Physics 12a Waves **Lecture 14** **Caltech, 11/14/19**

6 Reflection and Transmission

With these concrete examples in mind, we can move on to discuss some of their generic features, like transmission and reflection.

6.1 Impedance

We need to introduce an important notion before we discuss transmission and reflection: impedance. Impedance describes how big a wave can be generated with certain driving force. Roughly speaking, applying the same driving force to a system, a bigger wave is generated if impedance is small, a smaller wave is generated if impedance is large.

The physical situation we are going to consider is that of a half infinite system with one end being periodically driven while the other end is fixed.

Consider first the transversely vibrating string. If we shake one end of the string and send down a

traveling wave, the wave equation looks like

$$\psi(x,t) = A\cos(\omega t - kx + \varphi) \tag{1}$$

What is the force that is applied to generate the wave? If we suppose that the horizontal tension in the string is T, then the vertical part of the force (which is responsible for generating the wave) is

$$F = -T\frac{\partial \psi}{\partial x}\Big|_{x=0} \tag{2}$$

The velocity of the end point is

$$u = \frac{\partial \psi}{\partial t}\Big|_{x=0} \tag{3}$$

In a traveling wave, these two are proportional to each other and their proportionality constant is called the impedance

$$z = \frac{F}{u} = -T\frac{-k}{\omega} = \frac{T}{v} = \sqrt{T\rho} \tag{4}$$

which depends only on some intrinsic parameters of the string.

If we calculate the power input into the wave, it is

$$P = F \cdot u = zu^2 \tag{5}$$

which averages to a nonzero value over many cycles. Where does all the energy go? There will be energy output from the other end of the string. Also if there is friction, the energy may be dissipated along the way.

As a second example, let's consider the sound wave in a semi-infinite tube which is generated by pushing a piston at one end of the tube. If the piston moves as $\psi(0,t) = \psi_0 \cos(\omega t)$, then the wave traveling to the right takes the form

$$\psi(x,t) = \psi_0 \cos(\omega t - kx) \tag{6}$$

The velocity of the piston is

$$u = \frac{\partial \psi}{\partial t}\bigg|_{x=0} = -\psi_0 \omega \sin(\omega t) \tag{7}$$

The force applied to the piston is equal to

$$F = dP \cdot A = -\gamma P_0 A \frac{\partial \psi}{\partial x} = -\gamma k P_0 A \psi_0 \sin(\omega t) \tag{8}$$

where we have used P to denote pressure and P_0 is the equilibrium pressure.

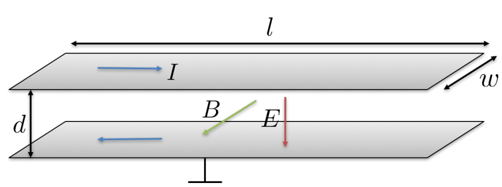
The impedance is then

$$z = \frac{F}{u} = \frac{\gamma k Pr_0 A}{\omega} = \frac{\gamma Pr_0 A}{v} = A\sqrt{\gamma Pr_0 \rho} \tag{9}$$

The power input is similarly given by

$$P = F \cdot u = zu^2 \tag{10}$$

As the third example, let's consider the parallel plate transmission line. There is no mechanical force here, but we can similarly define impedance as the ratio between the voltage at one end and the current at the same end.



In a traveling wave that moves to the right, the current oscillates as

$$I(x,t) = I_0 \cos(\omega t - kx) \tag{11}$$

The voltage is related to the charge density on the plate as

$$U = E \cdot d = \frac{\sigma d}{\epsilon_0} \tag{12}$$

The charge density is related to the current as

$$\frac{\partial \sigma}{\partial t} \cdot w = -\frac{\partial I}{\partial x} \tag{13}$$

In a traveling wave that moves to the right, the current oscillates as

$$I(x,t) = I_0 \cos(\omega t - kx) \tag{14}$$

Therefore, the charge density changes as

$$\sigma = \frac{I_0 k}{\omega w} \cos(\omega t - kx) \tag{15}$$

The voltage changes as

$$U = \frac{I_0 k d}{\omega w \epsilon_0} \cos(\omega t - kx) \tag{16}$$

The impedance is hence given by

$$z = \frac{U}{I} = \frac{kd}{w \omega \epsilon_0} = \frac{1}{C_0 v} = \sqrt{\frac{L_0}{C_0}} \tag{17}$$

The input power is

$$P = U \cdot I = z I^2 \tag{18}$$

The notion of impedance is going to play an important role in the discussion of transmission and reflection.

6.2 Transmission and reflection

Imagine that we hold one end of a string, shake it up and down as $\psi(0,t) = Ae^{i\omega t}$ and try to send a wave down the string. Depending on the boundary condition on the other end, we are going to get different results. If the string extends to infinity, then the wave can propagate forever

$$\psi(x,t) = Ae^{i(\omega t - kx)} \tag{19}$$

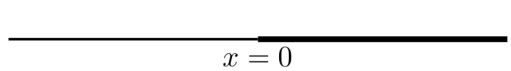
where k is determined from ω according to the dispersion relation of the string.

If the string is tied to a fixed point at $x = L$, then the wave cannot propagate beyond this point. Instead, the wave gets completely reflected so that the total wave is a superposition of the original wave and the reflected one

$$\psi(x,t) = Ae^{i(\omega t - kx)} - Ae^{i(\omega t + kx - 2kL)} \tag{20}$$

The phase shift of the second part makes sure that the fixed boundary condition is satisfied at $x = L$.

6.2.1 Example 1



$x = 0$

Now consider a more complicated situation where at $x = 0$, the first string is tied to a second string with a different density. The second string then extends to infinity. When the incoming waves arrives at $x = 0$, how much of it gets transmitted will depend on the density of the strings. If the two strings have the same density, the whole traveling wave will be transmitted with no reflection. If the second string has infinite density, the whole traveling wave will be reflected with no transmission. If the second string has different but finite density from the first one, there is partial transmission and partial reflection.

Suppose that the incoming wave (at $x < 0$) is given by

$$\psi_i(x,t) = A_i e^{i(\omega t - k_L x)} \tag{21}$$

where $k_L = \omega / v_L$. The reflected wave (at $x < 0$) is given by

$$\psi_r(x,t) = A_r e^{i(\omega t + k_L x)} \tag{22}$$

The transmitted wave (at $x > 0$) is given by

$$\psi_t(x,t) = A_t e^{i(\omega t - k_R x)} \tag{23}$$

where $k_R = \omega / v_R$. The total wave at $x < 0$ is then given by

$$\psi_L(x,t) = A_i e^{i(\omega t - k_L x)} + A_r e^{i(\omega t + k_L x)} \tag{24}$$

and the total wave at $x > 0$ is given by

$$\psi_R(x,t) = A_t e^{i(\omega t - k_R x)} \tag{25}$$

To figure out how much is transmitted and how much is reflected, we need to look at the boundary condition at $x = 0$. The two strings are connected in series, therefore they have the same displacement at $x = 0$.

$$\psi_L(0,t) = A_i e^{i\omega t} + A_r e^{i\omega t} = A_t e^{i\omega t} = \psi_R(0,t) \tag{26}$$

from which we find

$$A_i + A_r = A_t \tag{27}$$

On the other hand, the force at $x = 0$ coming from left and right should balance each other. As the tension in the two strings are the same, the vertical part of the force will balance each other if

$$\frac{\partial \psi_L}{\partial x}(0,t) = \frac{\partial \psi_R}{\partial x}(0,t) \tag{28}$$

from which we find

$$k_L(A_i - A_r) = k_R A_t \tag{29}$$

Combining these equations we find

$$\begin{cases} \frac{A_r}{A_i} = \frac{k_L - k_R}{k_L + k_R} \\ \frac{A_t}{A_i} = \frac{2k_L}{k_L + k_R} \end{cases} \tag{30}$$

Let's see what this result is saying.

a. If $\rho_L = \rho_R$, then $v_L = v_R$ and $k_L = k_R$. In this case, $A_r = 0$, $A_t = A_i$. So the input wave is completely transmitted as we expected.

b. If $\rho_L < \rho_R$, then $v_L = \sqrt{\frac{T}{\rho_L}} > v_R = \sqrt{\frac{T}{\rho_R}}$, $k_L < k_R$. This means $\frac{A_r}{A_i} < 0$, while $0 < \frac{A_t}{A_i} < 1$. The reflected wave has a π phase shift with the incoming one and therefore cancels out part of the incoming wave. The transmitted wave is smaller than the incoming one.

c. If $\rho_R = \infty$, then $v_R = 0$, $k_R = \infty$. In this case, $A_r = -A_i$, $A_t = 0$. The input wave is completely reflected (with a π phase shift) as we expected. When the incoming wave is superposed with the reflected wave, $x = 0$ becomes a node of the standing wave.

d. If $\rho_L > \rho_R$, then $k_L > k_R$. The reflected wave is in phase with the input wave and $\frac{A_t}{A_i} > 1$.

e. If $\rho_R = 0$, $k_R = 0$, $A_r = A_i$, $A_t = 2A_i$. The incoming wave and reflected wave has no phase shift at $x = 0$. They superpose into a standing wave with $x = 0$ being an anti-node (point with maximum amplitude in a standing wave).

In fact, because $k_L = \frac{\omega}{v_L} = \frac{\omega z_L}{T}$, $k_R = \frac{\omega}{v_R} = \frac{\omega z_R}{T}$, we have

$$\begin{cases} R = \frac{A_r}{A_i} = \frac{z_L - z_R}{z_L + z_R} \\ T = \frac{A_t}{A_i} = \frac{2z_L}{z_L + z_R} \end{cases} \tag{31}$$

When $z_L = z_R$, there is no reflected wave. The left and right hand side of the system are said to have impedance matching.