4 Oscillation in an Infinite System

4.7 The continuum limit

In the previous discussion, we have taken the infinite limit by keeping the lattice constant a a constant, taking system size N to infinity and correspondingly taking the total system length L to infinity. In a real physical medium that supports wave, usually the distance between the mass elements are so small that we can take the limit $a \to 0$, $N \to \infty$ while keeping $Na = L$ fixed.

In this way of thinking about waves, the spatial coordinate x also becomes continuous. Of course, if we have resolution down to the atomic scale, things are actually discrete. However, as long as we are interested in waves with large wavelength and small frequency, we can safely assume that $a \to 0$.

How to describe waves in the continuum limit?

In the discrete case, the equation of motion for a particular x is given by

$$m\frac{\partial^2}{\partial t^2}\psi(x,t) = -2k_h\psi(x,t)+k_h\psi(x-a,t)+k_h\psi(x+a,t) = k_h a\left[\frac{\psi(x+a,t)-\psi(x,t)}{a} - \frac{\psi(x,t)-\psi(x-a,t)}{a}\right] \tag{1}$$

In the limit of $a \to 0$, we can replace discrete subtraction with differentiation. That is, $\frac{\psi(x+a,t)-\psi(x,t)}{a}$ can be replaced by $\frac{\partial}{\partial x}\big|_x$, $\frac{\psi(x,t)-\psi(x-a,t)}{a}$ can be replaced by $\frac{\partial}{\partial x}\big|_{x-a}$. The right hand side then becomes a second order derivative

$$k_h a\left[\frac{\psi(x+a,t)-\psi(x,t)}{a} - \frac{\psi(x,t)-\psi(x-a,t)}{a}\right] \to k_h a^2\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{2}$$

The EOM relates the second order derivative in time to the second order derivative in space

$$m\frac{\partial^2}{\partial t^2}\psi = k_h a^2\frac{\partial^2}{\partial x^2}\psi \tag{3}$$

m and $k_h a^2$ are not good parameters for describing a continuous system. Instead we define

$$\rho = \frac{m}{a},\ \ \mu = k_h a \tag{4}$$

which represents respectively mass per unit length and spring constant per unit length (why?).

The EOM can then be written as

$$\rho\frac{\partial^2}{\partial t^2}\psi = \mu\frac{\partial^2}{\partial x^2}\psi \tag{5}$$

The eigenmodes are then given by

$$\psi(x,t) = |\alpha|\cos(\omega t - kx + \varphi) \tag{6}$$

where $-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$. As $a \to 0$, the range of k is basically infinite. The relation between ω and k can be found from the EOM to be

$$\rho\omega^2 = \mu k^2 \tag{7}$$

so that

$$\omega = \sqrt{\frac{\mu}{\rho}}|k| \tag{8}$$

This is exactly the linear dispersion relation we found previously in the small k limit. In the continuum formulation, the whole dispersion relation becomes linear, which implies that the continuum formulation only describes waves with wave length much longer than the atomic scale. For vibration on the atomic scale, this continuum description breaks down.

The velocity of the each mode is

$$v = \frac{\omega}{|k|} = \sqrt{\frac{\mu}{\rho}} \tag{9}$$

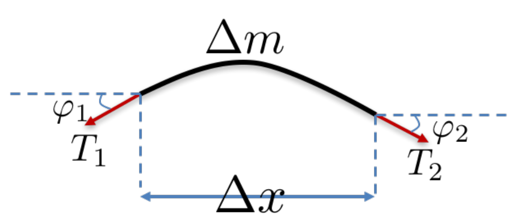
which is the same for all modes (in the long wavelength limit).

5 Various types of waves

Not every system can be described as mass blocks coupled by springs. For different types of waves, we need to look into the physical mechanism generating the wave to establish the EOM.

5.1 Transverse wave on a string

Consider the vibrational motion of a string in the transverse direction to its length. To establish the EOM, we break the string down to small segments as shown in the figure.



The small segment is being pulled by the segments on its two sides in the direction tangent to the string. The horizontal part of the two forces cancel each other

$$T_1 \cos(\varphi_1) = T_2 \cos(\varphi_2) = T \tag{10}$$

The vertical part adds up

$$-T_1 \sin(\varphi_1) - T_2 \sin\varphi_2 = -T(\tan(\varphi_1) + \tan(\varphi_2)) \tag{11}$$

On the other hand,

$$\tan(\varphi_1) = \frac{\partial\psi}{\partial x}\Big|_x, \ \tan(\varphi_2) = -\frac{\partial\psi}{\partial x}\Big|_{x+\Delta x} \tag{12}$$

Therefore we get

$$-T(\tan(\varphi_1) + \tan(\varphi_2)) = -T(-\frac{\partial^2\psi}{\partial x^2})\Delta x \tag{13}$$

And according to Newton's law

$$-T(-\frac{\partial^2\psi}{\partial x^2})\Delta x = \Delta m\frac{\partial^2\psi}{\partial t^2} \tag{14}$$

So we find the EOM to be

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2\psi}{\partial x^2} \tag{15}$$

where ρ is the mass density of the string.

This EOM takes exactly the same form as that in the mass coupled to spring example if we replace $\frac{\mu}{\rho}$ by $\frac{T}{\rho}$. The traveling wave modes are given by

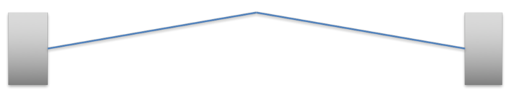
$$\psi(x,t) = |\alpha|\cos(\omega t - kx + \varphi) \tag{16}$$

and the dispersion relation is given by

$$\omega = \sqrt{\frac{T}{\rho}}|k| \tag{17}$$

The velocity of the wave in a string is hence inversely proportional to $\rho^{1/2}$.

Let's consider the problem of solving the motion of a plucked string, with initial configuration as shown in the figure. We want to find the motion of the string once it is let go.



With the fixed boundary condition of the system, we know that the possible modes are standing waves with wavelength $\lambda^{(n)} = \frac{2L}{n}$, where L is the total length of the string and n is integer. In each mode, the standing wave is described by

$$\psi^{(n)}(x,t) = |\alpha^{(n)}|\sin(k^{(n)}x)\cos(\omega^{(n)}t + \varphi^{(n)}) \tag{18}$$

where $k^{(n)} = \frac{2\pi}{\lambda^{(n)}} = \frac{n\pi}{L}$, $\omega^{(n)} = vk^{(n)}$ while $|\alpha^{(n)}|$ and $\varphi^{(n)}$ are free parameters. The general form of motion satisfying this boundary condition is then

$$\psi(x,t) = \sum_n |\alpha^{(n)}|\sin(k^{(n)}x)\cos(\omega^{(n)}t + \varphi^{(n)}) \tag{19}$$

The free parameters $|\alpha^{(n)}|$ and $\varphi^{(n)}$ can be determined from a general form of initial condition $\psi(x,0) = h(x)$, $\frac{\partial}{\partial t}\psi(x,0) = v(x)$ as

$$\begin{aligned} h(x) &= \textstyle\sum_n |\alpha_n|\sin(k_n x)\cos(\varphi_n) \\ v(x) &= \textstyle\sum_n -\omega_n|\alpha_n|\sin(k_n x)\sin(\varphi_n) \end{aligned} \tag{20}$$

From this set of equation we find

$$\begin{aligned} |\alpha_n|\cos(\varphi_n) &= \tfrac{2}{L}\textstyle\int dx h(x)\sin(k_n x) \\ -\omega_n|\alpha_n|\sin(\varphi_n) &= \tfrac{2}{L}\textstyle\int dx v(x)\sin(k_n x) \end{aligned} \tag{21}$$

5.2 Fourier Analysis

In the last step of the derivation, we have used the Fourier transform.

Let's recall the Fourier transform for real periodic functions:

Any reasonably continuous periodic function with period L can be expressed as an infinite sum of sines and cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right) \tag{22}$$

As the sin and cos functions are orthonormal in the following way

$$\begin{aligned} &\int_0^L dx \sin\left(\tfrac{2\pi n x}{L}\right) \sin\left(\tfrac{2\pi n' x}{L}\right) = \tfrac{L}{2}\delta_{nn'} \\ &\int_0^L dx \cos\left(\tfrac{2\pi n x}{L}\right) \cos\left(\tfrac{2\pi n' x}{L}\right) = \tfrac{L}{2}\delta_{nn'} \\ &\int_0^L dx \sin\left(\tfrac{2\pi n x}{L}\right) \cos\left(\tfrac{2\pi n' x}{L}\right) = 0 \end{aligned} \tag{23}$$

The coefficients a_n and b_n can be found from

$$\begin{aligned} a_n &= \tfrac{2}{L}\int_0^L dx f(x) \cos\left(\tfrac{2\pi n x}{L}\right) \\ b_n &= \tfrac{2}{L}\int_0^L dx f(x) \sin\left(\tfrac{2\pi n x}{L}\right) \end{aligned} \tag{24}$$

If $f(x)$ is a complex function, a very similar expansion hold

$$f(x) = \sum_{n=-N}^{N} C_n e^{ik_n x}, \quad k_n = \frac{2\pi n}{L} \tag{25}$$

If the function $f(x)$ is not necessarily periodic, the discrete sum becomes a continuous integral

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} dk \\ \tilde{f}(k) &= \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) e^{-kx} dx \end{aligned} \tag{26}$$

These two formulas are called the Fourier transform and the inverse Fourier transform respectively.

Some useful Fourier transform identifies

$$\begin{array}{ll} f(x) = 1 & \tilde{f}(k) = \delta(k) \\ f(x) = \delta(x) & \tilde{f}(k) = \frac{1}{2\pi} \\ f(x) = e^{iax} & \tilde{f}(k) = \delta(k-a) \\ f(x) = \cos(ax) & \tilde{f}(k) = \frac{1}{2}\left(\delta(k-a) + \delta(k+a)\right) \\ f(x) = \sin(ax) & \tilde{f}(k) = -\frac{i}{2}\left(\delta(k-a) - \delta(k+a)\right) \end{array} \tag{27}$$