1 Simple Harmonic Oscillator

1.4 General properties of Simple Harmonic Oscillator

1.4.5 Superposition of two independent SHO

Suppose we have two SHOs described by

\[ x_1 = A_1 \cos(\omega_1 t + \varphi_1) = Re(A_1 e^{i\omega_1 t + \varphi_1}) = Re(z_1) \] (1)

\[ x_2 = A_2 \cos(\omega_2 t + \varphi_2) = Re(A_2 e^{i\omega_2 t \varphi_2}) = Re(z_2) \] (2)

What if the two motions are happening at the same time on the same degree of freedom? What does the total motion look like? Let’s consider a couple interesting situations.

A. \( A_1 = A_2, \ \omega_1 = \omega_2, \ \text{but} \ \varphi_1 \neq \varphi_2 \)

To do the superposition of \( x_1 \) and \( x_2 \), we add \( z_1 \) and \( z_2 \) and take the real part

\[ z_1 + z_2 = A \left( e^{i(\omega_1 t + \varphi_1)} + e^{i(\omega_2 t + \varphi_2)} \right) \]
\[ = A e^{i\omega t} e^{i(\varphi_1 + \varphi_2)/2} \left[ e^{i(\varphi_1 - \varphi_2)/2} + e^{i(\varphi_2 - \varphi_1)/2} \right] \]
\[ = A e^{i(\omega t + \bar{\varphi})} 2 \cos(\delta \varphi) = 2A \cos(\delta \varphi) e^{i\omega t + \bar{\varphi}} \] (3)

\[ x_1 + x_2 = Re(z_1 + z_2) = 2A \cos(\delta \varphi) \cos(\omega t + \bar{\varphi}) \]

Therefore, the superposed motion is still of the simple sinusoidal form, with total amplitude \( |2A \cos(\delta \varphi)| \). When \( \delta \varphi = 0 \), the total amplitude reaches its maximum of \( 2A \); the two SHOs are said to be in phase. When \( \delta \varphi = \pi \), the total amplitude reaches its minimum of 0; the two SHOs are said to be out of phase and cancel each other. When two SHOs have the same frequency, they can interfere, either constructively or destructively or somewhere in between.

Question: What happens if \( \omega_1 = \omega_2 \) but \( A_1 \neq A_2 \)?

B. If \( A_1 = A_2, \ \varphi_1 = \varphi_2, \ \omega_1 \neq \omega_2, \ \text{but} \ \omega_1 \approx \omega_2 \)

Something interesting happens in this situation. Suppose that at some time \( t \), the two oscillations are in phase \( \delta \varphi = 0 \). On a short time scale, as the two have almost the same frequency, they interfere constructively. Some time later (on the scale of \( 1/(\omega_1 - \omega_2) \)), the two oscillations fall out of phase and may even become completely out of phase with \( \delta \varphi = \pi \). For some short period of time at this later point, the two would interfere destructively. Therefore, the total oscillation will alternate between very strong (large amplitude) and very weak (small amplitude) and this is the phenomena called ‘Beat’.

Mathematically, we use again the complex representation to do the superposition

\[ z_1 + z_2 = A e^{i\omega_1 t} + A e^{i\omega_2 t} \]
\[ = A e^{i(\omega_1 + \omega_2)t/2} \left[ e^{i(\omega_1 - \omega_2)t/2} + e^{i(\omega_2 - \omega_1)t/2} \right] \]
\[ = 2A \cos(\delta \omega t) e^{i\omega t} \] (4)
where we have defined \( \delta \omega = (\omega_1 - \omega_2)/2 \) and \( \bar{\omega} = (\omega_1 + \omega_2)/2 \). Taking the real part, we have

\[
x_1 + x_2 = 2A \cos(\delta \omega t) \cos \bar{\omega} t
\]

which can be interpreted as oscillation at frequency \( \bar{\omega} \), but with slowly time varying amplitude that changes with frequency \( \delta \omega \), that is, beat.

![Diagram showing oscillation with frequency \( \bar{\omega} \) and slowly varying amplitude due to \( \delta \omega \)]

The phenomena of beat provides a very useful way to calibrate / measure frequency against a frequency standard.

# 2 Damped and Forced Harmonic Oscillator

## 2.1 Damped Harmonic Oscillator

Now let’s consider the more realistic case where the system has dissipation so that the oscillatory motion cannot go on forever. We go back to the simple example of mass on the spring and consider now the situation where the surface is not friction free.

Friction exerts a force that’s in the opposite direction of motion and takes a simple form

\[
-m \Gamma \frac{dx}{dt}
\]

which is proportional to the mass of the block and its velocity. \( \Gamma \) is the friction constant. Note that it has the same dimension as \( \omega \).

The equation of motion now gets modified

\[
m \frac{d^2 x}{dt^2} + m \Gamma \frac{dx}{dt} + k x(t) = 0
\]

which can be reorganized into

\[
\frac{d^2 x}{dt^2} x(t) + \Gamma \frac{dx}{dt} x(t) + \omega_0^2 x(t) = 0
\]

![Diagram of a damped harmonic oscillator with mass, spring, and friction force]
Without solving the equation, we know that the solution should describe a decaying oscillation. Let’s see what the math say.

To solve this equation for the real function $x(t)$, we solve a corresponding equation for a complex function $z(t)$

$$\frac{d^2}{dt^2} z(t) + \Gamma \frac{d}{dt} z(t) + \omega_0^2 z(t) = 0 \quad (9)$$

and find $x(t)$ by taking the real part.

We guess the form of the solution to be $z(t) = A e^{-\alpha t}$, where $\alpha$ can in general be a complex number. Plugging this form of solution into the equation we find

$$\alpha^2 z(t) - \Gamma \alpha z(t) + \omega_0^2 z(t) = 0 \quad (10)$$

which leads to a quadratic equation for $\alpha$

$$\alpha^2 - \Gamma \alpha + \omega_0^2 = 0 \quad (11)$$

The solution of $\alpha$ depends on the relationship between $\Gamma$ and $\omega_0$. Let’s discuss various cases.

(1) $\Gamma^2 > 4 \omega_0^2$

This is the case where friction is very large, and we would expect a very quick decay of the oscillation.

The solution for $\alpha$ in this case is $\alpha_{\pm} = \frac{\Gamma \pm \sqrt{\Gamma^2 - 4 \omega_0^2}}{2}$, which are two real positive numbers. The generic solution of $z(t)$ is given by

$$z(t) = A_+ e^{-\alpha_{+} t} + A_- e^{-\alpha_{-} t} \quad (12)$$

As $z(t)$ is real, $x(t) = z(t) = A_+ e^{-\alpha_{+} t} + A_- e^{-\alpha_{-} t}$, which describes pure decay. Graphically, it looks something like this

This case is called over damping.

(2) $\Gamma^2 < 4 \omega_0^2$

When friction is small, it should result in a gradual decay of the oscillation.

The solution for $\alpha$ in this case is $\alpha_{\pm} = \frac{\Gamma \pm i \sqrt{4 \omega_0^2 - \Gamma^2}}{2} = \frac{\Gamma}{2} \pm i \frac{\sqrt{4 \omega_0^2 - \Gamma^2}}{2}$, which are complex numbers. The generic solution of $z(t)$ is given by

$$z(t) = A_+ e^{-\alpha_{+} t} + A_- e^{-\alpha_{-} t} = A_+ e^{-\Gamma t/2} e^{i \omega t} + A_- e^{-\Gamma t/2} e^{-i \omega t} \quad (13)$$
where \( \omega = \sqrt{\omega_0^2 - \Gamma^2/4} < \omega_0 \). Taking the real part we get

\[
x(t) = \text{Re}(z(t)) = A_1 e^{-\Gamma t/2} \cos(\omega t) + A_2 e^{-\Gamma t/2} \sin(\omega t)
\]  

which describes a decaying oscillation at frequency \( \omega \).

This case is called under damping.

(3) \( \Gamma^2 = 4\omega_0^2 \)

This is the case of critical damping and we will explore it in homework.

### 2.2 Driven Harmonic Oscillation

In order to have a persistent oscillatory motion in a system with dissipation, we can drive it by pushing on the system periodically. Consider again the classic example of mass on a spring, but now with a horizontal force \( F(t) \) as the drive.

Newton’s law now reads

\[
m \frac{d^2}{dt^2} x(t) = -k x(t) - m \Gamma \frac{dx(t)}{dt} + F(t)
\]  

Reorganizing the terms, we get

\[
\frac{d^2}{dt^2} x(t) + \omega_0^2 x(t) + \Gamma \frac{dx(t)}{dt} = F(t)/m
\]

Let’s consider first the case where \( F(t) = F_0 \cos(\omega_D t) \) is a periodic force, but with a frequency \( \omega_D \) that can be different from the intrinsic frequency \( \omega_0 \). To solve the equation, we solve instead its complex version

\[
\frac{d^2}{dt^2} z(t) + \Gamma \frac{dz(t)}{dt} + \omega_0^2 z(t) = \frac{F_0}{m} e^{-i\omega_D t}
\]

Let’s try a solution of the form \( z_D(t) = Ae^{i\omega_D t} \) which describes an oscillatory motion at the driving
frequency. Plugging this solution into the equation, we get

$$-\omega_D^2 A e^{-i\omega_D t} - i\Gamma A e^{-i\omega_D t} + \omega_0^2 A e^{-i\omega_D t} = \frac{F_0}{m} e^{-i\omega_D t}$$  \hspace{1cm} (18)

From which we find

$$A = \frac{F_0}{m(\omega_0^2 - i\Gamma \omega_D - \omega_D^2)}, \quad z_D(t) = \frac{F_0}{m(\omega_0^2 - i\Gamma \omega_D - \omega_D^2)} e^{-i\omega_D t}$$  \hspace{1cm} (19)