9 Wave packet and modulation

9.1 Wave packet in dispersive medium

Plane waves are nice; they travel and they are easy to analyze. But they do not carry any information. In order to send information, we need to send more complicated wave forms. For example, we can send a pulse or a sequence of pulses of the generic form $f(t)$. That is, we drive the left end of the system to oscillate as $f(t)$ and hope that the signal is carried to a receiver far away.

If the medium has a linear dispersion relation, i.e. $\omega = vk$, with a constant wave velocity v, then any wave form can travel undistorted. That is, if the oscillation at the left end has the form

$$f(t) = \int d\omega C(\omega) e^{i\omega t} \tag{1}$$

then the wave generated in the whole system looks like

$$\psi(x,t) = \int d\omega C(\omega) e^{i(\omega t - kx)} = \int d\omega C(\omega) e^{i\omega(t-x/v)} = f(t - x/v) \tag{2}$$

therefore, the oscillation at a generic location x exactly repeats what happens at $x = 0$ at a later time $t = x/v$. Any information carried by the wave form can be faithfully received.

However, a real medium may not have linear dispersion relation. In general, we can have a complicated dispersion relation $\omega(k)$. In this case, if the left end is made to oscillate as

$$f(t) = \int d\omega C(\omega) e^{i\omega t} \tag{3}$$

then the wave generated in the whole system looks like

$$\psi(x,t) = \int d\omega C(\omega) e^{i(\omega(k)t - kx)} \tag{4}$$

This is called a wave packet, which is a superposition of different plan waves with wave number k and wave frequency $\omega(k)$. But if $\frac{\omega(k)}{k}$ is not a constant, different components of the wave travels at different velocity. Over time, the shape of the wave packet gets distorted and eventually disappear. This is why the medium is called dispersive.

9.2 Modulation and Group Velocity

So how can we transmit signal reliably in a dispersive medium? To do that, we need to encode the signal on top of a carrier.

Consider the superposition of two waves with similar wave numbers and similar frequencies.

$$\psi(x,t) = e^{i(\omega_+ t - k_+ x)} + e^{i(\omega_- t - k_- x)} \tag{5}$$

When $\Delta k << k_0$, $\Delta\omega << \omega_0$, we assume $k_\pm = k_0 \pm \Delta k$, $\omega_\pm = \omega_0 \pm \Delta\omega$ and ω changes linearly with k, even though it may not be directly proportional to k. This assumption holds in any kind of medium with a smooth dispersion relation when frequency changes are small enough.

$$\psi(x,t) = e^{i(\omega_0 t - k_0 x)}\left(e^{i(\Delta\omega - \Delta k x)} + e^{-i(\Delta\omega - \Delta k x)}\right) = 2e^{i(\omega_0 t - k_0 x)}\cos(\Delta\omega t - \Delta k x) \tag{6}$$

That is, on top of a carrier wave with velocity

$$v_p = \frac{\omega_0}{k_0} \tag{7}$$

its amplitude changes as $\cos(\Delta\omega t - \Delta k x)$, which moves with velocity

$$v_g = \frac{\Delta\omega}{\Delta k} \sim \left.\frac{d\omega}{dk}\right|_{k=k_0} \tag{8}$$

v_p is called the phase velocity of the wave while v_g is called the group velocity.

By making superposition of such amplitude modulations, we can send signals of arbitrary shape. Suppose that the signal we want to send is $f(t)$. We can combine it on top of a carrier $e^{i\omega_0 t}$ and then input to the system as a boundary condition at $x = 0$.

$$\psi(0,t) = f(t)e^{i\omega_0 t} = \int d\omega C(\omega) e^{i(\omega+\omega_0)t} = \int d\omega C(\omega - \omega_0) e^{i\omega t} \tag{9}$$

Therefore, the full wave is

$$\psi(x,t) = \int d\omega C(\omega - \omega_0) e^{i(\omega t - kx)} \tag{10}$$

Suppose that the signal is center around a small frequency region so that $C(\omega - \omega_0) = 0$ when $|\omega - \omega_0| > \Delta\omega$ and

$$\omega(k) \approx \omega_0 + \frac{d\omega}{dk}(k - k_0) = \omega_0 + v_g(k - k_0) \tag{11}$$

Then the full wave is

$$\begin{aligned} \psi(x,t) &= \int d\omega C(\omega - \omega_0) e^{i(\omega_0 t - k_0 x + (\omega-\omega_0)t - (\omega-\omega_0)x/v_g)} \\ &= \int d\omega C(\omega - \omega_0) e^{i(\omega_0 t - k_0 x)} e^{i(\omega-\omega_0)(t - x/v_g)} \\ &= f(t - x/v_g) e^{i(\omega_0 t - k_0 x)} \end{aligned} \tag{12}$$

Therefore, on top of the carrier $e^{i(\omega_0 t - k_0 x)}$ which travels at phase velocity $v_p = \frac{\omega_0}{k_0}$, the signal travels at group velocity $v_g = \frac{d\omega}{dk}$, without changing its shape. Of course, this is still an approximate result which requires that the second order derivative $\frac{d^2\omega}{dk^2}$ can be ignored, but it is a much better approximation than assuming $\omega \sim v_p k$ and is usually satisfied to a good extend in common mediums.

Apart from amplitude modulation (AM), signal can also be added to the carrier through frequency modulation (FM), or phase modulation (PM).

10 A little bit of quantum mechanics

In quantum mechanics, wave is a fundamental part of the description. All the elementary particles have particle - wave duality. That is, they are particles and at the same time also waves. As particles, they are described by quantities like the mass m, location x, momentum $\vec{p}$, energy E etc. As wave, they are described by a complex wave function $\psi(x,t)$, and correspondingly quantities like the frequency ω and wave number $\vec{k}$.

The complex wave function does not correspond to the motion of some underlying degrees of freedom. Instead, it is interpreted as the 'probability amplitude' of the particle so that the probability for the particle to appear at location x at time t is given by

$$P(x,t) = |\psi(x,t)|^2 \tag{13}$$

'Probability amplitude' is not physical measurable while probability is. From this expression, we can see that the overall phase factor in $\psi(x,t)$ is unphysical in that it does not affect physically measurable results in any way. However, if there are two particles described by $\psi_1(x,t)$ and $\psi_2(x,t)$, then their relative phase difference is of physical relevance because it can affect how they interfere with each other.

The particle wave duality is reflected in the connection between these two sets of quantities:

$$E = \hbar\omega, \quad \vec{p} = \hbar k \tag{14}$$

where $\hbar = 6.63 \times 10^{-34}\mathrm{J}\cdot\mathrm{S}$ is the reduced Planck's constant. Therefore, a plane wave with fixed $\vec{k}$ and ω corresponds to a particle traveling at fixed $\vec{p}$ and E.

The wave picture provides a nice way to understand one of the most striking properties of quantum mechanical particles: the uncertainty principle. For example, the location and momentum of a quantum mechanical particle can not be exactly determined at the same time. The uncertainty in location and momentum satisfy

$$\Delta x \Delta p \geq \hbar/2 \tag{15}$$

As a wave, it is easy to see why location and momentum, or correspondingly wave number, cannot be simultaneously determined exactly. For example, a plane wave has a fixed wave number, and hence a fixed momentum. However, the probability for the particle to appear at location x at time t is given by

$$P(x,t) = |Ae^{i(\omega t - kx)}|^2 = |A|^2 \tag{16}$$

which is uniform over all x.

On the other hand, if the wave function is a Delta peak, $\psi(x,t) = \delta(x - vt)$, then the location of the particle at each time t is completely fixed. However, under Fourier transformation, the Delta peak decomposes into plane waves of all possible wave number

$$\psi(x,t) = \delta(x - vt) = \int dk e^{ik(x-vt)} \tag{17}$$

Therefore, the momentum of the wave is completely undetermined.

For more general forms of waves, uncertainty exists for both x and p. Of course, the fact that $\Delta x \Delta p$ is lower bounded by $\hbar/2$ is a result that is only possible in quantum mechanics.