Physics 129b **Homework 7** **Due 02/27/20 by 5pm**

1. Conjugacy classes of $SO(3)$

In this problem we are going to show that the conjugacy classes of $SO(3)$ consist of rotations around different axes but with the same angle. That is we are going to demonstrate the relation

$$e^{-i\phi\vec{n}_2\cdot\vec{J}}e^{i\theta\vec{n}_1\cdot\vec{J}}e^{i\phi\vec{n}_2\cdot\vec{J}} = e^{i\theta\vec{n}_3\cdot\vec{J}} \tag{1}$$

where $\vec{n}_3 = R_{\vec{n}_2}(\phi)\vec{n}_1$. $R_{\vec{n}_2}(\phi)$ is the three dimensional real orthogonal representation of rotation around $\vec{n}_2$ through angle ϕ and it acts on the three dimensional real vector space of $\vec{n}$.

(1) Define $S^x_y(\phi) = e^{-i\phi J_x}J_y e^{i\phi J_x}$, $S^x_z(\phi) = e^{-i\phi J_x}J_z e^{i\phi J_x}$. Take the derivative of $S^x_y(\phi)$ with respect to ϕ and show that $\frac{\partial}{\partial\phi}S^x_y(\phi) = S^x_z(\phi)$. (hint: $\frac{\partial}{\partial\phi}e^{i\phi J_x} = e^{i\phi J_x}iJ_x$).

(2) Similarly show that $\frac{\partial}{\partial\phi}S^x_z(\phi) = -S^x_y(\phi)$

(3) Combining these two equations and use the boundary condition that $S^x_y(0) = J_y$, $S^x_z(0) = J_z$ to obtain solutions $S^x_y(\phi) = \cos\phi J_y + \sin\phi J_z$, $S^x_z(\phi) = \cos\phi J_z - \sin\phi J_y$. That is, J_y and J_z rotate into each other under the conjugation of $e^{-i\phi J_x}$ as if they are the y and z component of a three dimensional vector transforming under rotation around x axis through angle ϕ.

(4) Taking into account the fact that $S^x_x(\phi) = e^{-i\phi J_x}J_x e^{i\phi J_x} = J_x$, show that $S^x_{\vec{n}}(\phi) = e^{-i\phi J_x}\left(\vec{n}\cdot\vec{J}\right)e^{i\phi J_x} = (R_x(\phi)\vec{n})\cdot\vec{J}$, where $R_x(\phi)$ is the three dimensional orthogonal matrix representing rotation around x axis through angle ϕ and it acts on the three dimensional vector $\vec{n}$.

Now convince yourself that this relation works not only for x axis rotation but for rotation around arbitrary axis as well. That is, $e^{-i\phi\vec{n}_2\cdot\vec{J}}\left(\vec{n}_1\cdot\vec{J}\right)e^{i\phi\vec{n}_2\cdot\vec{J}} = (R_{\vec{n}_2}(\phi)\vec{n}_1)\cdot\vec{J}$. You don't need to show work for this part, but if you want to work everything out, it is helpful to choose three vectors $\vec{n}_2$, $\vec{n}_2^a$, $\vec{n}_2^b$ which are orthogonal to each other and build up the relation from there.

(5) Now use the above relation and show that

$$e^{-i\phi\vec{n}_2\cdot\vec{J}}e^{i\theta\vec{n}_1\cdot\vec{J}}e^{i\phi\vec{n}_2\cdot\vec{J}} = e^{i\theta\vec{n}_3\cdot\vec{J}} \tag{2}$$

where $\vec{n}_3 = R_{\vec{n}_2}(\phi)\vec{n}_1$. (hint: decompose $e^{i\theta\vec{n}_1\cdot\vec{J}}$ as a polynomial series.)

2. The $SU(2)$ Group

In this problem, you will show that

$$\sigma_x = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \sigma_y = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \sigma_z = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3}$$

are the infinitesimal generators of the group of special unitary matrices of dimension two.

(1) For a two dimension complex matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, if it is unitary and has determinant 1, what conditions do a, b, c, d have to satisfy?

(2) Find a parameterization for all special unitary matrices of dimension two.

(3) Now take a linear combination of σ_x, σ_y and σ_z as $\sigma_{\vec{n}} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, where $\vec{n} = (n_x, n_y, n_z)$ is a real unit vector. Show that $(\sigma_{\vec{n}})^2 = \frac{1}{4}I_2$.

(4) Show that

$$R_{\vec{n}}(\theta) = e^{i\theta\sigma_{\vec{n}}} = \cos\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\sigma_{\vec{n}} \tag{4}$$

Hint: use the Taylor expansion of $e^{i\theta\sigma_{\vec{n}}}$.

(5) Show that the set of matrices you obtain in (2) coincide with the set of matrices you obtain in (4).

(6) Show that σ_x, σ_y and σ_z satisfy the same commutation relation as the infinitesimal generators J_x, J_y and J_z for $SO(3)$.

(7) Is the group $SU(2)$ isomorphic to $SO(3)$?

3. P orbital

The p orbital of an electron has three components p_x, p_y, p_z. The angular part of the wave function in these orbitals are given by

$$\psi_x(\theta,\phi) = \mathcal{N}\sin\theta\cos\phi, \quad \psi_y(\theta,\phi) = \mathcal{N}\sin\theta\sin\phi, \quad \psi_z(\theta,\phi) = \mathcal{N}\cos\theta \tag{5}$$

where θ and ϕ are the usual polar coordinates and $\mathcal{N}$ is some pre-factor independent of θ and ϕ.

(1) Show that the three wave functions are orthogonal to each other. Hint: inner product of wave functions in polar coordinates is given by $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\psi^*(\theta,\phi)\tilde{\psi}(\theta,\phi)\sin\theta d\theta d\phi$

(2) The angular momentum operator in z direction is given by $J_z = -i\frac{\partial}{\partial\phi}$. How is it represented as a three dimensional matrix in the Hilbert space spanned by ψ_x, ψ_y and ψ_z?

(3) Show that if the electron moves on a spherical shell of radius r, then ψ_x, ψ_y and ψ_z are proportional to the x, y, z coordinates of the electron respectively.

(4) Which irreducible representation do p_x, p_y and p_z orbitals form under $SO(3)$ rotation?