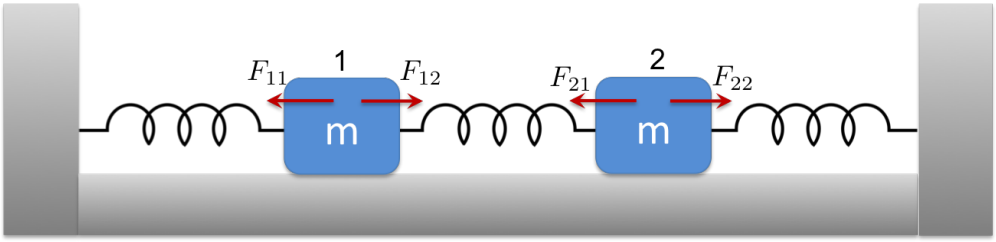
3 Coupled Harmonic Oscillators, Normal Modes

3.1 Two masses coupled by springs



In the last lecture, we solved the equation of motion for this system of two mass blocks coupled by three springs. The EOM reduced to an eigen equation of the form

$$M^{-1}KA = \omega^2 A \tag{1}$$

where

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \; M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \tag{2}$$

Solving this eigen equation, we get

$$\omega^{(1)} = \sqrt{\frac{k}{m}}, \; \omega^{(2)} = \sqrt{\frac{3k}{m}} \tag{3}$$

and correspondingly

$$A^{(1)} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; A^{(2)} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4}$$

which corresponds to two eigenmodes

$$X^{(1)}(t) = |a^{(1)}| \cos(\omega^{(1)} t + \varphi^{(1)}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; X^{(2)}(t) = |a^{(2)}| \cos(\omega^{(2)} t + \varphi^{(2)}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5}$$

The general motion of the two masses is the superposition of these two modes

$$X(t) = |a^{(1)}| \cos(\omega^{(1)} t + \varphi^{(1)}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + |a^{(2)}| \cos(\omega^{(2)} t + \varphi^{(2)}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6}$$

There are four free parameters in this solution, $|a^{(1)}|$, $\varphi(1)$, $|a^{(2)}|$ and $\varphi^{(2)}$ which can be determined from the initial conditions $x_1(0)$, $x_1'(0)$, $x_2(0)$, $x_2'(0)$.

Let's consider some special cases of initial conditions and see what kind of motion it starts.

(1) $x_1(0) = x_2(0) = x_0$, $x_1'(0) = x_2'(0) = 0$

The two blocks are displaced by the same amount and then released. In this case, only mode 1 is excited and we have

$$X(t) = x_0 \cos(\omega^{(1)} t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

(2) $x_1(0) = x_2(0) = 0$, $x_1'(0) = x_2'(0) = v_0$

The two blocks are set to move at the same initial speed, but with no initial displacement. Again, only mode 1 is excited and we have

$$X(t) = \frac{v_0}{\omega^{(1)}} \sin(\omega^{(1)} t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

(3) $x_1(0) = -x_2(0) = x_0$, $x_1'(0) = x_2'(0) = 0$

The two blocks are displaced by the opposite amount and then released. In this case, only mode 2 is excited and we have

$$X(t) = x_0 \cos(\omega^{(2)} t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9}$$

(4) $x_1(0) = x_0$, $x_2(0) = x_1'(0) = x_2'(0) = 0$

Even though only mass 1 is displaced initially, both blocks will move. The whole motion is given by

$$X(t) = \frac{1}{2} x_0 \cos(\omega^{(1)} t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} x_0 \cos(\omega^{(2)} t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10}$$

That is, both modes are excited, with the same amplitude. The motion of the two blocks is the superposition of two SHO with different frequencies.

Question: what if the middle spring has a different spring constant than the side ones?

We are going to answer this question in two ways: first by direct calculation and then using symmetry argument.

(1) Direct Calculation

Suppose that the middle spring has spring constant k'. For mass 1, the EOM is

$$m \frac{d^2 x_1}{dt^2} = -k x_1 + k'(x_2 - x_1) \tag{11}$$

For mass 2, the EOM is

$$m \frac{d^2 x_2}{dt^2} = -k x_2 - k'(x_2 - x_1) \tag{12}$$

In matrix form, this becomes

$$M^{-1} K A = \omega^2 A \tag{13}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, K = \begin{pmatrix} k + k' & -k' \\ -k' & k + k' \end{pmatrix} \tag{14}$$

Solving the eigenvalue equation for $M^{-1}K = \frac{1}{m}\begin{pmatrix} k+k' & -k' \\ -k' & k+k' \end{pmatrix}$, we get

$$\omega^{(1)} = \sqrt{\frac{k}{m}},\ \omega^{(2)} = \sqrt{\frac{k+2k'}{m}} \tag{15}$$

and correspondingly

$$A^{(1)} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ A^{(2)} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16}$$

Therefore, the eigenvectors do not change while the eigen frequencies do change.

(2) Symmetry Argument

Consider the reflection operation of $x_1 \to -x_2$, $x_2 \to -x_1$, which in matrix form is

$$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{17}$$

The system is setup in a way that is reflection symmetric. Put into physics language, this means that R commutes with $M^{-1}K$, which we can check explicitly. This is true no matter what the spring constant is for the middle spring.

Because R and $M^{-1}K$ commute, they have common eigenvectors. So instead of diagonalizing $M^{-1}K$, we can diagonalize R and find two eigenvectors

$$A^{(1)} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ A^{(2)} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{18}$$

which has eigenvalue -1 and 1 respectively under R.

Therefore, $A^{(1)}$ and $A^{(2)}$ are also eigenvectors of $M^{-1}K$, although from this symmetry argument we cannot determine the eigen frequencies of the eigen modes.

3.2 Driven dissipative coupled harmonic oscillator

What if the coupled harmonic oscillator system is also damped and driven? Well the EOM can become highly complicated. But there are also important simple physical features that we can identify out of the complicated equations. In particular, we are going to see that resonance can still happen when the driving frequency matches the eigen frequency of each mode.

Let's consider the two masses on spring example, now with friction and driving forces acting on the two masses. The EOM becomes

$$M\frac{d^2X}{dt^2} + M\Gamma\frac{dX}{dt} + KX = F \tag{19}$$

where $M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.

The complex version of the EOM is

$$\frac{d^2Z}{dt^2} + \Gamma\frac{dZ}{dt} + \frac{1}{m}KZ = \frac{1}{m}F_0 e^{i\omega_D t} \tag{20}$$

where we have assumed that F is a periodic driving force. Note that F_0 is again a two component vector.

Let's look for steady state solution of the form $Z = Ae^{i\omega_D t}$. Plugging it into the equation we get

$$-\omega_D^2 A + i\omega_D \Gamma A + \frac{1}{m} K A = \frac{F_0}{m} \tag{21}$$

This looks really complicated, but we can use our knowledge of the normal modes to decouple the motion and simplify the equation.

Take the eigenmodes $A^{(1)}$ and $A^{(2)}$ of the freely oscillating system which satisfy the condition

$$KA^{(1)} = m(\omega^{(1)})^2 A^{(1)},\ KA^{(2)} = m(\omega^{(2)})^2 A^{(2)} \tag{22}$$

As $A^{(1)}$ and $A^{(2)}$ are linearly independent, we can expand everything in terms of them.

$$A = a^{(1)}A^{(1)} + a^{(2)}A^{(2)}, F_0 = f^{(1)}A^{(1)} + f^{(2)}A^{(2)} \tag{23}$$

The EOM becomes

$$a^{(1)}(-\omega_D^2 + i\omega_D\Gamma + (\omega^{(1)})^2)A^{(1)} + a^{(2)}(-\omega_D^2 + i\omega_D\Gamma + (\omega^{(2)})^2)A^{(2)} = f^{(1)}A^{(1)} + f^{(2)}A^{(2)} \tag{24}$$

Because $A^{(1)}$, $A^{(2)}$ are linearly independent modes, the equation separates into two parts

$$a^{(1)}(-\omega_D^2 + i\omega_D\Gamma + (\omega^{(1)})^2) = f^{(1)}, a^{(2)}(-\omega_D^2 + i\omega_D\Gamma + (\omega^{(2)})^2) = f^{(2)} \tag{25}$$

which exactly matches the equation we get for the driven oscillation of a single DOF. Therefore, the full motion is the superposition of two driven oscillations, each can achieve resonance if 1. the driving frequency matches their eigen frequency and 2. if the driving force has a component along the eigen mode.

When we drive a generic system with many degrees of freedom, if we sweep the frequency, we are going to see multiple resonances. This is exactly what is being measured in, for example, molecular spectroscopy, where the location and width of the resonance peaks carry information regarding the structure of the molecule (hence how it vibrates). When a white light shines on a sample, the oscillating light drives the vibration of the molecules. When the light frequency matches the vibration eigen frequency of the molecule, the vibration goes into resonance and the light absorption reaches a peak. Therefore, after passing a white light through the sample, the absorption peak can tell us about the structure of the molecules in the sample.