1 Simple Harmonic Oscillator

1.3 Example III: LC circuit

Oscillation happens in all kinds of physical systems, not only mechanical. Consider the electrical circuit composed of a capacitor $C$ and an inductor $L$. We study the oscillation of the amount of charge across the capacitor $Q(t)$.

![Diagram of an LC circuit](image)

The voltage drop across the capacitor is given by

$$V_c = \frac{Q}{C}$$

(1)

The current $I$ flowing from the capacitor to the inductor is

$$I = \frac{dQ}{dt}$$

(2)

And the voltage drop across the inductor is given by

$$V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

(3)

As the voltage drop around the full circuit has to be zero, we have

$$0 = V_c + V_L = \frac{Q}{C} + L\frac{d^2Q}{dt^2}$$

(4)

That is,

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

(5)

which has a solution of the form

$$Q(t) = Q(0)\cos(\omega t) + \frac{1}{\omega}Q'(0)\sin(\omega t)$$

(6)

where $\omega = \frac{1}{\sqrt{LC}}$.

Demos. Questions: 1. What is the degree of freedom that is oscillating? 2. What is the restoring force?
1.4 General properties of Simple Harmonic Oscillator

Equation of motion

$$\frac{d^2X}{dt^2} = -\omega^2 X$$  \hspace{1cm} (7)

$X$ represents the small displacement from equilibrium position in the SHO. It can corresponds to $x$ in the mass on a spring problem, $\psi$ in the pendulum, or $Q$ in the LC circuit.

This equation of motion has a generic solution

$$X(t) = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t + \varphi)$$  \hspace{1cm} (8)

$C$ is called the amplitude of the oscillation, $\omega$ the angular frequency, $\varphi$ the phase.

The equation of motion has the following useful properties:

1.4.1 Linearity

If $x_1(t), x_2(t)$ are both solutions, then $c_1 x_1(t) + c_2 x_2(t)$ is also a solution.

Equations that contain terms with only one power of $x(t)$ are linear

$$\alpha x(t) + \beta \frac{dx(t)}{dt} + \gamma \frac{d^2x(t)}{dt^2} + \delta \frac{d^3x(t)}{dt^3} + ... = 0$$  \hspace{1cm} (9)

Equations like

$$x^2(t) + \frac{dx(t)}{dt} = 0$$  \hspace{1cm} (10)

or

$$\sin(x(t)) + \frac{d^2x(t)}{dt^2} = 0$$  \hspace{1cm} (11)

are not linear.

1.4.2 Energy conservation

In all three examples above, the total energy of the system is conserved.

In the mass on spring example, the system has a kinetic energy of $E_K = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2$ and a potential energy of $E_P = \frac{1}{2} k x^2 = \frac{1}{2} m \omega^2 x^2$. Plugging in the solution $x(t) = A \cos(\omega t) + B \sin(\omega t)$, the total energy is

$$E_t = E_K + E_P = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} m \omega^2 (A^2 + B^2)$$  \hspace{1cm} (12)

which is independent of time.

In the pendulum example, the system has a kinetic energy of $E_K = \frac{1}{2} m \left( \frac{d\psi}{dt} \right)^2 = \frac{1}{2} ml^2 \left( \frac{d\psi}{dt} \right)^2$ and a potential energy $E_P = mgl(1 - \cos(\psi)) = \frac{1}{2} mgl\psi^2$. Plugging in the solution $\psi(t) = A \cos(\omega t) +$
$B \sin(\omega t)$, the total energy is

$$E_t = E_K + E_P = \frac{1}{2} ml^2 \left(\frac{d\psi}{dt}\right)^2 + \frac{1}{2} mgl\psi^2 = \frac{1}{2} m\omega^2 l^2 (A^2 + B^2)$$

(13)

which is again independent of time.

An LC circuit is not a mechanical system but we can still define something like a kinetic energy, which is the energy stored in the inductor, $E_L = \frac{1}{2} L \left(\frac{dQ}{dt}\right)^2$ and something like a potential energy, which is the energy stored in the capacitor, $E_C = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} L\omega^2 Q^2$. Plugging in the solution $Q(t) = A \cos(\omega t) + B \sin(\omega t)$, the total energy is

$$E_t = E_L + E_C = \frac{1}{2} L \left(\frac{dQ}{dt}\right)^2 + \frac{1}{2} L\omega^2 Q^2 = \frac{1}{2} L\omega^2 (A^2 + B^2)$$

(14)

which is independent of time.

For all examples, we see that the total energy depends on system properties ($m$, $l$, $C$, $L$ etc.), angular frequency ($\omega$) and initial conditions ($A$, $B$), but not on time. Therefore, total energy is a constant of motion.

**1.4.3 Why is SHO so ubiquitous**

The SHO describes all small oscillations around equilibrium configurations in a physical system. Consider a system with a potential landscape as shown below

Point A is a local minimum in the potential landscape and an equilibrium position. The potential gradient is zero at this point.

$$\left.\frac{dV}{dx}\right|_A = 0$$

(15)

Close to A, the potential energy $V$ can be Taylor expanded into

$$V(x) = V(A) + \frac{1}{2} \left.\frac{d^2V}{dx^2}\right|_A (x - x_A)^2 + ....$$

(16)

If the system stays close enough to point A, we can ignore the higher order terms in ... and the potential energy takes the same square form as the three examples above. In particular, the restoring force is given by

$$F = -\left.\frac{dV(x)}{dx}\right|_A = -\left.\frac{d^2V}{dx^2}\right|_A (x - x_A)$$

(17)
which is proportional to displacement and in the opposite direction. The resulting motion will be the sinusoidal oscillation around the equilibrium point.

1.4.4 Relation to uniform circular motion

A SHO can be thought of as the projection of a uniform circular motion down to one dimension. Consider the circular motion around the origin described by

\[ x(t) = R \cos(\omega t + \varphi_0), \quad y(t) = R \sin(\omega t + \varphi_0) \]  

(18)

Projecting down to the \( x \) axis we get \( x(t) = R \cos(\omega t + \varphi_0) \), which describes the sinusoidal motion of a SHO.

This correspondence between circular motion and the SHO gives rise to the complex description of SHO. Instead of describing the circular motion with a vector, we use complex numbers to describe it.

\[ z(t) = x(t) + iy(t) = R \cos(\omega t + \varphi_0) + iR \sin(\omega t + \varphi_0) = Re^{i(\omega t + \varphi_0)} \]  

(19)

The SHO can then be described as the real part of this complex trajectory

\[ x(t) = Re(z(t)) = R \cos(\omega t + \varphi_0) \]  

(20)

The equation of motion of \( z(t) \) takes exactly the same form as \( x(t) \)

\[ \frac{d^2z}{dt^2} = -\omega^2 z \]  

(21)

which when expanded becomes

\[ \frac{d^2x}{dt^2} + i \frac{d^2y}{dt^2} = -\omega^2 x - i\omega^2 y \]  

(22)

If we match the real part on both sides, we get the equation of motion for the SHO.

It seems we are making things more complicated. Why bother? There are several reasons.
a. Complex number can make calculation much simpler, especially when there is dissipation. We are going to see some examples below.

b. In classical mechanics, the physics is in the real part. But in quantum mechanics, we actually care about the full complex number, not just the real part.

### 1.4.5 Superposition of two independent SHO

Suppose we have two SHOs described by

\[
x_1 = A_1 \cos(\omega_1 t + \varphi_1) = \text{Re}(A_1 e^{i\omega_1 t + \varphi_1}) = \text{Re}(z_1) \tag{23}
\]

\[
x_2 = A_2 \cos(\omega_2 t + \varphi_2) = \text{Re}(A_2 e^{i\omega_2 t + \varphi_2}) = \text{Re}(z_2) \tag{24}
\]

What if the two motions are happening at the same time on the same degree of freedom? What does the total motion look like? Let’s consider a couple interesting situations.

A. \( A_1 = A_2, \ \omega_1 = \omega_2, \ \text{but} \ \varphi_1 \neq \varphi_2 \)

To do the superposition of \( x_1 \) and \( x_2 \), we add \( z_1 \) and \( z_2 \) and take the real part

\[
z_1 + z_2 = A \left( e^{i(\omega_1 t + \varphi_1)} + e^{i(\omega_2 t + \varphi_2)} \right) = 2A e^{i(\omega_1 t + \varphi_1)/2} \left[ e^{i(\varphi_1 - \varphi_2)/2} + e^{i(\varphi_2 - \varphi_1)/2} \right] = 2A \cos(\delta \varphi) e^{i(\omega_1 t + \bar{\varphi})} \tag{25}
\]

\[
x_1 + x_2 = \text{Re}(z_1 + z_2) = 2A \cos(\delta \varphi) \cos(\omega_1 t + \bar{\varphi})
\]

Therefore, the superposed motion is still of the simple sinusoidal form, with total amplitude \( |2A \cos(\delta \varphi)| \). When \( \delta \varphi = 0 \), the total amplitude reaches its maximum of \( 2A \); the two SHOs are said to be in phase. When \( \delta \varphi = \pi \), the total amplitude reaches its minimum of 0; the two SHOs are said to be out of phase and cancel each other. When two SHOs have the same frequency, they can interfere, either constructively or destructively or somewhere in between.

**Question:** What happens if \( \omega_1 = \omega_2 \) but \( A_1 \neq A_2 \)?

B. If \( A_1 = A_2, \ \varphi_1 = \varphi_2, \ \omega_1 \neq \omega_2, \ \text{but} \ \omega_1 \approx \omega_2 \)

Something interesting happens in this situation. Suppose that at some time \( t \), the two oscillations are in phase \( \delta \varphi = 0 \). On a short time scale, as the two have almost the same frequency, they interfere constructively. Some time later (on the scale of \( 1/(\omega_1 - \omega_2) \)), the two oscillations fall out of phase and may even become completely out of phase with \( \delta \varphi = \pi \). For some short period of time at this later point, the two would interfere destructively. Therefore, the total oscillation will alternate between very strong (large amplitude) and very weak (small amplitude) and this is the phenomena called ‘Beat’.

Mathematically, we use again the complex representation to do the superposition

\[
z_1 + z_2 = A e^{i\omega_1 t} + A e^{i\omega_2 t} = 2A \cos(\delta \omega t) e^{i\bar{\omega}t} \tag{26}
\]
where we have defined $\delta \omega = (\omega_1 - \omega_2)/2$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$. Taking the real part, we have

$$x_1 + x_2 = 2A \cos(\delta \omega t) \cos \bar{\omega} t$$

which can be interpreted as oscillation at frequency $\bar{\omega}$, but with slowly time varying amplitude that changes with frequency $\delta \omega$, that is, beat.

![Diagram showing beat phenomenon](image)

The phenomena of beat provides a very useful way to calibrate / measure frequency against a frequency standard.