7 Continuous Group

7.2 $SU(2)$: the special unitary matrices of dimension two

Before I move on to talk about how the irreps of $SO(3)$ combine with each other (in direct product), I would like to digress and talk about $SU(2)$ first. $SU(2)$ is very similar to $SO(3)$ but also different in very important ways. It turns out that the irreps of $SO(3)$ is a subset of irreps of $SU(2)$ and when physicist study 'addition of angular momentum', what they really do is to study the direct product of irreps of $SU(2)$ instead of $SO(3)$. So let's first understand what $SU(2)$ is.

Instead of starting from the definition of $SU(2)$, let's start by considering the three Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

It is easy to check that (1) they are Hermitian finite dimensional matrices (2) they satisfy the commutation rule $[\sigma_a, \sigma_b] = i\epsilon_{abc}\sigma_c$. It seems that they fulfill the requirement of being the infinitesimal generator of $SO(3)$. Actually, not quite. You may notice that once exponentiated, they do not quite give rise to the $SO(3)$ group. In particular, consider the 2π rotation around a particular axis, say z

$$R_z^{(1/2)}(2\pi) = e^{-i2\pi\sigma_z} = -I_2 \tag{2}$$

That is, 2π rotation is not exactly doing nothing. Instead it adds a global phase factor of -1.

Therefore, the group generated by σ_x, σ_y, and σ_z is not quite the $SO(3)$ group, in which doing 2π rotation should be the same as doing nothing. Instead, it generates the $SU(2)$ group: the group of special unitary matrices of dimension two.

Let's make linear superpositions of the infinitesimal generators and take their exponential.

$$R_{\vec{n}}(\theta) = e^{-i\theta\sigma_{\vec{n}}} = e^{-i\theta(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)} \tag{3}$$

Because $(\sigma_{\vec{n}})^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 = I_2/4$, we have

$$R_{\vec{n}}(\theta) = \sum_{k=0}^{\infty} \frac{(-i\theta\sigma_{\vec{n}})^k}{k!} = \sum_{\text{even } k} \frac{(-i\theta/2)^k}{k!} + \sum_{\text{odd } k} \frac{(-i\theta/2)^k}{k!} 2\sigma_{\vec{n}} = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) 2\sigma_{\vec{n}} \tag{4}$$

This represents all possible 2×2 unitary matrices with determinant 1. (homework)

Although we started from three generators with the same commutation relation as those for $SO(3)$, the major difference between $SU(2)$ and $SO(3)$ is that the parameter takes value in $[0, 4\pi)$, not $[0, 2\pi)$. Only when $\theta = 4\pi$ does $R_{\vec{n}}(\theta)$ equal identity.

While $R_{\vec{n}}(2\pi)$ is not equal to identity, it is proportional to identity. Therefore, it commutes with all other group elements and generates the center of the group (recall the definition of the center), which is a C_2 group. As the center of a group is a normal subgroup as well, we can take the quotient of $SU(2)$ with respect to this C_2 group and we recover the $SO(3)$ group as the quotient group. We say that $SU(2)$ is a double cover of $SO(3)$.

You may wonder why we care about $SU(2)$ so much in physics. As it turns out, a very important property of electrons (and other fundamental particles) is their internal spin. This is not related to the orbital motion of the electron around a nucleus. Instead, it is something intrinsic to the electron. People realized that electron spin lives in a two dimensional Hilbert space. We can choose the basis state of this two dimensional Hilbert space as the eigenstates of σ_z.

$$\sigma_z \left|\frac{1}{2}\right\rangle = \frac{1}{2}\left|\frac{1}{2}\right\rangle, \sigma_z \left|-\frac{1}{2}\right\rangle = -\frac{1}{2}\left|-\frac{1}{2}\right\rangle \tag{5}$$

The raising and lowering operator maps between the two

$$\sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \sigma_x - i\sigma_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_+ \left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \sigma_- \left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle \tag{6}$$

This is exactly the same relation as those given in the previous lecture if we set $j = \frac{1}{2}$. Therefore, this spin 1/2 behaves in every way like its integer angular momentum cousins, with one difference. Under spatial rotation around axis $\vec{n}$ through angle θ, it transforms as

$$R_{\vec{n}}(\theta) = e^{-i\theta\sigma_{\vec{n}}} = e^{-i\theta(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)} \tag{7}$$

which does not form a representation of the $SO(3)$ group but the $SU(2)$ group. This is a special property of quantum mechanics. That is, we can have quantum mechanical wave functions transforming under symmetry operations up to a phase factor. Here the phase factor shows up as $R_{\vec{n}}(2\pi) = -I$. This is ok in quantum mechanics because global phase factor is not measurable. It is in some sense a redundancy of the wave function representation, but through this example we can see that this redundancy is absolutely crucial because without it, there cannot be a spin 1/2 representation of rotation symmetry!

In fact, $SU(2)$ has other irreducible representations as well. All the irreps of $SO(3)$ are irreps of $SU(2)$ as well, even though they are not quite faithful. Moreover, $SU(2)$ has one irrep in every even dimension which can be obtained in exactly the same way as the odd dimensional irreps for $SO(3)$ but starting from half integer j, $j = 1/2, 3/2, \ldots$. Of course, only the odd dimension irreps are irreps of $SO(3)$. The even dimension ones are irreps of $SO(3)$ only up to a phase factor, and we say that they are projective representations of $SO(3)$.

In physics, people often mix the notion of $SO(3)$ and $SU(2)$. It happens because in quantum mechanics both projective and nonprojective representations are allowed and they both show up in physical situations (like electron spin and orbital angular momentum) and can interact with each other.

7.3 Clebsch-Gordon Coefficients

Now let's take all the irreps of $SU(2)$ (or all the nonprojective and projective irreps of $SO(3)$) and see how they interact with each other. In particular, we want to know if we take the direct product between two irreps labeled by j and j' (both can be either integer or half integer), how does the composite representation decompose into a direct sum of irreducible blocks? In physics, this is called the 'addition of angular momentum' and it was known that

$$D^{(j_1)} \otimes D^{(j_2)} = \oplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)} \tag{8}$$

That is, the addition of angular momentum j_1 with angular momentum j_2 gives rise to angular momentum from $|j_1 - j_2|$ to $j_1 + j_2$.

One comment on terminology. When we talk about addition of angular momentum, we are actually taking the direct product of the corresponding irreps. We say that their angular momentum add, because the angular momentum operator of the direct product representation is the sum of the angular momentum operator of each of the irrep.

$$D^{(j_1)}_{\vec{n}}(\theta) \otimes D^{(j_2)}_{\vec{n}}(\theta) = e^{i\theta J^1_{\vec{n}}} \otimes e^{i\theta J^2_{\vec{n}}} = e^{i\theta(J^1_{\vec{n}} \otimes I_{j_2(j_2+1)} + I_{j_1(j_1+1)} \otimes J^2_{\vec{n}})} \tag{9}$$

Therefore, $J^{tot}_{\vec{n}} = J^1_{\vec{n}} \otimes I_{j_2(j_2+1)} + I_{j_1(j_1+1)} \otimes J^2_{\vec{n}}$, hence the name 'addition'. Note that $J^1_{\vec{n}} \otimes I_{j_2(j_2+1)}$ and $I_{j_1(j_1+1)} \otimes J^2_{\vec{n}}$ commute, therefore we can simply add them when multiplying their exponential. Usually we just use the short-hand notation $J^1_{\vec{n}}$ for $J^1_{\vec{n}} \otimes I_{j_2(j_2+1)}$ and $J^2_{\vec{n}}$ for $I_{j_1(j_1+1)} \otimes J^2_{\vec{n}}$. Hence we have the relation

$$J^{tot}_x = J^1_x + J^2_x, J^{tot}_y = J^1_y + J^2_y, J^{tot}_z = J^1_z + J^2_z \tag{10}$$

Now let's show the relation in Eq. 8 using the character of the irreps. Suppose that $j_1 \geq j_2$. The character of the direct product of $D^{(j_1)}$ and $D^{(j_2)}$ is

$$\chi^{(j_1)}(\theta)\chi^{(j_2)}(\theta) = \frac{\sin(j_1 + 1/2)\theta}{\sin(\theta/2)} \frac{\sin(j_2 + 1/2)\theta}{\sin(\theta/2)} \tag{11}$$

Using equivalent expressions for $\chi^{(j_1)}(\theta)$ and $\chi^{(j_2)}(\theta)$, we get

$$\begin{aligned} \chi^{(j_1)}(\theta)\chi^{(j_2)}(\theta) &= \frac{e^{i(j_1+1/2)\theta} - e^{-i(j_1+1/2)\theta}}{2i\sin(\theta/2)} \sum_{m=-j_2}^{j_2} e^{im\theta} \\ &= \frac{1}{2i\sin(\theta/2)} \sum_{m=-j_2}^{j_2} e^{i(j_1+m+1/2)\theta} - e^{-i(j_1-m+1/2)\theta} \end{aligned} \tag{12}$$

Because the summation over m is over $-j_2$ to j_2, we can change m to $-m$ in the second term so that j_1 and m always appear as a combination $j = j_1 + m$. Therefore, we can replace the summation as over $j = j_1 - j_2$ to $j = j_1 + j_2$.

$$\begin{aligned} \chi^{(j_1)}(\theta)\chi^{(j_2)}(\theta) &= \frac{1}{2i\sin(\theta/2)} \sum_{j=j_1-j_2}^{j_1+j_2} e^{i(j+1/2)\theta} - e^{-i(j+1/2)\theta} \\ &= \sum_{j=j_1-j_2}^{j_1+j_2} \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} = \sum_{j=j_1-j_2}^{j_1+j_2} \chi^{(j)}(\theta) \end{aligned} \tag{13}$$

□

In physics, people are not only interested in what $D^{(j)}$ are contained in the direct product of $D^{(j_1)} \otimes D^{(j_2)}$, but also how the basis states of $D^{(j)}$ can be obtained from the basis states of $D^{(j_1)}$ and $D^{(j_2)}$. As we have completely fixed the choice of basis for each of the irrep, there is a concrete process to obtain this relation.

Let's consider the simplest example of the direct product of two $j = 1/2$ irreps $D^{(1/2)} \otimes D^{(1/2)}$. Each $D^{(1/2)}$ is two dimensional and the direct product of two copies of them is four dimensional with basis states

$$\left|\frac{1}{2}\right\rangle \left|\frac{1}{2}\right\rangle, \left|\frac{1}{2}\right\rangle \left|-\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\right\rangle \left|\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\right\rangle \left|-\frac{1}{2}\right\rangle \tag{14}$$

the number in the $|\rangle$ labels the angular momentum m in z direction. In order to make it explicit that these states belong to the $j = 1/2$ irrep, we label the states as $|j_1, m_1; j_2, m_2\rangle$

$$\left|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right\rangle \tag{15}$$

According to the previous discussion

$$D^{(1/2)} \otimes D^{(1/2)} = D^{(0)} + D^{(1)} \tag{16}$$

$D^{(0)}$ is one dimensional and has one basis state with angular momentum 0. $D^{(1)}$ is three dimensional and has three basis states with z direction angular momentum 1, 0 and -1 respectively. To specify that they comes from the composition of two angular momentum 1/2 irreps, we label the state as $|(j_1, j_2)j, m\rangle$

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)0,0\right\rangle, \left|\left(\frac{1}{2},\frac{1}{2}\right)1,1\right\rangle, \left|\left(\frac{1}{2},\frac{1}{2}\right)1,0\right\rangle, \left|\left(\frac{1}{2},\frac{1}{2}\right)1,-1\right\rangle \tag{17}$$

The basis states listed in Eq. 14 and those listed in Eq. 17 are the basis states of the same four dimensional Hilbert space. Therefore, they are related by a unitary transformation. In particular, each state in Eq. 17 can be decomposed into a linear superposition of the basis states given in Eq. 14 and we want to find out the coefficient of these linear superpositions.

Let's start from the state $\left|\left(\frac{1}{2},\frac{1}{2}\right)1,1\right\rangle$. This state is special in that it has the largest angular momentum $m = 1$ in z direction among the four states in Eq. 17. Because

$$J_z^{tot} = J_z^1 + J_z^2 \tag{18}$$

we have

$$m = m_1 + m_2 \tag{19}$$

Among the four states listed in Eq. 14, only one satisfies $m_1 + m_2 = 1$: $\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle$. Therefore, these two states must be the same up to a phase factor. In physics, the convention is to choose the phase factor to be 1. That is

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)1,1\right\rangle = \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle \tag{20}$$

Starting from here, we can obtain all other states in $D^{(1)}$ by applying the lowering operator for the total angular momentum

$$J_-^{tot} = J_x^{tot} - iJ_y^{tot} = J_x^1 + J_x^2 - iJ_y^1 - iJ_y^2 = J_-^1 + J_-^2 \tag{21}$$

We find

$$\sqrt{2}\left|\left(\frac{1}{2},\frac{1}{2}\right)1,0\right\rangle = J_-^{tot}\left|\left(\frac{1}{2},\frac{1}{2}\right)1,1\right\rangle = \left(J_-^1 + J_-^2\right)\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle \tag{22}$$

That is

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)1,0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle\right) \tag{23}$$

We can see that this relation is consistent because $m = m_1 + m_2 = 0$.

If we apply the lowering operator again, we get

$$\sqrt{2}\left|\left(\tfrac{1}{2},\tfrac{1}{2}\right)1,-1\right\rangle = J_-^{tot}\left|\left(\tfrac{1}{2},\tfrac{1}{2}\right)1,0\right\rangle = \left(J_-^1 + J_-^2\right)\tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\right) = \sqrt{2}\left|\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \tag{24}$$

That is

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)1,-1\right\rangle = \left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle \tag{25}$$

which can also be obtained by comparing m with $m_1 + m_2$ in the basis states.

Now we have found all the basis states for $D^{(1)}$, we are left with only the basis state for $D^{(0)}$. In order to determine its decomposition in the basis states of Eq. 14, we only need to observe that $\left|\left(\frac{1}{2},\frac{1}{2}\right)0,0\right\rangle$ is a state with $m=0$, therefore, it has to be equal to some kind of superposition of $\left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle$ which both have $m_1+m_2=0$. We have already seen one such superposition which gave rise to

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)1,0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle\right) \tag{26}$$

As $\left|\left(\frac{1}{2},\frac{1}{2}\right)0,0\right\rangle$ is orthogonal to $\left|\left(\frac{1}{2},\frac{1}{2}\right)1,0\right\rangle$, it can only be

$$\left|\left(\frac{1}{2},\frac{1}{2}\right)0,0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}\right\rangle\right) \tag{27}$$

Now let's put our findings in a table

$j_1=\frac{1}{2}, j_2=\frac{1}{2}$	$m_1=\frac{1}{2}, m_2=\frac{1}{2}$	$m_1=-\frac{1}{2}, m_2=\frac{1}{2}$	$m_1=\frac{1}{2}, m_2=-\frac{1}{2}$	$m_1=-\frac{1}{2}, m_2=-\frac{1}{2}$
$j=1, m=1$	1	0	0	0
$j=1, m=0$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$j=1, m=-1$	0	0	0	1
$j=0, m=0$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0

The numbers in this table is called the Clebsch-Gordon coefficient or CG coefficient for short. It is the coefficient in the decomposition of total angular momentum basis in terms of the tensor product of component angular momentum basis. The CG coefficient of other j_1, j_2 and j can be obtained in a very similar way.