7 Continuous Group

Now we are going to move on to continuous groups. We have seen the simplest example of a continuous group, the circle group. Let's first review how that works and see how the idea can be generalized to more complicated groups.

7.1 $SO(2)$

Instead of saying "the circle group", we are going to call it by a more popular name: the $SO(2)$ group. It is a matrix group of two dimensional orthogonal matrices with +1 determinant. It represents rotation of a two dimensional vector space and is represented on this two dimensional space as

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

$\theta \in [0, 2\pi)$ and the group elements compose as

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2 \mod 2\pi) \tag{2}$$

Notice that here we are using a particular representation to define the group, but the group is a more general abstract notion. In particular the group can have other kinds of representations. This $2D$ representation is special though in that it is faithful. Other representations may not be faithful. It is a slight abuse of terminology to call the group $SO(2)$, but in most cases it should be clear enough whether we are talking about the abstract group or this particular two dimensional representation.

The continuity of the group elements comes from the continuity of the parameter θ. Moreover, the group has a nice property called compact, which roughly means that the parameter takes value in the bounded region of $[0, 2\pi)$.

This is an abelian group and the 2D representation actually decomposes into two 1D irreps through unitary transformation $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

$$SR(\theta)S^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{3}$$

Of course there are an infinite number of irreps given by $\{e^{in\theta}\}, n \in \mathbb{Z}$.

Because all irreps are $1D$, the character of the representation is just given by the irrep itself.

$$\chi^{(n)} = e^{in\theta}, \theta \in [0, 2\pi) \tag{4}$$

These characters satisfy an orthogonality condition similar to the finite group case. However, instead of summing over individual group elements, we need to perform an integration over them.

$$\langle \chi^{(n)}, \chi^{(n')} \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta} e^{in'\theta} = \delta_{nn'} \tag{5}$$

Notice that I have chosen a normalization for the inner product of characters so that each character have length 1.

We can use this orthogonality condition of characters in the same way as we have used it for finite groups. For example, we can check that the $2D$ rep given above decomposes into two $1D$ irreps. The character of the $2D$ irrep

$$\chi = 2\cos\theta = e^{i\theta} + e^{-i\theta} \tag{6}$$

Therefore

$$R(\theta) = D^{(1)}(\theta) \oplus D^{(-1)}(\theta) \tag{7}$$

as we have seen above.

The direct product of irreps goes as

$$D^{(n)} \otimes D^{(n')} = D^{(n+n')} \tag{8}$$

Therefore, under direct product, the irreps form a group which is isomorphic to the group of integers.

For finite groups, a useful notion is the generator of the group. Once we have identified the generators of a group and the relations between them, we knows which group it is. For continuous group, can we similarly find such generators? For the $SO(2)$ group, intuition says that the generator of the group is an infinitesimal rotation by a very small angle θ. But of course, no θ is small enough, we can always find a smaller one. What we define instead, is an infinitesimal generator

$$X = i\frac{dR(\theta)}{d\theta}\bigg|_{\theta=0} \tag{9}$$

for any representation R. Any group elements in the continuous group can then be obtained by taking the exponential of this infinitesimal generator.

$$R(\theta) = e^{-i\theta X} \tag{10}$$

The exponential of an operator is defined as $e^{-i\theta X} = \sum_{k=0}^{\infty} \frac{(-i\theta X)^k}{k!}$. If we can diagonalize X into $VXV^{-1} = D$, where $D = diag(d_1, d_2, ...)$, then $e^{-i\theta X} = V^{-1} diag(e^{-i\theta d_1}, e^{-i\theta d_2}, ...)V$.

For the irrep labeled by n, $D^{(n)}(\theta) = e^{in\theta}$, $X^{(n)} = -n$. For the $2D$ orthogonal representation, $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Note that X is Hermitian because R is unitary.

In physics, the Hermitian generator X is sometimes identified as the orbital angular momentum J_z around the rotation axis z (e.g. for electron orbits around a nucleus). Suppose that a wave function forms an irrep of the $SO(2)$ group. That is,

$$R(\theta)|\psi\rangle = e^{-in\theta}|\psi\rangle \tag{11}$$

The state is said to have orbital angular momentum $J_z = n$. In other situations, X maybe identified with the number of particles N in the system (e.g. for electrons in metals or insulators) and in this particular state $N = n$.

7.2 SO(3)

This is the group of three dimensional orthogonal matrices with +1 determinant. This set of matrices describe rotation of a three dimensional real vector space. Even though it is just one dimension up from $SO(2)$, it is much more complicated but also much more interesting!

First, $SO(3)$ is not abelian any more. Imagine we perform a rotation around the z axis first by an angle θ, the transformation of the $3D$ vector space is given by

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

Next let's perform a rotation around x axis by an angle θ', the transformation of the $3D$ vector space is given by

$$R_x(\theta') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta' & -\sin\theta' \\ 0 & \sin\theta' & \cos\theta' \end{pmatrix} \tag{13}$$

Direct calculation shows that $R_z(\theta)R_x(\theta') \neq R_x(\theta')R_z(\theta)$ for general θ and θ'. Similar to the case of $SO(2)$, these three dimensional matrices provide one particular representation of the $SO(3)$ group, but the group may have other representations. This three dimensional representation is special in that it is faithful and irreducible. As a nonabelian group, $SO(3)$ can have higher dimensional irreps, which we are going to discuss later.

Infinitesimal generators

First, let us try to understand what are the infinitesimal generators of $SO(3)$. Following the discussion of $SO(2)$, it is easy to see that to generate rotation around z axis, the infinitesimal generator is

$$X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

such that $R_z(\theta) = e^{-i\theta X_3}$. Similarly, we find that

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{15}$$

such that $R_x(\theta) = e^{-i\theta X_1}$, $R_y(\theta) = e^{-i\theta X_2}$. X_1, X_2, X_3 each generate a subgroup of rotation around x, y and z axes respectively.

Are X_1, X_2 and X_3 enough to generate all $SO(3)$ transformations?

Euler's rotation theorem says: any transformation in $SO(3)$ is equivalent to a single rotation about some axis for a certain angle. It can be shown (Jones page 102-103) that such rotation operation can be written in the form

$$R_{\vec{n}}(\theta) = e^{-i\theta(n_x X_1 + n_y X_2 + n_z X_3)} = e^{-i\theta X_{\vec{n}}} \tag{16}$$

Therefore, the linear combination of X_1, X_2 and X_3 gives the infinitesimal generator of all transformations in $SO(3)$. The inverse of this operation corresponds to rotation around the same axis but with opposite angle $R_{\vec{n}}^{-1}(\theta) = e^{i\theta X_{\vec{n}}}$.

In quantum mechanics, X_1, X_2 and X_3 correspond to angular momentum operator J_x, J_y, J_z and their linear combination $X_{\vec{n}} = n_x X_1 + n_y X_2 + n_z X_3$ corresponds to angular momentum in the $\vec{n}$ direction.

The vector space of linear combinations of X_1, X_2 and X_3 leads to the most important concept in describing the set of continuous groups we are interested in.

$SO(3)$ (and also $SO(2)$) is an example of a Lie group. Its infinitesimal generators form a Lie algebra. The Lie algebra is usually denoted with lower case letters of the name of the group. For example, the Lie algebra of the $SO(2)$ group is $so(2)$ and the Lie algebra of $SO(3)$ is $so(3)$.

There are two important structures of this algebra:

(1) it is a (real) vector space. That is, the linear combination of two infinitesimal generators is (linearly proportional to) an infinitesimal generator;

(2) The commutator of two infinitesimal generators $[X_i, X_j] = X_i X_j - X_j X_i$ is (linearly proportional to) an infinitesimal generator.

Comment:

1. By focusing on the infinitesimal generators, we reduce the study of a continuous group with an infinite number of elements to the study of a finite set, the basis of the Lie algebra.

2. In $SO(3)$, X_1, X_2, X_3 form the basis of the vector space. We have shown above that (1) is true for $SO(3)$. Let's now see that (2) is also true. First $[X_i, X_i] = 0$ which is the infinitesimal generator for doing nothing because $e^{i\theta 0} = I$.

$$[X_1, X_2] = iX_3, [X_2, X_3] = iX_1, [X_3, X_1] = iX_2 \tag{17}$$

From which we can show that for any two linear combinations of X_1, X_2, X_3, we have

$$[\vec{n}^a \cdot \vec{X}, \vec{n}^b \cdot \vec{X}] = i(\vec{n}^a \times \vec{n}^b) \cdot \vec{X} \tag{18}$$

3. Notice that because in general X_i's do not commute, $e^{i\theta \sum_i n_i X_i} \neq \prod_i e^{i\theta n_i X_i}$.

4. The commutator can be thought of as a composition rule between the infinitesimal generators, mapping two such generators to a third one. This composition rule is anti-commuting, $[X_i, X_j] = -[X_j, X_i]$. It satisfies the Jacobi Identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \tag{19}$$

5. The commutator between the infinitesimal generators is very useful in determining the conjugacy classes of the group. For $SO(3)$ we are going to find that rotation operations around different axes with the same angle are conjugate to each other. That is,

$$e^{i\phi \vec{n}_2 \cdot \vec{X}} e^{i\theta \vec{n}_1 \cdot \vec{X}} e^{-i\phi \vec{n}_2 \cdot \vec{X}} = e^{i\theta \vec{n}_3 \cdot \vec{X}} \tag{20}$$

Physically this is very intuitive. We can imagine that $e^{i\phi\vec{n}_2\cdot\vec{X}}$ and $e^{-i\phi\vec{n}_2\cdot\vec{X}}$ maps the vector $\vec{n}_1$ to $\vec{n}_3$ and back. Then under this mapping, the rotation around axis $\vec{n}_1$ for angle θ is mapped to rotation around axis $\vec{n}_3$ for angle θ.

We are not going to derive this result in class, but we are going to work it out in the homework.

6. The conjugacy classes of $SO(3)$ then consist of rotation through the same angle about different axes and can be labelled simply by that angle θ. Correspondingly, characters are just a function of θ. In the three dimensional special orthogonal representation, we have

$$\chi = 2\cos(\theta) + 1 \tag{21}$$

which can be verified directly for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$.

Irreducible representations

Now let's see what irreps the group $SO(3)$ has. There is an infinite number of them, as you might have expected from the continuous nature of this group. Instead of trying to find irreps of the group, we can just find irreps of the infinitesimal generators (the algebra). Then by taking exponentiation, we can recover the group elements.

That is, we are looking for irreducible representations of X_1, X_2, X_3 such that

1. X_1, X_2, X_3 are finite dimensional Hermitian operators.

2. they satisfy the relation $[X_a, X_b] = i\epsilon_{abc}X_c$, where $\epsilon_{abc} = 1$ if $\{ijk\}$ can be obtained from $\{xyz\}$ by a cyclic permutation, $\epsilon_{abc} = -1$ if $\{abc\}$ can be obtained from $\{xyz\}$ by a cyclic permutation and an exchange and $\epsilon_{abc} = 0$ otherwise.

3. once exponentiated, they give rise to the $SO(3)$ group. (This requirement may seem redundant, if the previous two are satisfied. But in fact it is not, as we are going to see later on.)

In physics, this exercise is called 'finding the orbit of an electron in a Hydrogen atom'. Rotation invariance around z axis in the Hydrogen atom implies that every orbit is labeled by a particular value of angular momentum J_z in the z direction. If we take into account the full rotation symmetry of the Hydrogen atom in three dimensional space, then the orbits should be labelled by irreps of the $SO(3)$ group, not just the $SO(2)$ group, and they transform under $SO(3)$ rotation as an irrep.