Physics 12a **Assignment 7** **Caltech, 11/14/19**

Web Page: http://www.its.caltech.edu/~xcchen/courses/physics12a.html

Problems: Due in the Ph12 IN box in Bridge Annex, 5:00pm, Thursday, 11/21/19.

1.) Traveling wave with friction.

Consider waves in a medium with uniform friction. Friction is proportional to velocity – the velocity of vibration at each point. Therefore, it contributes a term $\gamma \frac{\partial \psi}{\partial t}$ to the wave equation. The total wave equation becomes

$$\frac{\partial^2 \psi}{\partial t^2} + \gamma \frac{\partial \psi}{\partial t} = v^2 \frac{\partial^2 \psi}{\partial x^2} \tag{1}$$

where v is the velocity of the traveling wave without friction.

(1) We can still assume that the (complex) traveling wave solution to this equation takes the form

$$\psi(x,t) = Ae^{i(\omega t - kx)} \tag{2}$$

while allowing both k and ω to be potentially complex. What is the relation between ω and k? Note that A is also potentially complex.

(2) Consider a system with fixed boundary condition $\psi(0,t) = \psi(L,t) = 0$. A single traveling wave mode cannot satisfy this condition. Let's make a superposition of two of them

$$\psi(x,t) = A_+ e^{i(\omega t - kx)} + A_- e^{i(\omega t + kx)} \tag{3}$$

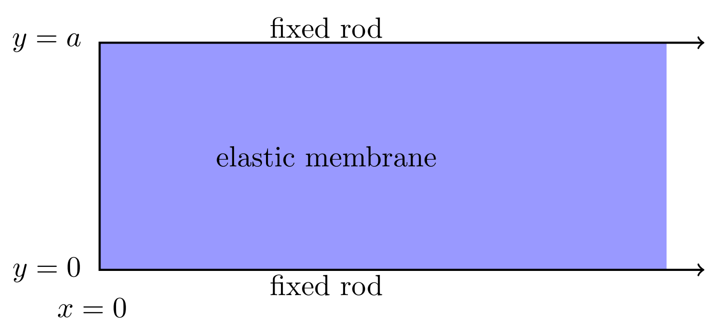
To satisfy this boundary condition, can k be complex? What are the possible values of k? What is the corresponding ω? What is the real part of ω and what is the imaginary part of ω? What does the value of ω imply about the standing wave motion? Note that based on physical principles, we can ignore the solution which corresponds to exponential growth with time.

(3) Now consider instead a driven oscillation in a semi-infinite system where the driving force is applied at the left end. The left end is driven so that it moves as

$$\psi(0,t) = B\cos(\omega t) \tag{4}$$

where ω is a real number. The driven oscillation in the whole system then has the same frequency. What is the corresponding k? What does the value of k imply about the shape of the wave as it propagates? Note that based on physical principles, we can ignore the solution which corresponds to exponential growth with space.

2.) Two-dimensional elastic membrane. A uniform semi-infinite membrane is stretched in the $z = 0$ plane, as shown in the figure below.



It is attached to fixed rods along $y = 0$, $z = 0$ and $y = a$, $z = 0$ from $x = 0$ to ∞. $\psi(x, y, t)$ is the z displacement of the point on the membrane with equilibrium position $(x, y, 0)$. For small oscillations, ψ satisfies the two-dimensional wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi.$$

a) If this system is extended to an infinite system by continuing it to negative x, show that the following equation gives normal modes of the system:

$$\psi(x, y) = A\sin(nk_0 y)e^{ikx},$$

where n is an integer. That is, show that

$$\psi(x, y, t) = A\sin(nk_0 y)e^{ikx}e^{i\omega t}$$

is a solution to the wave equation and determine the relation between ω and nk_0, k. Find the smallest k_0 that satisfies the boundary condition at $y = 0$ and $y = a$.

b) Back to the semi-infinite membrane. Suppose that the end of the membrane at $x = 0$ is driven as follows:

$$\psi(0, y, t) = [B\sin(3k_0 y) + C\sin(13k_0 y)]\cos(5vk_0 t).$$

The boundary condition at $x = \infty$ is such that there is no wave traveling in the $-x$ direction along the membrane. Find $\psi(x, y, t)$ in steady state.

c) Explain the following statement: For $\omega < 2vk_0$, the system acts like a one-dimensional wave carrier (i.e. the y direction decouples from the x and t directions in the wave equation) with the dispersion relation $\omega^2 = v^2k^2 + \omega_0^2$. What is ω_0?

3.) Reflections in transmission lines.

Consider two transmission lines with impedance z_1 and z_2. When the two lines are joined together, the boundary condition at their connection point is such that the total voltage and current on the left hand side are equal to the total voltage and current on the right hand side. Suppose that we send a traveling wave from left to right.

a) Write down the general form of the incoming wave, reflected wave and transmitted wave with their amplitude as free parameters to be determined from boundary condition. Specify the frequency and wave number for each of the waves.

b) Use the boundary conditions to find the ratio between the amplitudes of the reflected wave and the incoming wave, and between the transmitted wave and the incoming wave.

4.) Questions and Comments.