exact. However, we still cannot gap out our system completely. We are looking for four-Majorana terms that commute with each other; if they commute then they can be simultaneously diagonalized and each will lift the degeneracy two-fold. But if we want to make the perturbations commute, we can only choose two simultaneously: e.g. \(\gamma^1\gamma^2\gamma^3\gamma^4\) and \(\gamma^1\gamma^2\gamma^5\gamma^6\). (In order to commute the number of Majorana operators the terms share must be even.) There is no other perturbation we can add that commutes with both these two.

Defining

\[ n_1 = i\gamma^1\gamma^2, \quad n_2 = i\gamma^3\gamma^4, \quad n_3 = i\gamma^5\gamma^6 \]  

(9)

we note that our perturbations become

\[-n_1n_2, \quad -n_1n_3\]  

(10)

and the kets \(|n_1 = 1, n_2 = 1, n_3 = 1\rangle\) and \(|n_1 = -1, n_2 = -1, n_3 = -1\rangle\) are still degenerate.

Note that a perturbation of the form \(i\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\) is forbidden by TRS.

But we can define a perturbation that gaps the system out for \(M_0 = 8\). Consider the terms

\[ \gamma^1\gamma^2\gamma^3\gamma^4, \quad \gamma^1\gamma^2\gamma^5\gamma^6, \quad \gamma^1\gamma^2\gamma^7\gamma^8, \quad \gamma^1\gamma^3\gamma^5\gamma^7 \]  

(11)

These terms all commute. Our Hilbert space has dimension 16; let’s observe the effect of the first three terms. Define:

\[ n_1 = i\gamma^1\gamma^2, \quad n_2 = i\gamma^3\gamma^4, \quad n_3 = i\gamma^5\gamma^6, \quad n_4 = i\gamma^7\gamma^8 \]  

(12)

We are again left with a two-fold degeneracy

\[ |1\rangle = |n_1 = 1, n_2 = 1, n_3 = 1, n_4 = 1\rangle, \quad |-1\rangle = |n_1 = -1, n_2 = -1, n_3 = -1, n_4 = -1\rangle \]  

(13)

It remains to be shown that we can remove this degeneracy out with the final term. Let’s combine Majoranas into complex fermions
\[ c_1 = \frac{\gamma_1 + i\gamma_2}{2}, c_2 = \frac{\gamma_3 + i\gamma_4}{2}, c_3 = \frac{\gamma_5 + i\gamma_6}{2}, c_4 = \frac{\gamma_7 + i\gamma_8}{2} \tag{14} \]

Then \(|1\rangle\) is the state with all fermion sites occupied, as \(n_i = 2c_i^\dagger c_i - 1\), and \(|-1\rangle\) is the state with all fermion sites unoccupied.

The perturbation in fermion language is

\[ \gamma_1 \gamma_3 \gamma_5 \gamma_7 = (c_1 + c_1^\dagger)(c_2 + c_2^\dagger)(c_3 + c_3^\dagger)(c_4 + c_4^\dagger) \tag{15} \]

And this takes

\[ |-1\rangle \rightarrow |1\rangle, \quad |1\rangle \rightarrow |-1\rangle \tag{16} \]

Which is a \(\sigma^x\) term, and lifts the degeneracy, as the \(|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |-1\rangle)\) states are shifted up and down by \(\pm 1\) in energy. So we have fully gapped out the Hamiltonian.

(2) Now consider the system on a ring with periodic boundary condition. For \(M_0\) Majorana chains, design a gapped path of the Hamiltonian \(H(s), 0 \leq s \leq 1\) so that \(H(0)\) is the Hamiltonian of \(M_0\) Majorana chains

\[ H(0) = \sum_{m=1}^{M_0} \sum_j i\gamma_{2j}^{(m)}\gamma_{2j+1}^{(m)} \tag{17} \]

and \(H(1)\) reads

\[ H(1) = \sum_{m=1}^{M_0} \sum_j i\gamma_{2j-1}^{(m)}\gamma_{2j}^{(m)} \tag{18} \]

Show explicitly that the Hamiltonian is gapped for all \(0 \leq s \leq 1\). (hint: consider a path through the point where on each lattice site, the eight Majorana fermions on the left hand side couple to each other through the four fermion interaction terms and the eight on the right hand side couple to each other in a similar way.)

First consider the path

\[ H(\lambda) = \sum_{m=1}^{M_0} \sum_j [(1 - \lambda)i\gamma_{2j}^{(m)}\gamma_{2j+1}^{(m)} + \lambda\gamma_{2j-1}^{(m)}\gamma_{2j}^{(m)}] \tag{19} \]

This tunes between the two Hamiltonians as \(\lambda\) runs from 0 to 1. However, when \(\lambda = 1/2\) we run into the critical point where the system switches dimerization and the theory is gapless. Hence this will not work.

The key is to look at the 8 dimerized pairs

\[ H(0) = \sum_{m=1}^{M_0} \sum_j i\gamma_{2j}^{(m)}\gamma_{2j+1}^{(m)} \tag{20} \]
and try to dimerize them in a different fashion, namely, with the four-Majorana terms introduced in part (a). If this can be done without opening a gap, then it is obviously true that the four-Majorana terms can be re-deformed into the opposite dimerization without opening a gap.

Consider the Hamiltonians

\[ H(\lambda = 0) = \sum_{m=1}^{M_0} i \gamma_{2j}^{(m)} \gamma_{2j+1}^{(m)} \] (21)

and

\[ H(\lambda = 1/2) = H_{\text{int}} = \sum_i (\gamma_i^{(1)} \gamma_i^{(2)} \gamma_i^{(3)} \gamma_i^{(4)} + \gamma_i^{(1)} \gamma_i^{(2)} \gamma_i^{(5)} \gamma_i^{(6)} + \gamma_i^{(1)} \gamma_i^{(2)} \gamma_i^{(7)} \gamma_i^{(8)} + \gamma_i^{(1)} \gamma_i^{(3)} \gamma_i^{(5)} \gamma_i^{(7)}) \] (22)

Notice that it’s summed over all \( i \) (including both even and odd sites). By part (a), we showed that both \( H(0) \) and \( H_{\text{int}} \) are gapped. We want to show that there is a path connecting these two Hamiltonian without closing the gap. Since \( H(0) \) couples Majorana operators only at sites \( 2j \) and \( 2j + 1 \) and \( H_{\text{int}} \) couples only at the same site \( i \). The 16 Majorana operators \( \gamma_i^{(m)} \) (with \( m = 1, 2, \ldots, 8 \) and \( i = 2j, 2j + 1 \)) are decoupled with other Majorana operators in the systems. Therefore, we only need to focus on this finite dimension Hilbert space \( (d = 2^8 = 64) \), which is local with finite operators at finite sites. The total Hamiltonian is summation of these independent local Hamiltonian.

In this finite dimensional Hilbert space, we can rewrite the local Hamiltonians as

\[ H_{\text{local}}(\lambda = 0) = \sum_{n=1}^{d} E_n |a_n\rangle \langle a_n| \] (23)

and

\[ H_{\text{local}}(\lambda = 1/2) = \sum_{n=1}^{d} E'_n |a'_n\rangle \langle a'_n| \] (24)

where \( |a_n\rangle \) and \( |a'_n\rangle \) are eigenvectors of the local Hamiltonians.

First, we will perform a unitary rotation on \( H(\lambda = 0) \) that takes \( |a\rangle \rightarrow |a'\rangle \). This is unitary because it is simply a change of basis; we know it is possible because the unitary group is path connected. The gap never closes as we are simply rotating the eigenkets; the eigenvalues in a change of basis never change. Our rotated Hamiltonian (say this is \( H(\lambda = 1/4) \) is

\[ H(\lambda = 1/4) = \sum_a E_a |a'\rangle \langle a'| \] (25)
Now we need to change the energies $E_a$ to $E'_a$. This is simple: a convex combination

$$H(1/4 < \lambda < 1/2) = \sum_a (4(1/2 - \lambda)E_a + 4(\lambda - 1/4)E'_a) |a' \rangle \langle a'|$$

(26)

The gap at the beginning of this deformation is positive, and the gap at the end is positive. Because we have simply performed a convex combination the gap everywhere in between is still positive. Hence we have deformed $H(\lambda = 0)$ to $H(\lambda = 1/2)$ without closing the gap.

The argument for $H(\lambda = 1/2)$ to $H(\lambda = 1)$ is identical. Just shift over by one Majorana.

2. Entanglement Area Law in Matrix Product States

Consider a finite dimensional matrix product state with open boundary condition

$$|\psi\rangle = \sum_{i_1i_2...i_N} \langle l| A_{i_1}^1 A_{i_2}^2 ... A_{i_N}^N |r\rangle |i_1i_2...i_N\rangle$$

(27)

where $\langle l|$ and $|r\rangle$ are row and column vectors respectively. In this problem, we are going to show that the entanglement entropy between the left half of the chain (containing $n$ lattice sites) and the right half of the chain (containing $N - n$ lattice sites) is upper bounded by a constant.

(1) Show that the wave function can be written as

$$|\psi\rangle = \sum_m |\psi^m_L\rangle |\psi^m_R\rangle$$

(28)

where $m$ labels the inner dimensions of matrix $A$. Find the explicit form of $|\psi^m_L\rangle$ and $|\psi^m_R\rangle$.

$$|\psi\rangle = \sum_{m=1}^D \sum_{i_1i_2...i_N} \langle l| A_{i_1}^{1m} A_{i_2}^{2m} ... A_{i_N}^{Nm} |r\rangle |i_1i_2...i_N\rangle \otimes |i_{n+1}i_{n+2}...i_N\rangle$$

(29)

Therefore

$$|\psi^m_L\rangle = \sum_{i_1i_2...i_n} \langle l| A_{i_1}^{1m} A_{i_2}^{2m} ... A_{i_n}^{nm} |i_1i_2...i_n\rangle$$

(30)

and

$$|\psi^m_R\rangle = \sum_{i_{n+1}i_{n+2}...i_N} [A_{i_{n+1}}^{n+1m} A_{i_{n+2}}^{n+2m} ... A_{i_N}^{Nm} |r\rangle |i_{n+1}i_{n+2}...i_N\rangle$$

(31)

(2) Show that the dimension of the reduced density matrix $\rho_L$ (or $\rho_R$) is upper bounded by $m$.

Tracing out the right half of the chain, the reduced density matrix $\rho_L$ is supported on $\{ |\psi^m_L\rangle, m = 1, 2, ... D \}$. Therefore, the dimension of $\rho_L$ is upper bounded by $m$.

(3) Entanglement entropy between the left and right part of the chain is given by $S = -\text{Tr}\rho_L \ln \rho_L$. Show that $S$ is upper bounded by a constant that depends on $m$. 5
As a density matrix with maximum dimension $D$, the maximum entropy of $\rho_L$ (hence the entanglement entropy between the two halves of the chain) is reached when all the eigenvalues of $\rho_L$ are the same and equals $1/D$. Therefore,

$$S \leq -D \frac{1}{D} \ln \frac{1}{D} = \ln D \quad (32)$$

3. Toric Code

In this problem, we are going to explore the basic properties of the Toric Code model in 2D. Consider a square lattice with periodic boundary condition and one spin 1/2 degree of freedom (basis states $|0\rangle$ and $|1\rangle$) per each link.

The Hamiltonian contains two types of terms: one involving four $\sigma_z$’s around a vertex, one involving four $\sigma_x$’s around a plaquette.

$$H = -\sum_v \left( \prod_{v\in l} \sigma_z^l \right) - \sum_p \left( \prod_{l\in p} \sigma_x^l \right) \quad (33)$$

$\sigma_z$ and $\sigma_x$ acts as

$$\sigma_z |0\rangle = |0\rangle, \quad \sigma_z |1\rangle = -|1\rangle, \quad \sigma_x |0\rangle = |1\rangle, \quad \sigma_x |1\rangle = |0\rangle \quad (34)$$

(a) Show that all the terms in the Hamiltonian commute with each other.

Terms involving $\sigma_z$’s commute with each other, and terms involving $\sigma_x$ commute with each other. The only possible nontrivial case is between a $\sigma_z$ star term and $\sigma_x$ plaquette term. If they are well-separated then they commute, but if the star and plaquette overlap they will overlap on two sites, and the factor of $-1$ picked up by pushing a $\sigma_z$ operator past a $\sigma_z$ operator appears twice, so we have no overall factor. Hence all terms commute.

(b) As the Hamiltonian terms are all independent of each other, the ground state can be chosen to minimize energy of each of the Hamiltonian terms. We can think of the spin 1/2 degrees of freedom as a $Z_2$ string on each link. That is, the $|0\rangle$ state corresponds to no string on each link while the $|1\rangle$ state corresponds to the existence of a string on the link. To minimize the energy of the $-\prod_{v\in l} \sigma_z^l$ terms, what conclusion can you draw regarding the string configuration in the ground state? Draw three string configurations satisfying all the $-\prod_{v\in l} \sigma_z^l$ terms.
In order for the energy to be minimized the product of $\sigma_z$ terms around any star must be 1; that is, an even number of strings exist on the four links: either 0, 2, or 4. What this means is that a string cannot terminate (that is, occupy an odd number of links around a star) or else we are penalized in energy. Any configuration of closed string loops is allowed.

(c) Now within the subspace of states satisfying the $-\prod_{v \in I} \sigma^I_z$ terms, try to minimize the energy of all the $-\prod_{l \in p} \sigma^l_x$ terms. Write down the ground state wave function that contains the configuration which has no string (all $|0\rangle$ product state).

The plaquette term will create a string around plaquette $p$. If there already is a string present on all four sides it will remove it. If there is a string that shares edges with the plaquette the string will 'pulled' around the plaquette. What this means is that the $\sigma_x$ term takes a string configuration and maps it to another, different one. The ground state is the even superposition of all closed string configurations:

$$|\Omega\rangle = \sum |\text{all allowed string configurations}\rangle$$

as any closed string configuration on the plane can be deformed to the identity by repeated application of plaquette terms.

(d) For a square lattice on the torus, are there other ground states? What is the ground state degeneracy?

There are 4 degenerate ground states. We argued that the plane has 1 unique ground state $|\Omega\rangle$ that consists of the even superposition of all loops. But a torus is special - because of the periodic boundary conditions there are closed loops that wind around the torus that cannot be generated by plaquette terms - these are precisely the $S^1_x$, $S^2_x$ operators shown in the figure. $S^1_x |\Omega\rangle$, $S^2_x |\Omega\rangle$, $S^1_x S^2_x |\Omega\rangle$ are all different ground states, but applying two $S^1_x$ operators cancel each other out. Our ground state degeneracy is 4.

(e) Consider the string operator $S^1_x$, $S^2_x$, $S^1_z$, $S^2_z$ as shown in the figure. Show that they commute with the Hamiltonian. What is their action within the degenerate ground space?

Again, because the strings are closed if an $S_z$ operator overlaps with either a star term they share an even number of sites and they commute, similarly for $S_x$. Define:

$$|0\rangle = |\Omega\rangle$$
$$|1\rangle = S^1_x |\Omega\rangle$$
$$|2\rangle = S^2_x |\Omega\rangle$$
$$|3\rangle = S^1_x S^2_x |\Omega\rangle$$

Then the action of the operators become (let’s assume there are an even number of sites in both dimensions):
Note that $S^1_\tilde{z}$ anticommutes with $S^2_x$ and $S^2_z$ anticommutes with $S^1_x$, hence that is why we get factors of $-1$. In essence they measure the parity of the loops that wind around the torus.