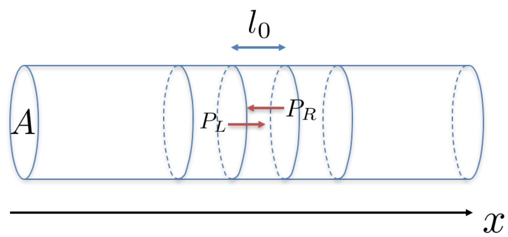
5 Various types of waves

5.2 Sound wave in air

Consider the air in a long tube. At equilibrium, the air particles is distributed uniformly, resulting in a uniform density and a uniform pressure. If we push a piston at one end of the tube, it is going to generate disturbance both in the density and in the pressure.



Let's divide the tube into small segments of length l_0. Suppose that at equilibrium each segment is centered at x and as sound wave propagates through the tube the center location changes as $\psi(x,t)$. The mass of air in each segment is $m_0 = \rho A l_0$, where A is the area of the cross section of the tube and ρ is equilibrium air density. The air in each segment is being pushed by the segment on its left and that on its right. The force it experiences is

$$F_L - F_R = (P_L - P_R)A \tag{1}$$

If we assume that the vibration in the air is happening so fast that there is no time for heat flow, the air is undergoing a so-called 'adiabatic' process where

$$PV^\gamma = \text{const.} \tag{2}$$

P is the pressure and V is the volume. Taking differentials on the two sides, we get

$$dPV^\gamma = -P\gamma V^{\gamma-1}dV \tag{3}$$

from which we find

$$dP = -\gamma P\frac{dV}{V} = -\gamma P\frac{dl}{l_0} \tag{4}$$

so that

$$dP_L = -\frac{\gamma P}{l_0}l_0\frac{d\psi(x,t)}{dx}\Big|_{x-l_0/2} = -\gamma P\frac{d\psi(x,t)}{dx}\Big|_{x-l_0/2}, dP_R = -\gamma P\frac{d\psi(x,t)}{dx}\Big|_{x+l_0/2} \tag{5}$$

The total force acting on the middle segment is

$$F = A(dP_L - dP_R) = A\gamma P l_0\frac{d^2\psi(x,t)}{dx^2} \tag{6}$$

Newton's law gives as

$$m_0\frac{d^2\psi(x,t)}{dt^2} = A\gamma P l_0\frac{d^2\psi(x,t)}{dx^2} \tag{7}$$

from which we get the wave equation

$$\frac{d^2\psi(x,t)}{dt^2} = \frac{\gamma P}{\rho}\frac{d^2\psi(x,t)}{dx^2} \tag{8}$$

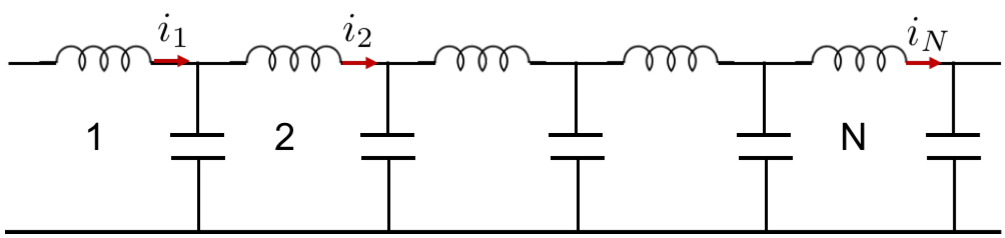
The wave velocity is hence given by $v = \sqrt{\frac{\gamma P}{\rho}}$. If we put in real numbers for air where $\gamma = 1.4$, $P = 1.01 \times 10^5$Pa, $\rho = 1.23$kg/m^3, we get $v = 339$m/s.

Now imagine blowing air into a pipe with a closed end and trying to make a sound. What is the pitch of the sound that can be generated? The closed end correspond to fixed boundary condition while the open end correspond to free boundary condition. The possible wave length that can be supported in the tube is

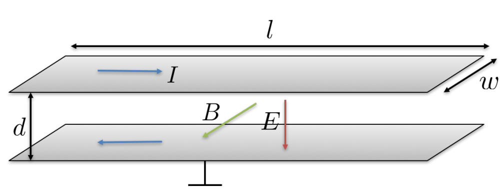
$$\lambda^{(n)} = \frac{4L}{2n+1} \tag{9}$$

where L is the total length of the tube, n are integers. Correspondingly $k^{(n)} = \frac{\pi(2n+1)}{2L}$ and $\omega^{(n)} = vk^{(n)} = \frac{v\pi(2n+1)}{2L}$. The lowest frequency of the sound wave that can be produced is hence inversely proportional to L.

5.3 Light wave in transmission lines



We studied in homework the oscillation in LC circuit networks. This is a discrete toy model for how light wave propagates. In the continuum limit, we can consider a pair of parallel plates which effectively replaces the top and bottom half of the LC circuit network.



If we take the continuum limit of the equation of motion we derived for the LC circuit network,

$$\ddot{I}_n + \frac{2I_n - I_{n+1} - I_{n-1}}{LC} = 0. \tag{10}$$

we find

$$\frac{\partial^2}{\partial t^2}I(x,t) = \frac{a^2}{LC}\frac{\partial^2}{\partial x^2}I(x,t) \tag{11}$$

so that the wave velocity is $v = \frac{a}{\sqrt{LC}}$

What does $\frac{a^2}{LC}$ correspond to in the continuum limit? For inductors connected in series, $\frac{L}{a}$ is inductance per unit length; for capacitors connected in parallel, $\frac{C}{a}$. Therefore,

$$\frac{a^2}{LC} = \frac{1}{L_0 C_0} \tag{12}$$

where L_0 and C_0 are inductance and capacitance per unit length.

Using what we learned in Electromagnetism, we can find L_0 and C_0 explicitly. Suppose that the total charge on one of the plate is Q. The total capacitance is given by

$$C_{tot} = \frac{Q}{V} = \frac{Q}{Ed} \tag{13}$$

where E is the electric field between the plates and d is the separation between the plates.

According to Gauss's law, the electric field between parallel plates with charge density σ is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{lw\epsilon_0} \tag{14}$$

Therefore,

$$C_0 = \frac{C_{tot}}{l} = \frac{Qlw\epsilon_0}{dlQ} = \frac{w\epsilon_0}{d} \tag{15}$$

On the other hand, the total inductance is given by

$$L_{tot} = \frac{\Phi}{I} = \frac{Bld}{I} \tag{16}$$

According to Ampere's law

$$B = \mu_0 \frac{I}{w} \tag{17}$$

Therefore,

$$L_0 = \frac{L_{tot}}{l} = \mu_0 \frac{Ild}{wIl} = \frac{\mu_0 d}{w} \tag{18}$$

Putting everything together we have

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{19}$$

where ϵ_0 is the permittivity of free space and μ_0 is the permeability of free space. If we put in the numbers, we find that

$$v = c = 3 \times 10^8 \mathrm{m/s} \tag{20}$$

which is the speed of light!

This is actually a rather surprising result if you think about it because it is saying that the propagation of electromagnetic wave between the parallel plates does not have much to do with the plates themselves – what their composition is, what's their electromagnetic properties, etc. Instead, it only depends on the medium in between. In fact, electromagnetic wave, or light wave, can still propagate if we get rid of the parallel plates and have only vacuum!