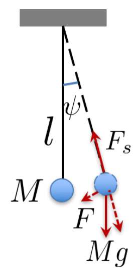
1 Simple Harmonic Oscillator

1.2 Example II: pendulum

A swinging pendulum can also be approximately described as a SHO as long as the motion is small enough. Let's examine the kinetics of the system and find the equation of motion.



The mass at the end of the pendulum experiences two forces: the gravitational force Mg and a pulling force F_s by the hanging string. As the mass has no motion along the string, the pulling force F_s is cancelled by the component of the gravitational force along the string

$$F_s = Mg\cos(\psi) \tag{1}$$

The force that drives the oscillation is the component of the gravitational force perpendicular to the string

$$Mg\sin(\psi) \approx Mg\psi \tag{2}$$

when ψ is small.

The equation of motion is then given by

$$Ma = M\frac{d^2(l\psi)}{dt^2} = Ml\frac{d^2\psi}{dt^2} = -Mg\psi \tag{3}$$

That is

$$\frac{d^2\psi}{dt^2} = -\frac{g}{l}\psi \tag{4}$$

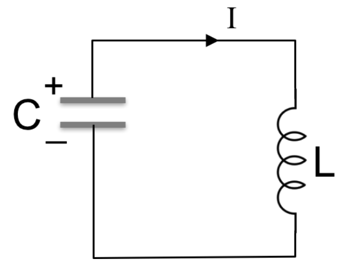
which has a solution of the form

$$\psi(t) = \psi(0)\cos(\omega t) + \frac{1}{\omega}\psi'(0)\sin(\omega t) \tag{5}$$

where $\omega = \sqrt{\frac{g}{l}}$.

1.3 Example III: LC circuit

Oscillation happens in all kinds of physical systems, not only mechanical. Consider the electrical circuit composed of a capacitor C and an inductor L. We study the oscillation of the amount of charge across the capacitor $Q(t)$.



The voltage drop across the capacitor is given by

$$V_c = \frac{Q}{C} \tag{6}$$

The current I flowing from the capacitor to the inductor is

$$I = -\frac{dQ}{dt} \tag{7}$$

And the voltage drop across the inductor is given by

$$V_L = -L\frac{dI}{dt} = L\frac{d^2Q}{dt^2} \tag{8}$$

As the voltage drop around the full circuit has to be zero, we have

$$0 = V_c + V_L = \frac{Q}{C} + L\frac{d^2Q}{dt^2} \tag{9}$$

That is,

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 \tag{10}$$

which has a solution of the form

$$Q(t) = Q(0)\cos(\omega t) + \frac{1}{\omega}Q'(0)\sin(\omega t) \tag{11}$$

where $\omega = \frac{1}{\sqrt{LC}}$.

1.4 General properties of Simple Harmonic Oscillator

Equation of motion

$$\frac{d^2X}{dt^2} = -\omega^2 X \tag{12}$$

X represents the small displacement from equilibrium position in the SHO. It can corresponds to x in the mass on a spring problem, ψ in the pendulum, or Q in the LC circuit.

This equation of motion has a generic solution

$$X(t) = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t + \varphi) \tag{13}$$

C is called the amplitude of the oscillation, ω the angular frequency, φ the phase.

The equation of motion has the following useful properties:

1.4.1 Linearity

Equations that contain terms with only one power of $x(t)$ are linear

$$\alpha x(t) + \beta \frac{dx(t)}{dt} + \gamma \frac{d^x(t)}{dt^2} + \delta \frac{d^3x(t)}{dt^3} + ... = 0 \tag{14}$$

Equations like

$$x^2(t) + \frac{dx(t)}{dt} = 0 \tag{15}$$

or

$$\sin(x(t)) + \frac{d^2x(t)}{dt^2} = 0 \tag{16}$$

are not linear.

If an equation is linear, then we can make superpositions of its solutions. That is, if $x_1(t)$, $x_2(t)$ are both solutions, then $c_1x_1(t) + c_2x_2(t)$ is also a solution.

1.4.2 Energy conservation

In all three examples above, the total energy of the system is conserved.

In the mass on spring example, the system has a kinetic energy of $E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$ and a potential energy of $E_P = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$. Plugging in the solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$, the total energy is

$$E_t = E_K + E_P = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2(A^2+B^2) \tag{17}$$

which is independent of time.

In the pendulum example, the system has a kinetic energy of $E_K = \frac{1}{2}m\left(\frac{d(l\psi)}{dt}\right)^2 = \frac{1}{2}ml^2\left(\frac{d\psi}{dt}\right)^2$ and a potential energy $E_P = mgl(1-\cos(\psi)) = \frac{1}{2}mgl\psi^2$. Plugging in the solution $\psi(t) = A\cos(\omega t) + B\sin(\omega t)$, the total energy is

$$E_t = E_K + E_P = \frac{1}{2}ml^2\left(\frac{d\psi}{dt}\right)^2 + \frac{1}{2}mgl\psi^2 = \frac{1}{2}m\omega^2l^2(A^2+B^2) \tag{18}$$

which is again independent of time.

An LC circuit is not a mechanical system but we can still define something like a kinetic energy, which is the energy stored in the inductor, $E_L = \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2$ and something like a potential energy, which is the energy stored in the capacitor, $E_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}L\omega^2Q^2$. Plugging in the solution $Q(t) = A\cos(\omega t) + B\sin(\omega t)$, the total energy is

$$E_t = E_L + E_C = \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2}L\omega^2Q^2 = \frac{1}{2}L\omega^2(A^2+B^2) \tag{19}$$

which is independent of time.

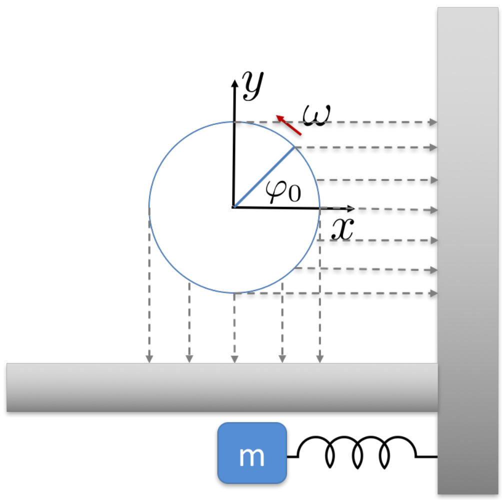
For all examples, we see that the total energy depends on system properties (m, l, C, L etc.), angular frequency (ω) and initial conditions (A, B), but not on time. Therefore, total energy is a constant of motion.

1.4.3 Relation to uniform circular motion

A SHO can be thought of as the projection of a uniform circular motion down to one dimension. Consider the circular motion around the origin described by

$$x(t) = R\cos(\omega t + \varphi_0),\ \ y(t) = R\sin(\omega t + \varphi_0) \tag{20}$$

Projecting down to the x axis we get $x(t) = R\cos(\omega t + \varphi_0)$, which describes the sinusoidal motion of a SHO.



This correspondence between circular motion and the SHO gives rise to the complex description of SHO. Instead of describing the circular motion with with a vector, we use complex numbers to describe it.

$$z(t) = x(t) + iy(t) = R\cos(\omega t + \varphi_0) + iR\sin(\omega t + \varphi_0) = Re^{i(\omega t + \varphi_0)} \tag{21}$$

The SHO can then be described as the real part of this complex trajectory

$$x(t) = Re(z(t)) = R\cos(\omega t + \varphi_0) \tag{22}$$

The equation of motion of $z(t)$ takes exactly the same form as $x(t)$

$$\frac{d^2 z}{dt^2} = -\omega^2 z \tag{23}$$

which when expanded becomes

$$\frac{d^2 x}{dt^2} + i\frac{d^2 y}{dt^2} = -\omega^2 x - i\omega^2 y \tag{24}$$

If we match the real part on both sides, we get the equation of motion for the SHO.

It seems we are making things more complicated. Why bother? There are several reasons.

a. Complex number can make calculation much simpler, especially when there is dissipation. We are going to see some examples below.

b. In classical mechanics, the physics is in the real part. But in quantum mechanics, we actually care about the full complex number, not just the real part.