Physics 12a **Assignment 2** **Caltech, 10/11/19**

Problems: Due in the Ph12a Inbox in Bridge Annex, 5:00pm, Friday, 10/18/19. You can use Mathematica to solve the following problems. Make sure to simplify your answer.

1) Small oscillations in a nonlinear potential.
A particle of mass m moves on the x axis with potential energy

$$V(x) = \frac{E_0}{a^4}\left(x^4 + 4ax^3 - 8a^2x^2\right) \ .$$

(a) Find the positions at which the particle is in stable equilibrium. That is, find the equilibrium positions where small displacement away from equilibrium leads to small oscillatory motion around that point.

(b) Find the angular frequencies of small oscillations about each equilibrium position.

(c) What do you mean by small oscillations (small compared to what)? Be quantitative and give a separate answer for each point of stable equilibrium.

2) Critical damping.
Consider a damped harmonic oscillator with critical damping. The equation of motion is

$$\frac{dx^2}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

where $\Gamma = 2\omega_0$.

(a) Try a solution of the form $x(t) = Ae^{-\alpha t}$. What is the possible value of α?

(b) Notice that there is only one solution of the above form. But we have a second order differential equation, so we should have two independent solutions. We missed one by assuming that the solution looks like $x(t) = Ae^{-\alpha t}$. Now show that there is a second solution of the form $x(t) = Bte^{-\alpha t}$ with the same value of α.

(c) The general solution of a critically damped harmonic oscillator then takes the form

$$x(t) = (A + Bt)e^{-\alpha t}$$

Suppose that the oscillator has initial condition $x(t = 0) = x_0$, $v(t = 0) = 0$. Determine A and B.

(d) Recall the general solution we found in class in the case of over damping.

$$x(t) = A_+e^{-\alpha_+ t} + A_-e^{-\alpha_- t}$$

where $\alpha_\pm = \frac{1}{2}\left(\Gamma \pm \sqrt{\Gamma^2 - 4\omega^2}\right)$ and $\Gamma > 2\omega_0$. With the same initial condition of $x(t = 0) = x_0$, $v(t = 0) = 0$, determine A_+ and A_-.

(f) Show that with the same initial condition, the trajectory $(x(t))$ in the over damping case approaches that in the critical damping case when $\Gamma \to 2\omega_0$.

(e) Plot $x(t)$ with $\Gamma = 4\omega_0$, $\Gamma = 3.5\omega_0$, $\Gamma = 3\omega_0$, $\Gamma = 2.5\omega_0$, and $\Gamma = 2\omega_0$. It should be straight forward to see that the oscillator takes the shortest time to return to equilibrium when the damping is critical.

3) Resonance peak.

From our discussion in the lecture, we see that the amplitude of a driven oscillation is given by

$$|A|^2 = \frac{F_0^2}{m^2\left[(\omega_0^2 - \omega_D^2)^2 + \Gamma^2\omega_D^2\right]}$$

(a) Show that $|A|^2$ reaches its maximum when $\omega_D = \sqrt{\omega_0^2 - \Gamma^2/2}$.

(b) What is the maximum value of $|A|^2$?

(c) What is the Full Width at Half Maximum of the resonance peak (as a function of $|A|^2$ vs. ω_D)? Show that your answer approach Γ when $\Gamma << \omega_0$.

4) Questions and Comments