III.1) **Normal modes of a CO$_2$ molecule.** French 5-9.

The CO$_2$ molecule can be crudely and classically modeled as a system with a central mass $m_2 = 12$ AMU connected by equal springs of spring constant $k$ to two masses $m_1 = m_3 = 16$ AMU, constrained to move only along the line joining their centers.

1. Set up and solve the equations for the two normal modes in which the masses oscillate along that line.
2. There are three masses but only two oscillatory normal modes. What’s the third mode?
3. What are the ratio of frequencies of the two normal modes of oscillation?

III.2) **Oscillations of two coupled LC circuits.** Crawford 1.21.

Find the two normal modes of oscillation of the coupled LC circuits shown in Crawford Fig. 1.12, with equations of motion given by

\[
L \frac{d^2 I_a}{dt^2} = -C^{-1} I_a + C^{-1} (I_b - I_a) \\
L \frac{d^2 I_b}{dt^2} = -C^{-1} (I_b - I_a) - C^{-1} I_b
\]

III.3) **Nonidentical coupled pendulums.** Crawford 1.19.
Consider two pendulums $a$ and $b$, with the same string length $l$ but with different bob masses $M_a$ and $M_b$. They are coupled by a spring of spring constant $K$ which is attached to the bobs. Show that the equations of motion (for small oscillations) are

$$ M_a \frac{d^2 \psi_a}{dt^2} = -M_a g \frac{l}{l} \psi_a + K(\psi_b - \psi_a) \quad (3) $$

$$ M_b \frac{d^2 \psi_b}{dt^2} = -M_b g \frac{l}{l} \psi_b - K(\psi_b - \psi_a) \quad (4) $$

(1) Solve these two equations for the two modes by the method of searching for normal coordinates. Show that $\psi_1 \equiv (M_a \psi_a + M_b \psi_b)/(M_a + M_b)$ and $\psi_2 \equiv \psi_a - \psi_b$ are normal coordinates. Find the frequencies and configurations of the modes. What is the physical significance of $\psi_1$? Of $\psi_2$?

(2) Find a superposition of the two modes which corresponds to the initial conditions at time $t = 0$ that both pendulums have zero velocity, that bob $a$ have amplitude $A$, and that bob $b$ have amplitude zero.

(3) Let $E$ be the total energy of bob $a$ at $t = 0$. Find an expression for $E_a(t)$ and for $E_b(t)$. Assume weak coupling. Does the energy of bob $a$ transfer completely to bob $b$ during a beat? Is it perhaps the case that if the pendulum which initially has all the energy is the heavy one, the energy is not completely transferred, but if it is the light one, the energy is completely transferred?

III.4) **Two coupled pendulums as a mechanical filter.** Crawford 3.28.

Consider two pendula, each with mass $M$ and length $l$, coupled by a single spring of spring constant $K$, with displacements from equilibrium of $\psi_a(t)$ and $\psi_b(t)$. Pendulum $a$ is being driven by a force $F(t) = F_0 \cos \omega t$. Neglect damping. Show that

$$ \psi_a(t) \approx \frac{F_0}{2M} \cos \omega t \left\{ \frac{1}{\omega_1^2 - \omega^2} + \frac{1}{\omega_2^2 - \omega^2} \right\}, $$

$$ \psi_b(t) \approx \frac{F_0}{2M} \cos \omega t \left\{ \frac{1}{\omega_1^2 - \omega^2} - \frac{1}{\omega_2^2 - \omega^2} \right\}, $$

$$ \frac{\psi_b}{\psi_a} \approx \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2 - 2\omega^2}, $$

where $\omega_1$ is the lower of the two mode frequencies, $\omega_2$ is the higher, and $\omega$ is the driving frequency. Plot this ratio vs $\omega$ (picking some sensible and simple values for the parameters). Explain how this works as a mechanical band-pass filter.

III.5) **Questions and Comments**