4 Group representations

4.3 Examples

Example 1: D_3 represented as 2×2 real matrices.

$$\begin{aligned} D(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, D(c^2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \\ D(b_1) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, D(b_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \; D(b_3) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned} \tag{1}$$

Example 2: Circle group as rotation of 2D real vector space

The elements of the circle group can be taken to be continuous rotations of the $2D$ vector space. The group element labeled by θ corresponds to the 2×2 matrix

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

This is again a real orthogonal representation.

Example 3: Circle group as rotation of 3D real vector space

We can also imagine that the circle group represent rotations of $3D$ real vector space around the z axis. The group element labeled by θ corresponds to 3×3 matrix

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Example 4: Circle group as phase factors

With real matrices, the smallest faithful representation for the circle group is two dimensional. However, if we are allowed to use complex numbers, the circle group has a one dimensional faithful representation

$$D(\theta) = e^{i\theta} \tag{4}$$

4.4 Properties

1. Equivalent representations

Definition: Two n dimensional representations $D^{(1)}$ and $D^{(2)}$ of a group G are equivalent if *all* the matrices $D^{(1)}(g)$ and $D^{(2)}(g)$ are related by the same similarity transformation

$$D^{(1)}(g) = SD^{(2)}(g)S^{-1} \tag{5}$$

Comments:

(1) Equivalent representations have the same dimension.

(2) S is independent of g and this relation has to be satisfied for all g.

(3) This equivalence relation is consistent with the group property of the two representations. That is, if $D^{(2)}(g_1)D^{(2)}(g_2) = D^{(2)}(g_1g_2)$, we have

$$D^{(1)}(g_1)D^{(1)}(g_2) = SD^{(2)}(g_1)S^{-1}SD^{(2)}(g_2)S^{-1} = SD^{(2)}(g_1g_2)S^{-1} = D^{(1)}(g_1g_2) \tag{6}$$

(4) The relation between equivalent representations basically amounts to a basis change of the underlying (real or complex) vector space. Equivalent representations are considered the same.

(5) Maschke's Theorem: For finite (or more generally compact) groups, representations are always equivalent to unitary representations.

2. Character

If we want to have a way to characterize representations such that equivalent representations are characterized the same way, we can use the character.

Definition: The character of a representation D of a group G is the set $\chi = \{\chi(g)|g \in G\}$, where $\chi(g)$ is the trace of the representation matrix $D(g)$

$$\chi(g) = Tr(D(g)) = \sum_{i=1}^{n} D(g)_{i,i} \tag{7}$$

Comments:

(1) Matrices related by similarity transformation have the same trace

(2) Equivalent representations have the same character.

(3) Two representations with the same character are equivalent. This is a highly nontrivial and highly useful fact. We are not going to prove it now, but we are going to see how it comes up when we talk about irreducible representations.

(4) $\chi(e) = n$

(5) $\chi(g) = \chi(hgh^{-1})$, that is, conjugate elements have the same trace.

There was a question in class regarding whether different conjugacy classes always have different traces in a faithful representation. The answer is no and here is a counterexample. For the group D_4 (the symmetry group of a regular square), there is a faithful two dimensional irreducible representation generated by

$$D(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, D(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, D(b_2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{8}$$

There are five conjugacy classes with traces $D(e) = 2$, $D(c) = D(c^3) = 0$, $D(b_1) = D(b_3) = 0$, $D(b_2) = D(b_4) = 0$, $D(c^2) = -2$.

3. Reducibility

Example: circle group in three dimensional real vector space

Recall that the representation of the circle group in three dimensional real vector space reads

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

All the $D(\theta)$ matrices take a block diagonal form. When they multiply, different blocks do not talk to each other. For the circle group, this is related to the fact that the z axis is invariant under the rotation operation, while vectors in the $x-y$ plane get rotated into each other. Therefore, this representation effectively 'decomposes' into two separate representations, a $2D$ representation $D^{(2)}$ acting on the $x-y$ coordinates and the $1D$ representation $D^{(1)}$ acting on the z coordinate, which are completely independent.

We write

$$D(\theta) = D^{(1)}(\theta) \oplus D^{(2)}(\theta) \tag{10}$$

$\oplus$ denotes direct sum, which is to combine matrices in block diagonal form.

Definition: A representation of dimension $n+m$ is said to be (completely) reducible or decomposible if there exists a basis transformation S such that $SD(g)S^{-1}$ is of the form

$$SD(g)S^{-1} = \begin{pmatrix} A(g) & O \\ O & B(g) \end{pmatrix} \tag{11}$$

for all $g \in G$, where A, B are sub-matrices of dimension $m \times m$ and $n \times n$ respectively. O is a null matrix (all entries 0) of dimension $n \times m$ and $m \times n$.

Comment: (1) $A(g)$ and $B(g)$ each form a representation of G. $A(g_1)A(g_2) = A(g_1g_2)$, $B(g_1)B(g_2) = B(g_1g_2)$.

(2) A and B could be decomposible themselves and we can continue the process until every block is irreducible.

(3) If we think of representation matrices as linear transformations of vector space, then blocks in a reducible representation correspond to closed subspaces under the linear transformations.

The irreducible representations provide building blocks for general decomposible representations and this is what we are going to focus on when we try to study in more detail about representations.

5 Irreducible Representations

Nick name: irreps.

5.1 Examples

(1) $C_2 = \{e, c\}$: the cyclic group of order two has two irreducible representations.

$$D(e) = 1, D(c) = 1 \text{ and } D(e) = 1, D(c) = -1 \tag{12}$$

Comments:

a. The first one is trivial (everything mapped to 1) while the second one is nontrivial.

b. Both of them are one dimensional.

c. If we consider the two irreps as two two-component vectors, they are orthogonal to each other.

(2) $D_3 = \{e, c, c^2, b_1, b_2, b_3\}$:

The dihedral group of order six has three irreps. The first one is trivial (everything mapped to 1). The two nontrivial ones are:

$$D(c) = 1, D(b_1) = -1 \text{ and } D(c) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, D(b_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

Comments:

a. the D_3 group has two $1D$ irreps and one $2D$ irrep. In particular, the $2D$ representation cannot be decomposed into two $1D$ representations, because no vector in the $2D$ vector space remains invariant under the transformation of the whole group.

b. We have specified the irreps just by the matrices of their generators. If we write out the matrices for all group elements, they are

irreps	e	c	c^2	b_1	b_2	b_3
$D^{(1)}$	1	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1	-1
$D^{(3)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

(14)

c. In $D^{(1)}$ and $D^{(2)}$, different group elements can be mapped to the same matrix (number). In $D^{(3)}$, different group elements are always mapped to different matrices. $D^{(3)}$ is said to be a faithful representation, while $D^{(1)}$ and $D^{(2)}$ are not faithful. The kernel of $D^{(1)}$ is the whole group, the kernel of $D^{(2)}$ is the rotation subgroup (a normal subgroup), and the kernel of $D^{(3)}$ is the identity element.

d. $D^{(1)}$ and $D^{(2)}$ are orthogonal six-component vectors. Is $D^{(3)}$ also orthogonal to them in some way?

The answer is yes. Let's take the element at position i,j $(i,j = 1,2)$ from each of the matrices in $D^{(3)}$.

$$\begin{array}{ccccccc} & e & c & c^2 & b_1 & b_2 & b_3 \\ D^{(3)}[1,1] & 1 & -\frac{1}{2} & -\frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} \\ D^{(3)}[1,2] & 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ D^{(3)}[2,1] & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ D^{(3)}[2,2] & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array} \tag{15}$$

The $D^{(3)}[i,j]$'s are pair-wise orthogonal. Moreover, they are orthogonal to $D^{(1)}$ and $D^{(2)}$ as well!