Physics 129b  

Homework 7  

Due 02/27/20 by 5pm

1. Conjugacy classes of $SO(3)$

In this problem we are going to show that the conjugacy classes of $SO(3)$ consist of rotations around different axes but with the same angle. That is, we are going to demonstrate the relation

$$e^{-i\theta \vec{n}_2 \cdot \vec{J}} e^{i\theta \vec{n}_1 \cdot \vec{J}} e^{i\phi \vec{n}_2 \cdot \vec{J}} = e^{i\theta \vec{n}_3 \cdot \vec{J}}$$  \hspace{1cm} (1)$$

where $\vec{n}_3 = R_{\vec{n}_2}(\phi) \vec{n}_1$. $R_{\vec{n}_2}(\phi)$ is the three dimensional real orthogonal representation of rotation around $\vec{n}_2$ through angle $\phi$ and it acts on the three dimensional real vector space of $\vec{n}$.

(1) Define $S_y^x(\phi) = e^{-i\phi J_y} J_x e^{i\phi J_y}$, $S_z^x(\phi) = e^{-i\phi J_z} J_x e^{i\phi J_z}$. Take the derivative of $S_y^x(\phi)$ with respect to $\phi$ and show that $\partial_{\phi} S_y^x(\phi) = S_z^x(\phi)$. (hint: $\frac{\partial}{\partial \phi} e^{i\phi J_x} = e^{i\phi J_x} i J_x$).

(2) Similarly show that $\partial_{\phi} S_z^x(\phi) = -S_y^x(\phi)$

(3) Combining these two equations and use the boundary condition that $S_y^x(0) = J_y$, $S_z^x(0) = J_z$ to obtain solutions $S_y^x(\phi) = \cos \phi J_y + \sin \phi J_z$, $S_z^x(\phi) = \cos \phi J_z - \sin \phi J_y$. That is, $J_y$ and $J_z$ rotate into each other under the conjugation of $e^{-i\phi J_z}$ as if they are the $y$ and $z$ component of a three dimensional vector transforming under rotation around $x$ axis through angle $\phi$.

(4) Taking into account the fact that $S_z^x(\phi) = e^{-i\phi J_z} J_x e^{i\phi J_z} = J_x$, show that $S_z^x(\phi) = e^{-i\phi J_z} \left( \vec{n} \cdot \vec{J} \right) e^{i\phi J_z} = (R_x(\phi) \vec{n}) \cdot \vec{J}$, where $R_x(\phi)$ is the three dimensional orthogonal matrix representing rotation around $x$ axis through angle $\phi$ and it acts on the three dimensional vector $\vec{n}$.

Now convince yourself that this relation works not only for $x$ axis rotation but for rotation around arbitrary axis as well. That is, $e^{-i\theta \vec{n}_2 \cdot \vec{J}} \left( \vec{n}_1 \cdot \vec{J} \right) e^{i\phi \vec{n}_2 \cdot \vec{J}} = (R_{\vec{n}_2}(\phi) \vec{n}_1) \cdot \vec{J}$. You don’t need to show work for this part, but if you want to work everything out, it is helpful to choose three vectors $\vec{n}_2$, $\vec{n}_3$, $\vec{n}_3$ which are orthogonal to each other and build up the relation from there.

(5) Now use the above relation and show that

$$e^{-i\theta \vec{n}_2 \cdot \vec{J}} e^{i\theta \vec{n}_1 \cdot \vec{J}} e^{i\phi \vec{n}_2 \cdot \vec{J}} = e^{i\theta \vec{n}_3 \cdot \vec{J}}$$ \hspace{1cm} (2)$$

where $\vec{n}_3 = R_{\vec{n}_2}(\phi) \vec{n}_1$. (hint: decompose $e^{i\theta \vec{n}_1 \cdot \vec{J}}$ as a polynomial series.)

2. The $SU(2)$ Group

In this problem, you will show that

$$\sigma_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (3)$$

are the infinitesimal generators of the group of special unitary matrices of dimension two.

(1) For a two dimension complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if it is unitary and has determinant 1, what conditions do $a, b, c, d$ have to satisfy?

(2) Find a parameterization for all special unitary matrices of dimension two.
(3) Now take a linear combination of $\sigma_x$, $\sigma_y$ and $\sigma_z$ as $\sigma_{\vec{n}} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$, where $\vec{n} = (n_x, n_y, n_z)$ is a real unit vector. Show that $(\sigma_{\vec{n}})^2 = \frac{1}{4} I_2$.

(4) Show that

$$R_{\vec{n}}(\theta) = e^{i\theta \sigma_{\vec{n}}} = \cos \left( \frac{\theta}{2} \right) + i 2 \sin \left( \frac{\theta}{2} \right) \sigma_{\vec{n}}$$

Hint: use the Taylor expansion of $e^{i\theta \sigma_{\vec{n}}}$.

(5) Show that the set of matrices you obtain in (2) coincide with the set of matrices you obtain in (4).

(6) Show that $\sigma_x$, $\sigma_y$ and $\sigma_z$ satisfy the same commutation relation as the infinitesimal generators $J_x$, $J_y$ and $J_z$ for $SO(3)$.

(7) Is the group $SU(2)$ isomorphic to $SO(3)$?

3. P orbital

The $p$ orbital of an electron has three components $p_x, p_y, p_z$. The angular part of the wave function in these orbitals are given by

$$\psi_x(\theta, \phi) = N \sin \theta \cos \phi, \quad \psi_y(\theta, \phi) = N \sin \theta \sin \phi, \quad \psi_z(\theta, \phi) = N \cos \theta$$

where $\theta$ and $\phi$ are the usual polar coordinates and $N$ is some pre-factor independent of $\theta$ and $\phi$.

(1) Show that the three wave functions are orthogonal to each other. Hint: inner product of wave functions in polar coordinates is given by

$$\int_0^\pi \int_0^{2\pi} \psi_x^*(\theta, \phi) \psi(\theta, \phi) \sin \theta d\theta d\phi$$

(2) The angular momentum operator in $z$ direction is given by $J_z = -i \frac{\partial}{\partial \phi}$. How is it represented as a three dimensional matrix in the Hilbert space spanned by $\psi_x, \psi_y$ and $\psi_z$?

(3) Show that if the electron moves on a spherical shell of radius $r$, then $\psi_x, \psi_y$ and $\psi_z$ are proportional to the $x, y, z$ coordinates of the electron respectively.

(4) Which irreducible representation do $p_x, p_y$ and $p_z$ orbitals form under $SO(3)$ rotation?