7 Interference and Diffraction

In our discussion of multiple reflection, we have already seen a situation where the total (reflected) wave is the sum of many components. The different components can have different phase shifts between them, resulting in different amplitude for the total wave. If all the components are in phase, then they interfere constructively, giving rise to a large total wave; if the components are out of phase with each other, they interfere destructively, giving rise to a small total wave. In general, this kind of phenomena is called interference. In this section, we are going to discuss how interference can lead to different interference and diffraction patterns.

7.1 Cylindrical and spherical wave

Recall that in 1D, to generate a wave that travels in the $+x$ direction, we can drive the $x = 0$ point to oscillate as $Ae^{i\omega t}$. The wave generated is then $Ae^{i(\omega t - kx)}$. Similarly, in 2D and 3D, to generate a plane wave that travels in the $+x$ direction, we can drive the $x = 0$ line / plane to oscillate as $Ae^{i\omega t}$ and the wave generated would be $Ae^{i(\omega t - kx)}$, where $k = \omega/v$. More generally, we can drive the $x = 0$ line / plane as $Ae^{i(\omega t - k_y y)}$ / $Ae^{i(\omega t - k_y y - k_z z)}$. The wave generated is still a plane wave, taking the form $Ae^{i(\omega t - k_x x - k_y y)}$ / $Ae^{i(\omega t - k_x x - k_y y - k_z z)}$, where $k_x = \sqrt{k^2 - k_y^2}$ / $k_x = \sqrt{k^2 - k_y^2 - k_z^2}$, $k = \omega/v$. At a particular t, if we collect all the points with the same phase, we get parallel planes with distance $\lambda = \frac{2\pi}{k}$ between them. Such planes are called wavefronts in the plane wave.

What if we drive only one point $x = 0, y = 0, (z = 0)$ as $Ae^{i\omega t}$? What kind of wave is generated?

In 2D, if we drive $x = 0, y = 0$ as $Ae^{i\omega t}$, a cylindrical wave is generated which takes the form

$$\psi(\vec{x}, t) = \frac{A}{\sqrt{r}} e^{i(\omega t - kr)} \tag{1}$$

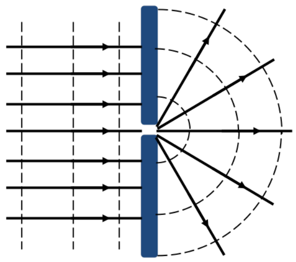
where $r = \sqrt{x^2 + y^2}$. The wavefront are circles with distance $\lambda = \frac{2\pi}{k}$ between them.

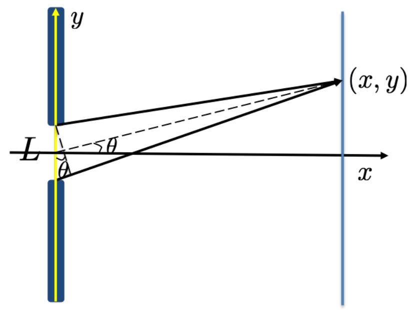
In 3D, if we drive $x = 0, y = 0, z = 0$ as $Ae^{i\omega t}$, a spherical wave is generated which takes the form

$$\psi(\vec{x}, t) = \frac{A}{r} e^{i(\omega t - kr)} \tag{2}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The wavefronts are spheres with distance λ between them.

The $\frac{1}{\sqrt{r}}$ and $\frac{1}{r}$ scaling of the amplitude is to satisfy energy conservation: the energy flux (energy flow per unit time and unit area) is proportional to amplitude squared in a wave. The surface area across which energy flows scales as r in 2D and r^2 in 3D. Therefore, to ensure energy conservation, the amplitude scales as $\frac{1}{\sqrt{r}}$ in 2D and $\frac{1}{r}$ in 3D.





7.2 Single slit diffraction

Consider a wall with a slit. Suppose we send a plane wave in the $+x$ direction. It hits the wall at $x = 0$. If the slit is infinitely small, it forms a point source (in 2D) for wave in the region $x > 0$. The wave generated is, at $x > 0$,

$$\psi(x, y, t) = \frac{A}{\sqrt{r}} e^{i(\omega t - kr)} \tag{3}$$

The intensity is uniform in all directions!

This is the phenomena of diffraction. Light after passing through the slit does not travel purely in the $+x$ direction any more. This is a strong manifestation of the wave nature of light.

In reality , we do expect that the angular spread of the wave after passing through the slit to be finite. This is because every slit has a finite width L. What is the intensity distribution pattern for diffraction at a finite width slit?

Imagine dividing the slit into N pieces. As $N \to \infty$, each piece generates a cylindrical wave. The total wave is then a superposition of waves from each piece of the slit.

$$\psi(x, y, t) = \int_{-\frac{L}{2}}^{\frac{L}{2}} A d\tilde{y} \frac{1}{\sqrt{r_{\tilde{y}}}} e^{i(\omega t - k r_{\tilde{y}})} \tag{4}$$

Here $(0, \tilde{y})$ is the location of the piece in the slit, (x, y) is where we are measuring the wave, $r_{\tilde{y}} = \sqrt{x^2 + (y - \tilde{y})^2}$.

This is a complicated integral, but let's consider the case of $x >> y >> \tilde{y}$. Correspondingly, $r = \sqrt{x^2 + y^2} >> y >> \tilde{y}$,

$$r_{\tilde{y}} \approx \sqrt{x^2 + y^2 - 2y\tilde{y}} = r\sqrt{1 - \frac{2y\tilde{y}}{r^2}} \approx r(1 - \frac{y\tilde{y}}{r^2}) \approx r - \tilde{y}\frac{y}{x} \tag{5}$$

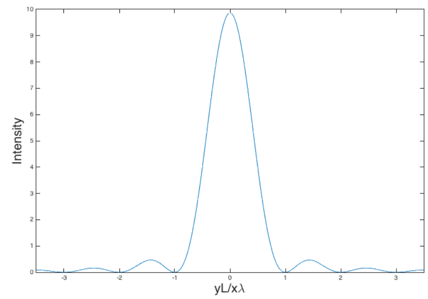
The variation in amplitude due to the factor $\frac{1}{\sqrt{r_{\tilde{y}}}}$ is not important, we can take it simply to be a uniform factor $\frac{1}{\sqrt{r}}$. The difference in phase due to $k r_{\tilde{y}}$, on the other hand, is very important and cannot be ignored.

With these simplifications, the integral reduces to

$$\begin{aligned} \psi(x, y, t) &= \int_{-\frac{L}{2}}^{\frac{L}{2}} A d\tilde{y} \tfrac{1}{\sqrt{r}} e^{i(\omega t - kr)} e^{ik\tilde{y}y/x} \\ &= \tfrac{A}{\sqrt{r}(iky/x)} e^{i(\omega t - kr)} e^{ik\tilde{y}yx} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} \\ &= \tfrac{2Ax}{\sqrt{r}k} e^{i(\omega t - kr)} \tfrac{\sin\left(\frac{kL}{2x} y\right)}{y} \end{aligned} \tag{6}$$

The time averaged intensity is then proportional to

$$I \propto \frac{\sin^2\left(\frac{kL}{2x}y\right)}{y^2} \tag{7}$$



From this formula (and the plot), we see that $y = 0$ is an intensity maximum, $I_{max} \propto \frac{k^2L^2}{4x^2}$. The intensity becomes zero at regularly spaced points

$$\frac{kL}{2x}y = n\pi,\ (n \neq 0) \Rightarrow\ y = \frac{2n\pi x}{kL} = n\lambda\frac{x}{L},\ (n \neq 0) \tag{8}$$

This is the so-called diffraction pattern of a single slit. The central peak is twice as wide as that of the side peaks. At large y, the envelope of the diffraction patterns decays very quickly as $\frac{1}{y^2}$. As $L \to 0$, the central peak becomes infinitely wide and we recover the point source limit.

In this analysis of single slit diffraction pattern, we have used the Huygens principle, which states that every point on a wave front (the slit) is itself the source of cylindrical / spherical waves. The total wave is then the superposition of all the component waves. We are going to use this principle to analyze more complicated situations in the following.