8 Polarization

Recall that in a light wave, there are six oscillating degrees of freedom: $E_x, E_y, E_z, B_x, B_y, B_z$, so it is much more complicated than the other 'scalar' waves (only one degree of freedom) that we have been talking about. In order to give a full description of the wave, we need to specify how each of the components oscillate. In a plane wave propagating in the z direction, the E_z and B_z components are zero – the EM field is transversal. Moreover, the transverse B field is determined by the transverse E field. Therefore, in order to completely describe this plane wave, we only need to specify the E field in the xy plane.

$$\vec{E}(x,y,z,t) = (E_x\hat{x} + E_y\hat{y})\, e^{i(\omega t - kz)} \tag{1}$$

This freedom in E_x and E_y is called polarization.

8.1 Polarization

8.1.1 Linear polarization

If E_x and E_y are in phase with each other, the light wave is linearly polarized. There can be different situations. If $E_x = 0$, the wave is linearly polarized in the y direction.

$$\vec{E}(z,t) = |E_y|e^{i\phi}\hat{y}e^{i(\omega t - kz)} \tag{2}$$

If we take the real part of this expression and find the physical electric field, we find

$$\vec{E}_R(z,t) = |E_y|\hat{y}\cos(\omega t - kz + \phi) \tag{3}$$

The electric field points only in the y direction and oscillates with space and time.

If $E_y = 0$, the wave is linearly polarized in the x direction.

$$\vec{E}(z,t) = |E_x|e^{i\phi}\hat{x}e^{i(\omega t - kz)} \tag{4}$$

If we take the real part of this expression and find the physical electric field, we find

$$\vec{E}_R(z,t) = |E_x|\hat{x}\cos(\omega t - kz + \phi) \tag{5}$$

The electric field points only in the x direction and oscillates with space and time.

More generally, the wave can be polarized along any direction in the xy plane.

$$\vec{E}(z,t) = |E|e^{i\phi}\,(\cos\theta\hat{x} + \sin\theta\hat{y})\, e^{i(\omega t - kz)} \tag{6}$$

If we take the real part of this expression and find the physical electric field, we find

$$\vec{E}_R(z,t) = |E|\,(\cos\theta\hat{x} + \sin\theta\hat{y})\cos(\omega t - kz + \phi) \tag{7}$$

The electric field points only in the $(\cos\theta\hat{x} + \sin\theta\hat{y})$ direction and oscillates with space and time. All linearly polarized wave propagating in the z direction can be written in this way. That is, any linearly polarized wave in the xy plane can be decomposed into a linearly polarized wave in the x direction and one in the y direction and the components have the same phase ϕ.

What if they have different phases?

8.1.2 Circular polarization

Let's consider the combination of linearly polarized wave in x, y directions with the same amplitude but phase difference of $\pi/2$.

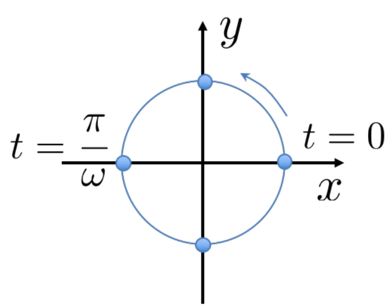
$$\vec{E}(z,t) = \left(E\hat{x} + Ee^{i\pi/2}\hat{y}\right)e^{i(\omega t - kz)} \tag{8}$$

How does every point on the way move?

At $z = 0$, $\vec{E}(0,t) = \left(E\hat{x} + Ee^{i\pi/2}\hat{y}\right)e^{i(\omega t)}$. Of course, to find out the real electric field, we need to take the real part of this expression, which becomes

$$\vec{E}_R(0,t) = E\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right) \tag{9}$$

That is, the electric field at $z = 0$ is rotating in circle.



The electric field at all the other spatial locations in the wave also move in circles, but with a phase difference kz. If we look into the ray from the $+\hat{z}$ direction, at every spatial point, the electric field rotates in the counter-clockwise direction. This is called 'left-circular polarization'.

On the other hand, if

$$\vec{E}(z,t) = \left(E\hat{x} + Ee^{-i\pi/2}\hat{y}\right)e^{i(\omega t - kz)} \tag{10}$$

At every point, the electric field rotates in the clockwise direction. This is called 'right-circular polarization'.

8.1.3 Elliptical polarization

In general, the E_x and E_y components can have different amplitudes and arbitrary phase difference.

$$\vec{E}(z,t) = \left(|E_x|e^{i\phi_x}\hat{x} + |E_y|e^{i\phi_y}\hat{y}\right)e^{i(\omega t - kz)} \tag{11}$$

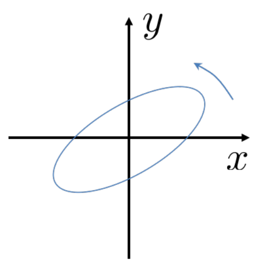
At $z = 0$,

$$\vec{E}(0,t) = \left(|E_x|e^{i\phi_x}\hat{x} + |E_y|e^{i\phi_y}\hat{y}\right)e^{i\omega t} \tag{12}$$

If we take the real part it becomes

$$\vec{E}_R(0,t) = |E_x|\cos(\omega t + \phi_x)\hat{x} + |E_y|\cos(\omega t + \phi_y)\hat{y} \tag{13}$$

If we track the trajectory of the $\vec{E}_R$ vector in the xy plane, it traces out an oval shape. In fact, at



every z point in the wave, the electric field traces out the same oval shape, but with a phase shift from one point to another.

8.1.4 Unpolarized wave

A monochromatic plane wave is always polarized. That is, for a plane wave with a single frequency

$$\vec{E}(z,t) = (E_x\hat{x} + E_y\hat{y})e^{i(\omega t - kz)} \tag{14}$$

At every point in space, the x component of the electric field is oscillating with frequency ω and so is the y component. These two components have a fixed phase difference. Such waves are said to be coherent.

However, in reality, waves always have a finite distribution over frequency.

$$\vec{E}(z,t) = \int d\omega\,(E_x(\omega)\hat{x} + E_y(\omega)\hat{y})\,e^{i(\omega t - kz)} \tag{15}$$

For a generic choice of $E_x(\omega)$ and $E_y(\omega)$, the polarization of the wave will change from time to time and the wave is incoherent.

Consider the example of a wave composed of four components with frequencies $\omega_1 \approx \omega_2 \approx \omega_3 \approx \omega_4$. Suppose that the four components are all linearly polarized: 1,2 in x direction and 3,4 in y direction. They have the same amplitude but different phases.

$$E_x = Ae^{i\phi_1}e^{i(\omega_1 t - k_1 z)} + Ae^{i\phi_2}e^{i(\omega_2 t - k_2 z)}, \quad E_y = Ae^{i\phi_3}e^{i(\omega_3 t - k_3 z)} + Ae^{i\phi_4}e^{i(\omega_4 t - k_4 z)} \tag{16}$$

At $z = 0$, taking the real part of the expression we get

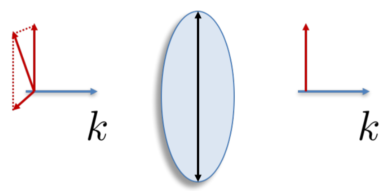
$$E_x^R(0,t) = 2A\cos(\bar{\omega}_1 t - \bar{\phi}_1)\cos(\Delta\omega_1 t - \Delta\phi_1), \;\; E_y^R(0,t) = 2A\cos(\bar{\omega}_2 t - \bar{\phi}_2)\cos(\Delta\omega_2 t - \Delta\phi_2) \tag{17}$$

where $\bar{\omega}_1 = (\omega_1 + \omega_2)/2$, $\Delta\omega_1 = (\omega_1 - \omega_2)/2$, $\bar{\omega}_2 = (\omega_3 + \omega_4)/2$, $\Delta\omega_2 = (\omega_3 - \omega_4)/2$.

$E_x^R(0,t)$ and $E_y^R(0,t)$ can be thought of as two beats. On a time scale $t << \frac{1}{\Delta\omega}$, E_x^R and E_y^R oscillates with roughly the same frequency, fixed amplitude and their phase difference remains roughly constant. Therefore, over a short period of time, the wave is coherent. However, when we look at the wave over a longer period of time, the frequency of the two oscillations differ, their amplitude changes, and their phase difference changes. The total $\vec{E}$ vector will wander around in the xy plane, can point in any direction, and the wave becomes incoherent. $\frac{1}{\Delta\omega}$ is hence called the coherence time of the wave.

8.2 Polarizer and Wave Plate

A polarizer allows waves polarized in a particular direction to pass but absorbs waves polarized in the perpendicular direction. The direction that wave can pass is called the easy axis of the polarizer.

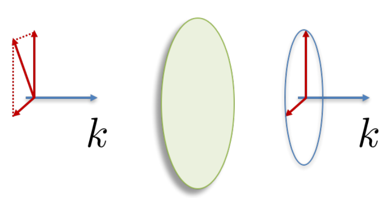


If we denote the polarized light as a two dimensional vector $\begin{pmatrix} E_x \\ E_y \end{pmatrix}$, the polarizer acts as a projector P. If the easy axis is along the x direction, $P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If the easy axis is along the y direction, $P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. If the easy axis is along a general direction that makes an angle θ with the x axis,

$$P_\theta = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \tag{18}$$

We can verify that as a projector $P_\theta P_\theta = P_\theta$. That is, repeated application of the same polarizer has the effect of just one of them.

A wave plate can change the relative phase of the two polarization components. That is, if we send in a linearly polarized light, we can get a circularly / elliptically polarized light at the output. This



is possible because the refraction index (speed of light in vacuum divided by the speed of light in the medium) is different for the two linearly polarized components. Suppose that the refraction indices are n_x and n_y respectively, then the phase difference induced by a wave plate of thickness L is

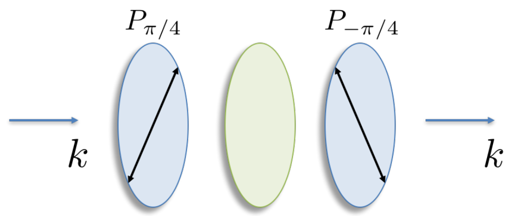
$$\Delta\phi = k_x L - k_y L = L\omega\frac{n_x - n_y}{c} \tag{19}$$

In matrix form, the effect of the wave plate can be written as

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{pmatrix} \tag{20}$$

(what if we rotate the wave plate by an angle θ around the z axis?)

Putting a wave plate between two polarizers, we can generate polarized colored light from unpolarized white light.



If there is no wave plate in between, then because the two polarizers are orthogonal to each other, no light can pass through. Now with the wave plate, after the first polarizer, the light vector is $(1, 1)$ in the xy plane. After the wave plate, it becomes $(1, e^{i\Delta\phi})$, where $\Delta\phi$ is generally ω dependent. When $\Delta\phi = 0$, no light passes through the second polarizer. When $\Delta\phi = \pi$, the light can pass through. In between , the light is partially transmitted. Therefore, the output light has color and each wavelength component is linearly polarized.

The combined effect of the polarizers and the wave plate can be found by multiplying their corresponding matrices

$$P_{-\pi/4}QP_{\pi/4} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - e^{i\Delta\phi} & 1 - e^{i\Delta\phi} \\ -1 + e^{i\Delta\phi} & -1 + e^{i\Delta\phi} \end{pmatrix} \tag{21}$$