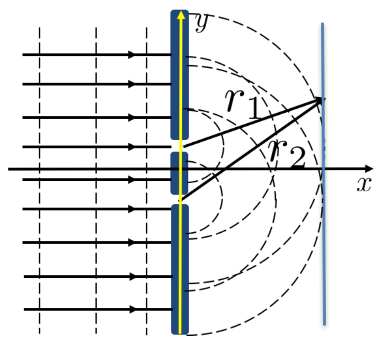
7 Interference and Diffraction

7.3 Double slits

7.3.1 Infinitely thin double slits

Now consider two small slits on the plane $x = 0$, with a distance a between them. If we shine a plane wave in the x direction, what would we observe on a plane at $x = x_0$?



Each slit makes a cylindrical wave. At a point (x, y), the total wave is a superposition of the two

$$\frac{A}{\sqrt{r_1}} e^{i(\omega t - kr_1)} + \frac{A}{\sqrt{r_r}} e^{i(\omega t - kr_2)} \tag{1}$$

We can ignore the difference in amplitude, but the difference in phase is important. If we make the approximation that

$$r_1 = r + \frac{y}{x}\frac{a}{2}, r_2 = r - \frac{y}{x}\frac{a}{2} \tag{2}$$

Then the total wave becomes

$$\frac{A}{\sqrt{r}} e^{i(\omega t - kr)} \left(e^{ik\frac{y}{x}\frac{a}{2}} + e^{-ik\frac{y}{x}\frac{a}{2}} \right) = \frac{2A}{\sqrt{r}} e^{i(\omega t - kr)} \cos\left(k\frac{y}{x}\frac{a}{2} \right) \tag{3}$$

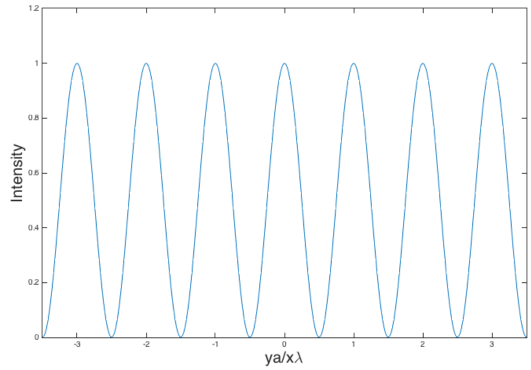
The time averaged intensity is then proportional to

$$I \propto \cos^2\left(\frac{ka}{2x} y \right) \tag{4}$$

In this interference pattern, the intensity reaches its peak at $y = \frac{n\lambda x}{a}$. This is easy to understand: at the peaks, the optical path difference of the two waves are

$$a\frac{y}{x} = n\lambda \tag{5}$$

resulting in a phase difference of $n2\pi$. The two waves interference constructively and we see a peak in the intensity.



Half way between the peaks, where $y = \frac{(2n+1)\lambda x}{2a}$, the two waves have an optical path difference of half wave length, resulting in a π phase difference. The two wave interference destructively and we see a zero in the intensity.

All peaks are regularly spaced, with the same width and the same height.

7.3.2 Finite width double slits

What if each of the double slits has finite width L? For a slit with width L, the wave that emanates from each slit is not a simple cylindrical wave. Instead, it makes the diffraction pattern of the form

$$\psi(x, y, t) = \frac{2Ax}{\sqrt{r}Lk} e^{i(\omega t - kr)} \frac{\sin\left(\frac{kL}{2x} y\right)}{y} \tag{6}$$

When we have two slits, centered at $\pm\frac{a}{2}$ respectively, they interfere as

$$\psi_+(x, y, t) + \psi_-(x, y, t) = \frac{2Ax}{\sqrt{r_+}Lk} e^{i(\omega t - kr_+)} \frac{\sin\left(\frac{kL}{2x}(y - a/2)\right)}{y - a/2} + \frac{2Ax}{\sqrt{r_-}Lk} e^{i(\omega t - kr_-)} \frac{\sin\left(\frac{kL}{2x}(y + a/2)\right)}{y + a/2} \tag{7}$$

When $x >> L, a$, we can approximate r_+ as $r - \frac{a}{2}\frac{y}{x}$ and r_- as $r + \frac{a}{2}\frac{y}{x}$. Similar to the previous case, we ignore the difference in amplitude of the two waves so that

$$\frac{2Ax}{\sqrt{r_+}Lk} \frac{\sin\left(\frac{kL}{2x}(y - a/2)\right)}{y - a/2} \approx \frac{2Ax}{\sqrt{r_-}Lk} \frac{\sin\left(\frac{kL}{2x}(y + a/2)\right)}{y + a/2} \approx \frac{2Ax}{\sqrt{r}Lk} \frac{\sin\left(\frac{kL}{2x}(y)\right)}{y} \tag{8}$$

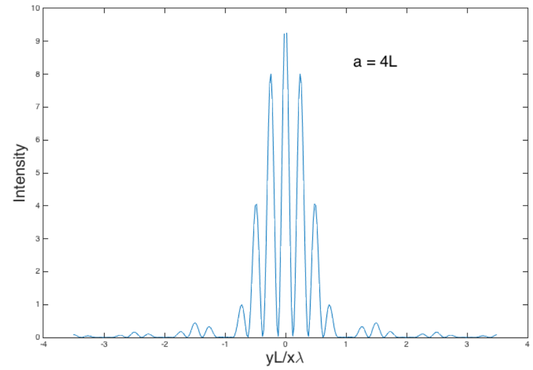
But we need to take into account the difference in phase factor so that

$$\psi_+(x, y, t) + \psi_-(x, y, t) \approx \frac{2Ax}{\sqrt{r}Lk} \frac{\sin\left(\frac{kL}{2x}(y)\right)}{y} e^{i(\omega t - kr)} \cos\left(\frac{ka}{2x} y\right) \tag{9}$$

The total intensity is then proportional to

$$I \sim \frac{\sin^2\left(\frac{kL}{2x}(y)\right)}{y^2} \cos^2\left(\frac{ka}{2x} y\right) \tag{10}$$

That is, the interference pattern is 'modulated' by the diffraction pattern.



7.3.3 Many slits

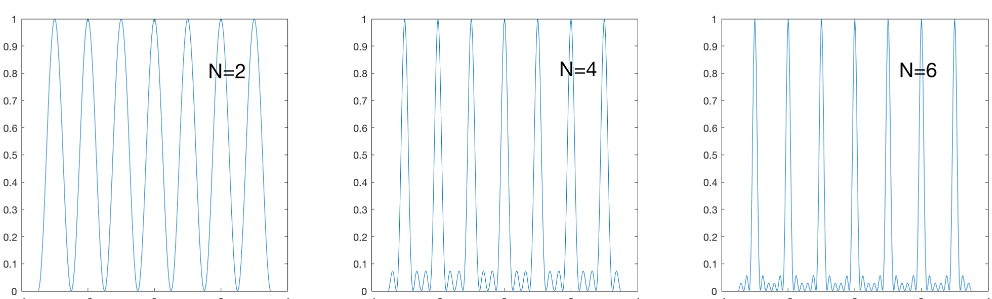
If there are more than 2 slits, what is the interference pattern like?

First let's assume that the slits are infinitely narrow and ignore diffraction. When the path difference between different slits is integer multiples of λ, then the waves from different slits interfere constructively with each other, giving rise to a peak in the interference pattern. If we ignore the difference in the amplitude of the waves, then the intensity distribution on the screen is proportional to

$$I \propto \left(e^{ikay/2x} + e^{-ikay/2x} + e^{3ikay/2x} + e^{-3ikay/2x} + ... + e^{(2N-1)ikay/2x} + e^{-(2N-1)ikay/2x}\right)^2 \tag{11}$$

where a is the distance between neighboring slits and there are in total $2N$ slits.

If we make a plot of the intensity vs. $ya/x\lambda$, we see that



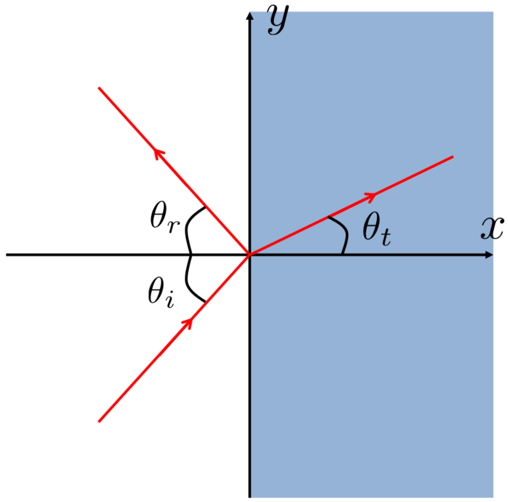
the (highest) peak locations are the same, but with more slits, the peaks become sharper.

A useful optical device is made by cutting a large number of very thin slits into an opaque plane. This is called the diffraction grating. An important function of diffraction grating is to separate light of different wavelength. In particular, the location of the second, third ... maximums depend (is proportional to) on the wavelength λ. Therefore, if white light shines perpendicularly on the grating, the central peak is still white, but the second, third etc. peak would be rainbow like: different wave lengths have peak at different locations.

Now if we take into consideration the fact that the slits have finite width, the interference pattern shown above is going to be modulated by the diffraction pattern of each individual slit.

6 Reflection and Transmission

6.4 Non-perpendicular incident wave, refraction



Now consider the more complicated case of transverse wave on a two dimensional membrane. Suppose that we have two such membranes in the xy plane with wave equations

$$\frac{\partial^2\psi}{\partial t^2} = v_1^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), \quad \frac{\partial^2\psi}{\partial t^2} = v_2^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{12}$$

respectively and they are connected along the $x = 0$ axis.

Plane wave of the form

$$\psi(x,t) = Ae^{i(\omega t - \vec{k}\cdot\vec{x})} \tag{13}$$

can exist in both membranes. When the plane wave crosses the boundary at $x = 0$ from one membrane to another, there can be reflection and the transmitted wave may not proceed in the same direction as the incident wave. This is the phenomena of refraction and we are going to find out below the direction of the reflected and transmitted wave.

Consider an incident wave of the form

$$\psi_i(\vec{x},t) = A_i e^{\omega t - \vec{k}_i\cdot\vec{x}} \tag{14}$$

where $\omega = v_1|\vec{k}_i|$. Suppose that the reflected wave and transmitted wave take the form

$$\psi_r(\vec{x},t) = A_r e^{\omega t - \vec{k}_r\cdot\vec{x}}, \quad \psi_t(\vec{x},t) = A_t e^{\omega t - \vec{k}_t\cdot\vec{x}} \tag{15}$$

where $\omega = v_1|\vec{k}_r| = v_2|\vec{k}_t|$.

To satisfy continuity along the whole boundary of $x = 0$, we require that

$$\psi_i(0,y,t) + \psi_r(0,y,t) = \psi_t(0,y,t) \tag{16}$$

That is

$$A_i e^{-ik_i^y y} + A_r e^{-ik_r^y y} = A_t e^{-ik_t^y y} \tag{17}$$

In order for this to be true for any y, we must have

$$k_i^y = k_r^y = k_t^y \tag{18}$$

and at the same time

$$A_i + A_r = A_t \tag{19}$$

From this we can find the direction of the reflected and transmitted wave. Suppose that the wave vector of the incident wave is $\vec{k}_i = (k_i^x, k_i^y)$. It makes an angle θ_i with the normal direction of the interface

$$\tan\theta_i = \frac{k_i^x}{k_i^y} \tag{20}$$

The wave vector of the reflected wave has the same magnitude ω/v_1 and the same y direction component. Therefore, $k_r^x = -k_i^x$.

$$\tan\theta_r = \frac{|k_r^x|}{k_r^y} = \tan\theta_i \tag{21}$$

The wave vector of the transmitted wave has a different magnitude $|\vec{k}_t| = \omega/v_2$. Therefore, θ_t satisfies

$$\sin\theta_t = \frac{k_t^y}{|\vec{k}_t|} = \frac{k_i^y}{|\vec{k}_i|}\frac{v_2}{v_1} = \sin\theta_i\frac{v_2}{v_1} \tag{22}$$

That is

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_t}{v_2} \tag{23}$$

If $v_2 < v_1$, for a given θ_i, we can always find a θ_t that satisfies this condition. However, if $v_2 < v_1$, this may not be the case. In particular if $\frac{v_2}{v_1}\sin\theta_i > 1$, or equivalently

$$\theta_i > \theta_c = \arcsin\frac{v_1}{v_2} \tag{24}$$

then θ_t does not exist. What happens in this case is that the incident wave is completely reflected, with no transmitted wave at all. This is the phenomena of total internal reflection. Total internal reflection is very useful for keeping a wave propagating along a waveguide without leaking out of the wave guide.