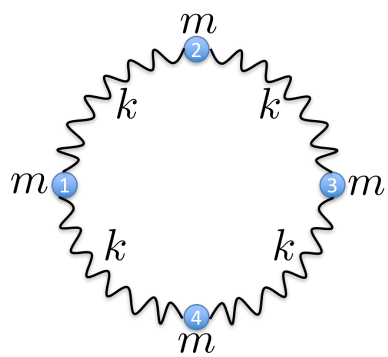
4 Oscillation in an Infinite System

4.1 Translation invariant, Locally interacting system

Recall our discussion from last lecture about four particles coupled along a 1D chain in a translation invariant way.



The problem is reduced to an eigenvalue equation

$$\frac{1}{m}KA = \omega^2 A \tag{1}$$

where $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$, $K = \begin{pmatrix} 2k & -k & 0 & -k \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ -k & 0 & -k & 2k \end{pmatrix}$.

We found the four eigenmodes to be

$$\begin{aligned} \omega^{(1)} = 0, \qquad A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \omega^{(2)} = \sqrt{\tfrac{2k}{m}}, \quad A^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\ \omega^{(3)} = \sqrt{\tfrac{2k}{m}}, \quad A^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \omega^{(4)} = \sqrt{\tfrac{4k}{m}}, \quad A^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \end{aligned} \tag{2}$$

The first mode is not an oscillating mode. The motion is described by

$$X^{(1)}(t) = Re(Z^{(1)}(t)) = |a| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{3}$$

Actually we missed a solution, as we have discussed about the $\omega = 0$ case before. The full solution in this case is

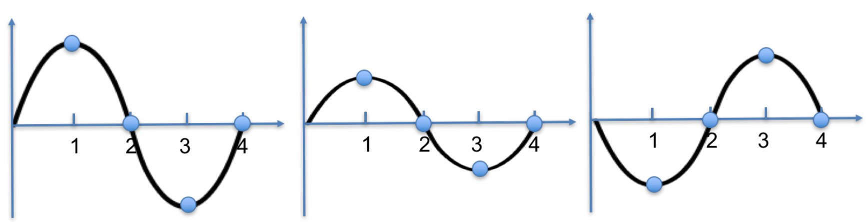
$$X^{(1)}(t) = (|a| + |b|t)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \tag{4}$$

which describes the same motion for all four particles around the circle at constant velocity. This is not an oscillatory motion, but rather the center of mass motion.

In the second mode, the first and third particle have opposite displacement while the second the fourth particle do not move. The motion in this mode is then described as

$$X(t) = Re(Z(t)) = |\alpha|\cos(\omega t + \varphi)\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \tag{5}$$

If we plot the instantaneous shots of this motion at different time points, it would look like



The third mode is very similar. The second and fourth particle having opposite displacement while the first and third particle do not move. These two modes have the same frequency and are said to be degenerate.

In the fourth mode, all particles move, with the amplitude change between 1 and -1 from particle to particle. This mode has the highest frequency.

Now let's solve the problem in a different way. Instead of directly solving this eigen equation, we can make use of the symmetry conditions of the system. The system is translation invariant where translation symmetry is implemented as

$$S = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix} \tag{6}$$

because

$$SX = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_2\\x_3\\x_4\\x_1\end{pmatrix} \tag{7}$$

The EOM is translation invariant because

$$SK = KS \tag{8}$$

Because S and K commute, we can find the eigen vectors of $\frac{1}{m}K$ from the eigen vectors of S. That is, we look for A's that satisfy

$$SA = \beta A \tag{9}$$

where β is a number. We find four sets of solutions

$$\beta = 1, i, -i, -1 \tag{10}$$

and the eigen vectors

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \tag{11}$$

Let's look at the four eigenmodes separately

(1) $\beta^{(1)} = 1,\ A^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Plugging this into the eigen equation for $\frac{1}{m}K$, we find $\left(\omega^{(1)}\right)^2 = 0$.

This is the center of mass motion with

$$X^{(1)}(t) = (|a| + |b|t) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{12}$$

This mode is invariant under translation

$$SX^{(1)}(t) = X^{(1)}(t) \tag{13}$$

(2') $\beta^{(2)} = i,\ A^{(2)} = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$

The corresponding eigenvalue for $\frac{1}{m}K$ is $(\omega^{(2)})^2 = \frac{2k}{m}$.

The motion is described by

$$Z^{(2)}(t) = \alpha \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} e^{i\omega t} \tag{14}$$

so that

$$\begin{aligned} x_1^{(2)}(t) &= |\alpha| \cos(\omega t + \varphi) \\ x_2^{(2)}(t) &= |\alpha| \cos(\omega t + \varphi + \tfrac{\pi}{2}) \\ x_3^{(2)}(t) &= |\alpha| \cos(\omega t + \varphi + \pi) \\ x_4^{(2)}(t) &= |\alpha| \cos(\omega t + \varphi + \tfrac{3\pi}{2}) \end{aligned} \tag{15}$$

where φ is the phase of α. The four particles are all oscillating, with the same amplitude, the same frequency, but with a phase shift of $\frac{\pi}{2}$ from one to the next.

Because of the phase shift, the motion is not completely invariant under translation, but it transforms in a nice way. In particular,

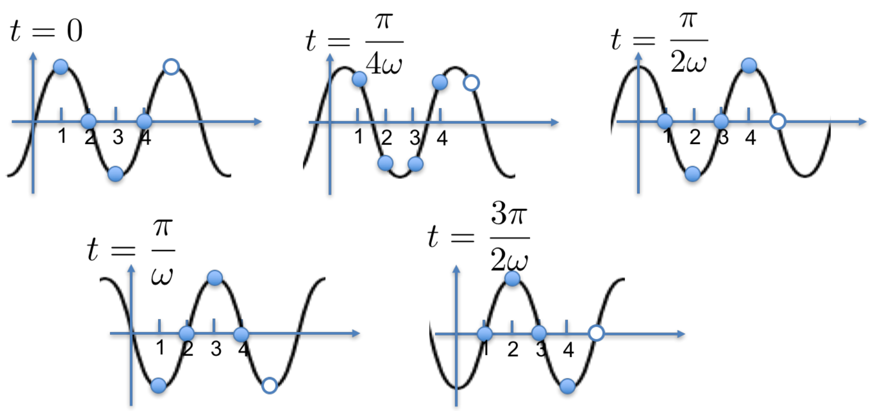
$$SA^{(2)} = iA^{(2)} \tag{16}$$

and correspondingly

$$SX^{(2)}(t) = X^{(2)}(t - \frac{\pi}{2\omega}) \tag{17}$$

That is, spatial translation is related to time translation in a straight forward way.

Let's take some instantaneous shots of the motion in solution (2). At $t = 0$, $\frac{\pi}{4\omega}$, $\frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$, the displacement of the four particles looks like



We have connected the four dots with a sinusoidal curve so that it becomes clear that from one frame to the next, the sinusoidal curve is moving to the left. That is, the wave form in the system is 'traveling'. It is moving with a velocity

$$v = \frac{a}{\pi/(2\omega)} = \frac{2a\omega}{\pi} \tag{18}$$

Therefore, this solution describes a Traveling Wave. This can also be seen from expression of the solution

$$x_j(t) = |\alpha| \cos\left[\omega t + \varphi + \frac{\pi}{2}(j-1)\right] = |\alpha| \cos\left[\frac{\pi}{2a}(a(j-1) + vt) + \varphi\right] \tag{19}$$

where spatial coordinate and time coordinate are combined in a way as $a(j-1)+vt$ and the solution is a function of this combination only. This is the hallmark of having a traveling wave.

(3') $\beta^{(3)} = -i$, $A^{(3)} = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$

The corresponding eigenfrequency of this mode is the same as the previous case $(\omega^{(3)})^2 = \frac{2k}{m}$.

The motion is described by

$$Z^{(3)}(t) = \alpha \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} e^{i\omega t} \tag{20}$$

so that

$$\begin{aligned} x_1^{(3)}(t) &= |\alpha| \cos(\omega t + \varphi) \\ x_2^{(3)}(t) &= |\alpha| \cos(\omega t + \varphi - \tfrac{\pi}{2}) \\ x_3^{(3)}(t) &= |\alpha| \cos(\omega t + \varphi - \pi) \\ x_4^{(3)}(t) &= |\alpha| \cos(\omega t + \varphi - \tfrac{3\pi}{2}) \end{aligned} \tag{21}$$

where φ is the phase of α. The four particles are all oscillating, with the same amplitude, the same frequency, but with a phase shift of $-\frac{\pi}{2}$ from one to the next.

Again, the solution transforms under translation symmetry in a nice way. In particular,

$$SA^{(3)} = -iA^{(3)} \tag{22}$$

and correspondingly

$$SX^{(3)}(t) = X^{(3)}(t + \frac{\pi}{2\omega}) \tag{23}$$

where again spatial translation is related to time translation in a straight forward way.

(4) $\beta^{(4)} = -1$, $A^{(4)} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

The eigen-frequency of this mode is $(\omega^{(4)})^2 = \frac{4k}{m}$.

The motion is described by

$$Z^{(4)}(t) = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} e^{i\omega t} \tag{24}$$

so that

$$\begin{aligned} x_1^{(4)}(t) &= |\alpha| \cos(\omega t + \varphi) \\ x_2^{(4)}(t) &= |\alpha| \cos(\omega t + \varphi + \pi) \\ x_3^{(4)}(t) &= |\alpha| \cos(\omega t + \varphi) \\ x_4^{(4)}(t) &= |\alpha| \cos(\omega t + \varphi + \pi) \end{aligned} \tag{25}$$

where φ is the phase of α. The four particles are all oscillating, with the same amplitude, the same frequency, but with a phase shift of π from one to the next.

The solution transforms under translation symmetry as

$$SA^{(4)} = -A^{(4)} \tag{26}$$

and correspondingly

$$SX^{(4)}(t) = X^{(4)}(t + \frac{\pi}{\omega}) \tag{27}$$

Solutions (3') and (4) also describing traveling waves.

Why are solution (2') and (3') different from solution (2) and (3)? In fact, (2) and (3) are eigen modes of the same eigenvalue $(\omega)^2 = \frac{2k}{m}$, so we can take different linear superpositions of them and we still have eigenmodes. In particular, we can make superpositions of (2) and (3) into (2') and (3') as

(2')

$$A = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{28}$$

(3')

$$A = \frac{-i}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{29}$$

That is, if we make superposition of two traveling waves with the same frequency but moving in opposite directions we get a wave that does not move – a Standing Wave. In the standing wave, some particles do not move at all. They are called the nodes of the wave.