Physics 129b **Homework 9**

(You don't need to turn in this homework. Solution to this homework has been posted.)

1. $SL(2,\mathbb{C})$ as double cover of $SO(3,1)$

Consider the $(1/2,0)$ representation of the Lie algebra of $SO(3,1)$.

(1) What are the generators X_a^+ and X_a^- which form two separate $su(2)$ sub-algebra?

(2) What are the spatial rotation and boost generators? ($X_a = X_a^+ + X_a^-$, $Y_a = -i(X_a^+ - X_a^-)$)

(3) Now take the exponential of linear combinations of X_a and Y_a with real coefficients. Show that the transformations generated in this way are special linear complex matrices of dimension two. (hint: the Pauli matrices σ_x, σ_y and σ_z form a basis of all traceless 2×2 matrices.) Comments: It is also true that all special linear complex matrices of dimension two can be generated in this way, but it is more complicated to show and not required.

2. Electromagnetic tensor

The electromagnetic tensor $F^{\mu\nu}$ is antisymmetric. Written in matrix form

$$F = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

Under Lorentz transformation, F transforms as

$$F' = \Lambda F \Lambda^T \tag{2}$$

(1) We can take out the six independent nonzero components of F: E_x/c, E_y/c, E_z/c, B_x, B_y, B_z and write the Lorentz transformation as 6×6 matrices. What is the form of the matrix for spatial rotation around x, y and z direction? What is the form of the matrix for Lorentz boost in x, y and z direction? (You can use Mathematica to help with the calculation. Also you can do the calculation for one direction only and then guess the form of the transformation in other directions.)

(2) What are the 6×6 infinitesimal generators for spatial rotation around x, y and z direction and Lorentz boost in x, y and z direction?

(3) Is this representation of $SO(3,1)$ reducible? If so, which irreducible blocks does it contain? (Find the Casimir operator of the two $su(2)$ sub-algebras and diagonalize them.)

3. Nonabelian gauge theory

Consider a particle forming a representation j under $SU(2)$ and coupled to the corresponding gauge field W_μ. The Schrodinger equation for the particle is

$$i\frac{\partial}{\partial t}\psi = \left(\frac{1}{2m}\sum_{s=1}^{3}\left(-i\frac{\partial}{\partial x_s} - W_s\right)^2 + W_0\right)\psi \tag{3}$$

Here ψ is a $2j+1$ dimensional vector and W are $(2j+1)\times(2j+1)$ matrices $W=\sum_k w^k J_k$.

Show that the form of the equation remains invariant under transformation

$$\psi \to U\psi, W_\mu \to UW_\mu U^{-1} - i(\partial_\mu U)U^{-1} \tag{4}$$

where $\mu = 0,1,2,3$, $\partial_0 = -\partial/\partial t$, $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_3 = \partial/\partial z$, $U = e^{i\sum_k a_k J_k}$.