Physics 129b **Lecture 17** **Caltech, 03/03/20**

Reference: Jones, "Groups, Representations, and Physics", Chapter 10.

9 Lorentz Group and Special Relativity

Special relativity says, physics laws should look the same for different observers in different inertial reference frames.

In the non-relativistic setting, the coordinates of different reference frames are related by the Euclidean transformation. In particular, if two reference frames S and S' coincide at $t = 0$ and are moving with relative velocity $\vec{v} = (v, 0, 0)$, then the relation between the coordinates of an event in the two reference frames is

$$t' = t, \vec{r}' = \vec{r} - \vec{v}t \left(x' = x - vt, y' = y, z' = z\right) \tag{1}$$

Under such a transformation the spatial distance between two points (at the same time) remains invariant

$$(\vec{r}_1' - \vec{r}_2')^2 = (\vec{r}_1 - \vec{r}_2)^2 \tag{2}$$

In special relativity however, we need to use the Lorentz transformation and replace the above relation with

$$t' = \gamma(t - xv/c^2), x' = \gamma(x - vt), y' = y, z' = z \tag{3}$$

where $\gamma = \frac{c}{\sqrt{c^2 - \vec{v}^2}}$. This particular transformation induced by a relative velocity is called a boost.

The transformation may look complicated, but it is designed so that the velocity of light remains invariant in all inertial reference frames. Suppose that we send out a light signal from the origin at $t = 0$ in the x direction. In reference frame S, at a later time t, the signal has travelled to point $x = ct, y = 0, z = 0$. Transformed to the reference frame of S', we find that the coordinate of the corresponding signal is

$$t' = t\sqrt{\frac{c-v}{c+v}}, x' = ct\sqrt{\frac{c-v}{c+v}}, y' = 0, z' = 0 \tag{4}$$

Therefore the velocity of light in the frame of S' is also c.

This is just one particular example of the whole group of Lorentz transformation, which we are going to study in detail below.

9.1 Coordinate four vector

The fact that time and space gets mixed together under Lorentz transformation motivates the definition of the space-time four vector $\tilde{x}_\mu$, $\mu = 0, 1, 2, 3$ for the Minkowski space

$$\tilde{x}_0 = ct, \tilde{x}_1 = x, \tilde{x}_2 = y, \tilde{x}_3 = z \tag{5}$$

Notice that the components of the four vector all have the dimension of length.

Lorentz transformation leaves the 'interval'

$$|\tilde{x}|^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{6}$$

invariant. (This ensures that the speed of light remains invariant in all reference frames.)

We define a metric tensor, $g^{\mu\nu}$ such that $g^{\mu\nu} = 0$ if $\mu \neq \nu$ and $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$. That is,

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7}$$

Then we can define the 'length' of the four vector as

$$|\tilde{x}|^2 = \tilde{x}^T g \tilde{x} \tag{8}$$

This is very different from the metric we are used to. In the usual Euclidean space, $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is positive definite. The metric for the Minkowski space is not positive definite and will result in some special properties of the Lorentz group.

Suppose that under a Lorentz transformation, the four vector transforms as

$$\tilde{x}' = \Lambda \tilde{x} \tag{9}$$

Then the invariance of the length of the vector requires that

$$|\tilde{x}'|^2 = \tilde{x}'^T g \tilde{x}' = \tilde{x} \Lambda^T g \Lambda \tilde{x} = \tilde{x}^T g \tilde{x} = |\tilde{x}|^2 \tag{10}$$

Because this is true for all x, we have $\Lambda^T g \Lambda = g$.

Notice that if $g = I_3$, this condition reduces to the orthogonality condition of three dimensional rotation transformations which form the group $SO(3)$. The Lorentz group can be thought of as the group of 'orthogonal' transformations on a space with metric $g = \mathrm{diag}(-1, 1, 1, 1)$ and it is denoted as $SO(3, 1)$.

9.2 Lorentz Transformations

What kind of Λ satisfies the above condition?

First, any spatial rotation involving $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ keeps the length of the four vector invariant. Therefore, the spatial rotation transformations $\in SO(3)$ forms a subgroup of the Lorentz group. The transformation matrices take the form

$$\Lambda^r_{\vec{n}}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_{\vec{n}}(\theta) & \\ 0 & & & \end{pmatrix} \tag{11}$$

where $R_{\vec{n}}(\theta)$ is the three dimensional special orthogonal matrix representing the spatial rotation. This subgroup of transformations is generated by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

A different kind of Lorentz transformations which do involve time are the 'boosts'. Boost in the x direction gives rise to the transformation

$$\Lambda_x^b(v) = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

where $\gamma = \frac{c}{\sqrt{c^2-v^2}}$. If we define $\gamma = \cosh\zeta = \frac{e^{\zeta}+e^{-\zeta}}{2}$, so that $\tanh\zeta = \frac{\sinh\zeta}{\cosh\zeta} = \frac{v}{c}$, then $\Lambda_x^b(v)$ can be re-written as

$$\Lambda_x^b(\zeta) = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

One can explicitly check that $\left(\Lambda_x^b(\zeta)\right)^T g \Lambda_x^b(\zeta) = g$.

The infinitesimal generator for x direction boost is

$$Y_1 = i\frac{d\Lambda_x^b(\zeta)}{d\zeta}\Big|_{\zeta=0} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

By exponentiating Y_1, we can recover $\Lambda_x^b(\zeta)$

$$\Lambda_x^b(\zeta) = e^{-i\zeta Y_1} \tag{16}$$

However, notice one important difference with the generator for $SO(3)$: ζ is not bounded. As v approaches c, ζ approaches $+\infty$. Therefore, the Lorentz group $SO(3,1)$ is not compact. This has a series of consequence. One of them being that the finite dimensional representations of $SO(3,1)$ are no longer unitary.

Similarly, boosts in y and z directions are generated by

$$Y_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Boosts in an arbitrary direction $\vec{n}$ can be obtained first by rotating $\vec{n}$ to x axis, applying the boost in x direction and then rotating back.

In general, an arbitrary Lorentz transformation contains both spatial rotation and boost. The whole group is generated from X_1, X_2, X_3 and Y_1, Y_2, Y_3. The Lie algebra of $SO(3,1)$ is a six dimensional real vector space with commutators

$$[X_a, X_b] = i\epsilon_{abc}X_c, [X_a, Y_b] = i\epsilon_{abc}Y_c, [Y_a, Y_b] = -i\epsilon_{abc}X_c \tag{18}$$

Comments:

(1) While X_1, X_2, X_3 is closed under commutation, Y_1, Y_2, Y_3 are not. Therefore, the boosts do not form a subgroup.

(2) While X_1, X_2, X_3 are Hermitian, Y_1, Y_2, Y_3 are anti-Hermitian. Therefore, the representation is not unitary (the boost transformation is not unitary).

(3) The first and second commutation relation says that X_1, X_2, X_3 transform as a vector under $SO(3)$, so do Y_1, Y_2, Y_3.

(4) We can make linear combinations between X and Y

$$X_a^{(\pm)} = \frac{1}{2}(X_a \pm iY_a) \tag{19}$$

In terms of $X_a^{\pm}$, the commutation relations become

$$[X_a^{(+)}, X_b^{(+)}] = i\epsilon_{abc}X_c^{(+)}, [X_a^{(-)}, X_b^{(-)}] = i\epsilon_{abc}X_c^{(-)}, [X_a^{(+)}, X_b^{(-)}] = 0 \tag{20}$$

That is, the set of six generators break up into two subsets, such that each subset is equivalent to the Lie algebra of $SU(2)$ and the two subsets are independent of each other.

9.3 Irreducible representations

The four dimensional matrices Λ provides one possible representation of $SO(3,1)$ while the group is abstractly defined as that satisfying the same composition rule as the Λ's. In terms of Lie algebra, the group is defined as that with a six dimensional Lie algebra, satisfying the commutation relation

$$[X_a, X_b] = i\epsilon_{abc}X_c, [X_a, Y_b] = i\epsilon_{abc}Y_c, [Y_a, Y_b] = -i\epsilon_{abc}X_c \tag{21}$$

or

$$[X_a^{(+)}, X_b^{(+)}] = i\epsilon_{abc}X_c^{(+)}, [X_a^{(-)}, X_b^{(-)}] = i\epsilon_{abc}X_c^{(-)}, [X_a^{(+)}, X_b^{(-)}] = 0 \tag{22}$$

Now we can try to see what irreducible representations do $SO(3,1)$ have. Following our analysis of $SO(3)$, in order to find irreps for a Lie group, we can try to find the irreps for its Lie algebra, but with the danger that we get the irrep of the covering group ($SU(2)$ for $SO(3)$). Things work in a very similar way for $SO(3,1)$.

We see that the Lie algebra of $SO(3,1)$ contains two $SU(2)$ part. Therefore, its irrep can be labelled by (j_1, j_2), where j_1, j_2 are integer or half-integer. The representation is then $(2j_1+1)(2j_2+1)$ dimensional. The generators are

$$X_a^+ = J_a^{j_1} \otimes I_{2j_2+1}, X_a^- = I_{2j_1+1} \otimes J_a^{j_2} \tag{23}$$

From which we get

$$X_a = J_a^{j_1} \otimes I_{2j_2+1} + I_{2j_1+1} \otimes J_a^{j_2}, Y_a = -i(J_a^{j_1} \otimes I_{2j_2+1} - I_{2j_1+1} \otimes J_a^{j_2}) \tag{24}$$

If we then take the exponential, we can recover the group (or its covering group).

Let's see some example irreps.

(1) $j_1 = 0$, $j_2 = 0$

This is the trivial representation. It is one dimensional, with all the generators being 0 and all the group elements being represented by 1. In quantum field theory, this representation is carried by a relativistic scalar field (e.g. Higgs field).

(2) $j_1 = 1/2$, $j_2 = 0$

This is called a spinor representation. It is two dimensional.

$$X_1^+ = \sigma_x, X_2^+ = \sigma_y, X_3^+ = \sigma_z, X_1^- = 0, X_2^- = 0, X_3^- = 0 \tag{25}$$

Correspondingly

$$X_1 = \sigma_x, X_2 = \sigma_y, X_3 = \sigma_z, Y_1 = -i\sigma_x, Y_2 = -i\sigma_y, Y_3 = -i\sigma_z \tag{26}$$

The Lorentz transformations are then parameterized by six real numbers θ_1, θ_2, θ_3, ϕ_1, ϕ_2, ϕ_3

$$\Lambda(\vec{\theta}, \vec{\phi}) = e^{i(\vec{\theta}\cdot\vec{X}+\vec{\phi}\cdot\vec{Y})} = e^{i(\vec{\theta}-i\vec{\phi})\cdot\vec{\sigma}} \tag{27}$$

Note that this is different from the $SU(2)$ group which contains matrices $e^{i\vec{\theta}\cdot\vec{\sigma}}$. In fact, the set of matrices generated are the group of special (determinant 1) linear (invertible) matrices of dimension 2, $SL(2,\mathbb{C})$. $SL(2,\mathbb{C})$ is a covering group of the $SO(3,1)$ group as 2π spatial rotation results in $-I$ instead of I.

In quantum field theory, this representation is carried by the Weyl fermion.

(3) $j_1 = 0$, $j_2 = 1/2$

This is another spinor representation with generators

$$X_1 = \sigma_x, X_2 = \sigma_y, X_3 = \sigma_z, Y_1 = i\sigma_x, Y_2 = i\sigma_y, Y_3 = i\sigma_z \tag{28}$$

This is inequivalent to the previous representation because they cannot be related by a basis transformation.

In quantum field theory, the direct sum of the $j_1 = 1/2$, $j_2 = 0$ representation and the $j_1 = 0$, $j_2 = 1/2$ representation is carried by the Dirac fermion.

(4) $j_1 = 1/2$, $j_2 = 1/2$

This is a four dimensional representation. Actually, it is exactly the four dimensional representation which we used to define $SO(3,1)$. That is, the space time four vector transforms with this representation.