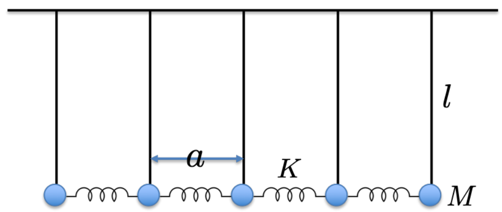
Physics 12a **Assignment 6** **Caltech, 11/7/19**

Web Page: http://www.its.caltech.edu/∼xcchen/courses/physics12a.html

Problems: Due in the Ph12 IN box in Bridge Annex, 5:00pm, Thursday, 11/14/19.

1) System of coupled pendula.



Consider a linear array of N pendulums (each of length l and bob mass M coupled in a translation invariant way. Each bob is separated in x from neighboring bobs by an equilibrium distance a so that the nth bob is located at $x = (n - 1/2)a$. The bobs are coupled to nearest neighbors on either side by springs with spring constant K.

a) Show that the equation of motion for the nth pendulum bob displacement from equilibrium $\psi_n(t)$ is given (for small oscillations) by

$$\frac{d^2\psi_n}{dt^2} = -\frac{g}{l}\psi_n + \frac{aK}{M}\left(\frac{\psi_{n+1} - \psi_n}{a}\right) - \frac{aK}{M}\left(\frac{\psi_n - \psi_{n-1}}{a}\right).$$

b) Go to the continuum limit and show that the equation of motion becomes a wave equation of the form

$$\frac{\partial^2\psi}{\partial t^2} = -\omega_0^2\psi + v_0^2\frac{\partial^2\psi}{\partial x^2}.$$

What are ω_0 and v_0?

c) Write down the traveling wave solution to this equation.

d) Show that the dispersion relation is

$$\omega^2 = \frac{g}{l} + \frac{Ka^2}{M}k^2.$$

2) Fourier Analysis.

Consider the periodic function

$$f(x) = \begin{cases} 1, & m < x \le m + 0.5 \\ -1, & m + 0.5 < x \le m + 1 \end{cases} \tag{1}$$

where m is any integer.

$f(x)$ can be decomposed into Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\right) \tag{2}$$

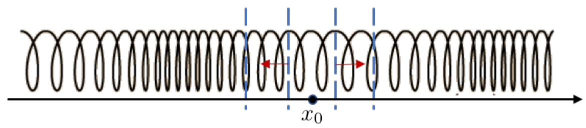
(1) Find the first few coefficients a_0, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 using the formula given in class. Use Mathematica to do the integration.

(2) Plot

$$\begin{aligned}
f_0(x) &= \tfrac{a_0}{2}, \\
f_1(x) &= \tfrac{a_0}{2} + a_1 \cos(2\pi n x) + b_1 \sin(2\pi n x), \\
f_2(x) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{2} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right), \\
f_3(x) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{3} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right), \\
f_4(x) &= \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{4} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\right)
\end{aligned} \tag{3}$$

and see how they approach the original function.

3) Longitudinal wave in a massive spring.



Consider a massive spring which at equilibrium has mass density ρ (per unit length) and Hooke's constant per unit length μ. If we push on one end of the spring, there can be longitudinal vibrations along the spring as shown in the figure. Let's try to establish the equation of motion for the spring. We use ψ to denote longitudinal displacement along the spring, x the equilibrium spatial location, and t the time.

(1) Consider a small segment of the spring of length Δx at equilibrium and with center location x_0. What is the mass of this segment?

(2) Now consider the segment immediately to the left. This segment exerts a force F_L on the original segment. At equilibrium, this segment also has length Δx. Once in motion, the total length changes. How is the total length related to $\frac{\partial \psi}{\partial x}$?

(3) What is F_L? And in a similar way, find F_R. (Ignore second order terms in $\frac{\partial \psi}{\partial x}$ if there is any.)

(4) Now use Newton's second law on the middle segment, take the continuum limit, and find the equation of motion.

(5) What is the dispersion relation of longitudinal wave in the massive spring?

4) Questions and Comments.