Throughput and Delay Scaling of General Cognitive Networks

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Abstract—There has been recent interest within the networking research community to understand how performance scales in cognitive networks with overlapping \( n \) primary nodes and \( m \) secondary nodes. Two important metrics, i.e., throughput and delay, are studied in this paper. We first propose a simple and extendable decision model, i.e., the hybrid protocol model, for the secondary nodes to exploit spatial gap among primary transmissions for frequency reuse. Then a framework for general cognitive networks is established based on it to analyze the occurrence of transmission opportunities for secondary nodes. We show that in the case that the transmission range of the secondary network is smaller than that of the primary network in order, as long as the primary network operates in a round-robin TDMA fashion or employs a routing scheme that flows independently choose relays, the hybrid protocol model suffice to guide the secondary network to achieve the same throughput and delay scaling as a standalone network, without harming the transmissions of the primary network. Our approach is general in the sense that we only make a few weak assumptions on both networks, and therefore obtain a wide variety of results. We show secondary networks can obtain the same order of throughput and delay as standalone networks when primary networks are classic static networks, networks with random walk mobility, hybrid networks, CSMA networks or networks with general mobility. Our work presents a relatively complete picture of the performance scaling of cognitive networks and provides fundamental insight on the design of them.

I. INTRODUCTION

The electromagnetic radio spectrum is a natural resource, the use of which by transmitters and receivers is licensed by governments. Today, as wireless applications demand ever more bandwidth, efficient usage of spectrum is becoming necessary. However, recent measurement [1] observed a severe under-utilization of the licensed spectrum, implying the non-optimality of the current scheme of spectra management. As a remedy, the Federal Communications Commission (FCC) has recently recommended [1], [2] more flexibility in spectrum assignment so that new regulations would allow for devices which are able to sense and adapt to their spectral environment, such as cognitive radios, to become secondary or cognitive users. Cognitive users could opportunistically access the spectrum originally licensed to primary users, in a manner that their transmissions will not affect the performance of primary users. Primary users have a higher priority to the spectrum; they may be legacy devices and may not cooperate with secondary users. The overlapping primary network and secondary network together form the cognitive network.

This paper focuses on the performance scaling analysis of cognitive networks. The scaling behaviors of wireless networks have attracted tremendous interest in the networking community for long. They provide fundamental insight into whether a system is feasible for large scale deployment and how well the performance will tend to be as more users join. This track of research is initiated by Gupta and Kumar, whose landmark paper [3] showed the per-node throughput capacity of a wireless ad hoc network with \( n \) users scales as \( O(1/\sqrt{n}) \). Following works have covered a wide variety of ad hoc networks with different features, such as mobile ad hoc networks (MANETs) [5], [6], hybrid networks [7], [8], networks that implement distributed CSMA protocol [9], etc. Performance metrics other than capacity are also studied, among which delay and its optimal tradeoff with throughput are of critical importance [4], [10].

As most related works, under Gaussian channel model, Jeon et al. [11] considered the capacity scaling of a cognitive network where the number of secondary users, \( m \), is larger than \( n \) in order. Under similar assumption, Yin et al. [12] developed the throughput-delay tradeoff of both primary and secondary networks and Wang et al. [13] studied the cases of multicast traffic pattern. Interestingly, all these works showed that both the primary network and the secondary network can achieve the same performance bounds as they are standalone networks.

All previous works on cognitive networks [11], [12], [13] considered some particular scenarios. They first assumed some particular primary networks with specific scheduling and routing protocols, then proposed the communication schemes for secondary users accordingly, and lastly showed such schemes suffice to achieve the same performance bounds as standalone networks. However, a key principle of cognitive networks is that primary users are spectrum license holders and may operate at their own will without considering secondary nodes. Therefore, though assuming a specific primary network can simplify the problem, the results will heavily depend on the communication schemes of the primary network, which is often unmanageable.

That motivates us to study a general cognitive network in...
this paper. Our major contributions are three folds. First, we characterize the regime that cognitive networks can achieve the same order of throughput and delay scaling as standalone networks. Secondly, we propose a simple decision model for secondary users to identify transmission opportunities and based on it establish a framework with which schemes of standalone networks can be readily extended to secondary networks. Thirdly, we apply the framework to various specific scenarios and show that secondary networks can obtain the same order of throughput and delay scaling as standalone networks when primary networks are classic static networks, networks with random walk mobility, hybrid networks, CSMA networks or networks with general mobility.

In particular, the following conditions are sufficient for a general cognitive network to achieve the same throughput and delay bounds as standalone networks.

A1) The cognitive network is subject to the physical interference model. The primary network operates at a SINR level larger than the threshold for successful reception by some small allowance.

A2) The primary network is scheduled in a round-robin TDMA manner or traffic flows of the primary network choose relays independently for routing.

A3) Scheduling schemes of secondary network follow \( r_{\text{max}} = o(R_{\text{min}}) \) and \( r_{\gamma}^{\text{max}} = o(R_{\text{min}}^{\gamma}/R_{\text{max}}^{\gamma}) \) with high probability, where \( R \) and \( r \) are the transmission ranges of primary and secondary networks, and \( \gamma \) is the path loss exponent. Intuitively, condition A1 ensures that primary transmission links are neither too dense nor too vulnerable so that there exist opportunities for secondary users. Such opportunities will frequently appear, as a consequence of A2. The first condition of A3 is the generalization of the condition \( m = \omega(n) \) in related works. The second equation is more technical. It characterizes the case that the scheduling of primary networks is somewhat “homogeneous” such that there exists a simple rule for opportunity decision. A1 is based on the physical model, and in the last part of this paper we also extend it to Gaussian channel model.

We note this paper is not merely a generalization of results from previous works. Our work shows the fact that cognitive networks, and especially secondary networks, can achieve the same throughput and delay scaling as standalone networks, is mainly determined by the underlying interference model, and only weakly relies on the specific settings such as scheduling and routing protocols of primary networks. Such insight is fundamental and implies that for quite general cases, “cognitive” will not be a handicap to performance scaling.

The paper is organized as follows. In Section 2 we introduce system models and formalize the operation rules of cognitive networks. We propose the hybrid protocol model and establish its physical feasibility in Section 3. Section 4 identifies the conditions under which the secondary network will have plenty transmission opportunities if scheduled according to the hybrid protocol model. We present our final results in Section 5, and Section 6 concludes the paper.

II. System Model

Throughout this paper we denote the probability of an event \( E \) as \( \Pr(E) \) and say \( E \) happens with high probability (w.h.p.) if \( \lim_{n \to \infty} \Pr(E) = 1 \). A number of parameters and constants will be needed and by convention we use \( c_i \) to denote some positive constants and \( \{C_i\} \) some parameters dependent on \( n \).

A. Network Topology

We define the network extension \( \mathcal{O} \) to be a unit square. The size normalization is a technical assumption commonly adopted in previous works [3], [5]. Two kinds of nodes, i.e., the primary nodes and the secondary nodes, overlap in \( \mathcal{O} \). They share the same time, space and frequency dimensions. In particular, we assume \( n \) primary nodes are independently and identically distributed (i.i.d.) in \( \mathcal{O} \) according to uniform distribution, and so do the \( m \) secondary users. Their positions are \( \{X_i\}_{i=1}^n \) and \( \{Y_j\}_{j=1}^m \), \( \forall i, j, X_i \neq Y_j \). At times we may denote a node by its position, i.e., we refer to primary node \( i \) and secondary node \( j \) as \( X_i \) and \( Y_j \), and let \( |X_i - Y_j| \) be the distance between them. Two types of nodes form their respective networks, the primary network and the secondary network. In each network nodes are randomly grouped into source-destination (S-D) pairs, such that every node is both source and destination, with traffic rate \( \lambda \). Equivalently we can describe the traffic pattern in matrix form \( \mathbf{A} \), where \( \mathbf{A} = [\mathbf{A}_{sd}] \) is a random permutation matrix \(^2\) with \( \mathbf{A}_{sd} \in \{0, 1\} \). Note that we do not consider cross-network traffic. We use index \( p \) and \( s \) to distinguish quantities between primary nodes and secondary nodes when needed, for example, \( \lambda_p \) and \( \lambda_s \).

B. Communication Model

We assume all nodes share a wireless channel with bandwidth \( W \) bps. Assume that path loss exponent is \( \gamma > 2 \), then the normalized channel gain is \( G(|X - Y|) = |X - Y|^{-\gamma} \). Besides, wireless transmission may be subject to failures or collisions caused by noise or interference. To judge whether a direct wireless link is feasible, we have the following physical model, whose well-known prototype is proposed in [3]:

The Physical Model: Let \( \{X_i; i \in T^{(p)}\} \) and \( \{Y_j; j \in T^{(s)}\} \) be the subsets of nodes simultaneously transmitting at some time instant. Let \( P \) be the uniform power level of primary network, and \( P_j \) be the power chosen by secondary node \( Y_j \), for \( j \in T^{(s)} \). Then, For primary network, the transmission from node \( X_i \) is successfully received by node \( X_j \) if

\[
\frac{PG(|X_i - X_j|)}{N + \sum_{k \in T^{(p)} \setminus \{i\}} PG(|X_k - X_j|) + \sum_{l \in T^{(s)}} P_l G(|Y_l - X_j|)} \geq \alpha \tag{1}
\]

where \( N \) is ambient noise and constant \( \alpha \) characterize the minimum signal-to-interference-plus-noise ratio (SINR) necessary for successful receptions for primary nodes. For secondary network, the transmission from node \( Y_i \) is successfully received

\(^2\) \( \mathbf{A} = [\mathbf{A}_{sd}] \) is a permutation matrix if \( \forall s, d, \mathbf{A}_{sd} \in \{0, 1\}; \forall d, \sum_s \mathbf{A}_{sd} = 1; \forall s, \sum_d \mathbf{A}_{sd} = 1 \)
Definition

The feasible family of primary decision model is denoted as $\mathcal{D}(\alpha, \beta)$. The feasible family of hybrid protocol model is denoted as $\mathcal{H}(\Delta_p, \Delta_{ps}, \Delta_{sp}, \Delta_s)$. The feasible family of physical model is denoted as $\mathcal{P}(\alpha, \beta, \epsilon)$. Therefore the assumptions made about the aggregate interference and SINR, but the following lemma relates it to a simpler pairwise model. This alternative model is known as protocol model in literature and often plays a key role in network design.

### C. Operation Rules

The operation rules are the key that make cognitive networks different from normal ad hoc networks. Though primary and secondary users overlap and share the channel, they are different essentially because of their behavior. In principle, primary nodes are spectrum license holders and have the priority to access the channel. It is followed by two important implications. First, primary nodes may operate at their own will without considering secondary nodes. They may be legacy devices running on legacy protocols, which are fixed and unmanageable. Therefore the assumptions made about primary networks should be as few and general as possible. Besides, the secondary network, which is opportunistic in nature, should control its interference to the primary network and prevent deteriorating the performance of primary users.

The challenge is, primary scheduler may not alter its protocol due to the existence of secondary network and its decision model could be different from physical model (1), i.e., the interference term from secondary network in the denominator is not available. But in order to leave some margin for secondary nodes, it is necessary for the decision model to operate at a SINR larger than $\alpha$ by an allowance $\epsilon$.

**Operation Rule 1. Decision model for primary network:** The primary scheduler considers the transmission from $X_i$ to $X_j$ to be feasible if:

$$\frac{PG(|X_i - X_j|)}{N + \sum_{k \neq i}^{T(p)} PG(|X_k - X_j|)} \geq \alpha + \epsilon$$

The feasible family of primary decision model is denoted as $\mathcal{D}(\alpha + \epsilon)$.

Then, as the operation rule, secondary nodes should guarantee that feasible state under decision model $\mathcal{D}$ above should be indeed feasible under physical model.

**Operation Rule 2.** Let $S^{(p)}$ and $S^{(s)}$ be the sets of active primary links and active secondary links. If $S^{(p)} \in \mathcal{D}(\alpha + \epsilon)$, then $S^{(p)} \cup S^{(s)} \in \mathcal{D}(\alpha, \beta)$, w.h.p.

### TABLE I: Important Notations

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$X_i$</td>
<td>position of primary user $i$</td>
</tr>
<tr>
<td>$Y_j$</td>
<td>position of secondary user $j$</td>
</tr>
<tr>
<td>$\mathcal{D}(\alpha, \beta)$</td>
<td>feasible family of physical model</td>
</tr>
<tr>
<td>$\mathcal{D}(\alpha + \epsilon)$</td>
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<tr>
<td>$\mathcal{P}(\Delta_p, \Delta_{ps}, \Delta_{sp}, \Delta_s)$</td>
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<td>$\mathcal{H}(\Delta_p, \Delta_{ps}, \Delta_{sp}, \Delta_s)$</td>
<td>feasible family of hybrid protocol model</td>
</tr>
<tr>
<td>$S^{(p)}$</td>
<td>set of active primary links</td>
</tr>
<tr>
<td>$S^{(s)}$</td>
<td>set of active secondary links</td>
</tr>
<tr>
<td>$S$</td>
<td>$S^{(p)} \cup S^{(s)}$</td>
</tr>
<tr>
<td>$r_i$</td>
<td>Tx range of active link $(X_i, X_{Rx(i)})$</td>
</tr>
<tr>
<td>$r_j$</td>
<td>Tx range of active link $(Y_j, Y_{Rx(j)})$</td>
</tr>
<tr>
<td>$P$</td>
<td>Tx power of primary network</td>
</tr>
<tr>
<td>$P_j$</td>
<td>Tx power of link $(Y_j, Y_{Rx(j)})$</td>
</tr>
</tbody>
</table>

### D. Capacity Definition

**Definition 1.** Feasible throughput: Per-node throughput $g(n)$ of primary network is said to be feasible if there is a spatial and temporal scheme for scheduling transmissions, such that by operating the primary network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission opportunities, every primary source can send $g(n)$ bps to its destination.

**Definition 2.** Asymptotic per-node capacity $\lambda_p(n)$ of the primary network is said to be $\Theta(g(n))$ if there exist two positive constant $c$ and $c'$ such that:

$$\lim_{n \to \infty} \Pr \{ \lambda_p(n) = cg(n) \text{ is feasible} \} = 1$$

$$\lim_{n \to \infty} \Pr \{ \lambda_p(n) = c'g(n) \text{ is feasible} \} < 1$$

Similarly we can define the asymptotic per-node capacity of secondary network $\lambda_s(m)$.

### III. IDENTIFYING OPPORTUNITIES: THE HYBRID PROTOCOL MODEL

In this section we consider the problem of how to schedule links in the cognitive network under interference constraint. Recall from operation rules that primary nodes are unmanageable, so in fact the key issue is the schedule strategy for the secondary network. In specific, we will face two challenges: first, how to ensure that secondary transmissions are harmless to the primary network and; secondly, how to establish a secondary link given uncontrollable interference from the primary network. Our goal is to design a practical decision model for the secondary users to address these two seemingly contradictory challenges at the same time. Intuitively, that is to say we should find simple rules for secondary nodes to hunt and exploit opportunities in the network.

### A. Hybrid Protocol Model

Since we assume the primary network to be a general network which operates according to decision model $\mathcal{D}(\alpha + \epsilon)$, it is our starting point. $\mathcal{D}$ is of physical concern and cares about the aggregate interference and SINR, but the following lemma relates it to a simpler pairwise model. This alternative model is known as protocol model in literature and often plays
the role as interference model. But here we use it as a tool to characterize the relative position of active primary nodes.

**Definition 3. Protocol Model** for primary network: A transmission from $X_i$ to $X_j$ is feasible if

$$|X_k - X_j| \geq (1 + \Delta_p)|X_i - X_j|, \quad \forall k \in T^{(p)}$$

where $\Delta_p$ defines the guard zone for the primary network. The corresponding feasible family is noted as $\mathcal{P}_p(\Delta_p)$. Likewise we define protocol model $\mathcal{D}_s(\Delta_s)$ for the secondary network.

**Lemma 1.** If $S^{(p)} \in \mathcal{P}(\alpha + \epsilon)$ and $\Delta_p \leq (\alpha + \epsilon)^\frac{1}{2} - 1$, then $S^{(p)} \in \mathcal{P}_p(\Delta_p)$.

**Proof:** Let $S^{(p)} \in \mathcal{P}$, $\forall (X_i, X_j) \in S^{(p)}$ and $\forall k \neq i, k \in T^{(p)}$, holds,

$$\frac{|X_i - X_j| - \gamma}{|X_k - X_j| - \gamma} \geq \frac{P|X_i - X_j| - \gamma}{N + P \sum_{i \in T^{(p)}} |X_i - X_j| - \gamma} \geq \alpha + \epsilon$$

therefore $|X_k - X_j| \geq (\alpha + \epsilon)^\frac{1}{2}|X_i - X_j|$, set $\Delta_p \leq (\alpha + \epsilon)^\frac{1}{2} - 1$, then $S^{(p)} \in \mathcal{P}_p(\Delta_p)$.

Since $\mathcal{P}_p \supseteq \mathcal{P}$, i.e., $\mathcal{P}_p$ captures all degrees of freedom of the primary network, and considering the simplicity of the form of protocol model, it motivates us to define a new **hybrid protocol model** $\mathcal{H}$ based on $\mathcal{P}_p$ and $\mathcal{D}_s$, to be the decision model for secondary network.

**Definition 4. The Hybrid Protocol Model** with feasible family $\mathcal{H}(\Delta_p, \Delta_{ps}, \Delta_{sp}, \Delta_s)$: $\forall S \in \mathcal{H}$, let $S^{(p)} = \{(X_i, X_j) \in S\}$ and $S^{(s)} = \{(Y_i, Y_j) \in S\}$, then $S^{(p)} \in \mathcal{P}_p(\Delta_p)$, $S^{(s)} \in \mathcal{D}_s(\Delta_s)$. Further, $\forall (X_i, X_j) \in S^{(p)}$,

$$|Y_k - X_j| \geq (1 + \Delta_{sp})|X_i - X_j|, \forall k \in T^{(s)}$$

and $\forall (Y_i, Y_j) \in S^{(s)}$,

$$|X_k - Y_j| \geq (1 + \Delta_{ps})|Y_i - Y_j|, \forall k \in T^{(p)}$$

where $\Delta_{sp}$ and $\Delta_{ps}$ define the inter-network guard zone.

The hybrid protocol model only depends on pairwise distance between transmitters and receivers. Such simplicity will facilitate our analysis in the next section. Besides, it is compatible with the classic protocol interference model. Thus rich communication schemes and results based on protocol model can be easily extended to cognitive networks, as will be shown in Section 5.

In the following we should prove that if $\mathcal{H}$ is used as decision model for secondary nodes, it will comply with Operation Rule 2. This involves correctly tuning the parameters $\Delta_p$, $\Delta_{ps}$, $\Delta_{sp}$, $\Delta_s$ and $\{P_j\} \in T^{(i)}$.

**B. Interference at Primary Nodes**

We first address the challenge that primary transmissions should not be interrupted by secondary nodes. The main task is to bound the interference from the secondary network. We start with a useful property of the hybrid protocol model.

**Lemma 2.** Given arbitrary $Z_i$, $Z_j$, $Z_k$, $Z_l \in \mathcal{O}$, if $(Z_i, Z_j)$, $(Z_k, Z_l)$ are active links (primary or secondary), and $|Z_k - Z_j| \geq (1 + \Delta_1)|Z_i - Z_j|$, $|Z_l - Z_i| \geq (1 + \Delta_2)|Z_k - Z_l|$, then the $\Delta_1|Z_i - Z_j|/2$ and $\Delta_2|Z_k - Z_l|/2$ neighborhood of the line segment joining $Z_i, Z_j$ and $Z_k, Z_l$ are disjoint.

**Proof:** Let $E$ and $F$ be two arbitrary points on line segment $Z_iZ_j$ and $Z_kZ_l$ by triangle inequity,

$$|Z_i - E| + |E - F| + |F - Z_l| + |Z_k - F| + |F - E| + |E - Z_j| \geq |Z_i - Z_l| + |Z_k - Z_j|$$

Since $Z_i, E, Z_j$ are collinear points,

$$|Z_i - Z_j| + |Z_k - Z_l| + 2|E - F| \geq |Z_i - Z_l| + |Z_k - Z_j|$$

Substituting the lemma condition to the R.H.S.,

$$|Z_i - Z_j| + |Z_k - Z_l| + 2|E - F| \geq |Z_i - Z_l| + |Z_k - Z_j|$$

therefore,

$$|E - F| \geq \frac{\Delta_1}{2}|Z_i - Z_l| + \frac{\Delta_2}{2}|Z_k - Z_l|$$

Now we prove the lemma by contradiction. Suppose the two neighborhoods overlap, then there exist point $X$ on $Z_iZ_j$, $Y$ on $Z_kZ_l$ and $Z$ such that $|Z - X| < \frac{\Delta_1}{2}|Z_i - Z_l|$ and $|Z - Y| < \frac{\Delta_2}{2}|Z_k - Z_l|$. Then,

$$|X - Y| < |Z - X| + |Z - Y| < \frac{\Delta_1}{2}|Z_i - Z_l| + \frac{\Delta_2}{2}|Z_k - Z_l|$$

which is a contradiction.

**Corollary 1.** Under hybrid protocol model,

- If $(X_i, X_j)$ and $(X_k, X_l)$ are active primary links, the $\Delta_p|X_i - X_j|/2$ neighborhood of line segment $X_iX_j$ and $\Delta_p|X_k - X_l|/2$ neighborhood of $X_kX_l$ are disjoint.
- If $(Y_i, Y_j)$ and $(Y_k, Y_l)$ are active secondary links, the $\Delta_s|Y_i - Y_j|/2$ neighborhood of line segment $Y_iY_j$ and $\Delta_s|Y_k - Y_l|/2$ neighborhood of $Y_kY_l$ are disjoint.
- If $(X_i, X_j)$ is active primary link and $(Y_k, Y_l)$ is active secondary link, the $\Delta_{sp}|X_i - X_j|/2$ neighborhood of line segment $X_iX_j$ and $\Delta_{ps}|Y_k - Y_l|/2$ neighborhood of $Y_kY_l$ are disjoint.

For active link $(X_i, X_{R_{RX(i)}})$ and $(Y_j, Y_{R_{RX(j)}})$, where function $Rx$ indicates the index of receiver, let $R_i = |X_i - X_{R_{RX(i)}}|$ and $r_j = |Y_j - Y_{R_{RX(j)}}|$. Let $R_{max} = \max R_i$, $R_{min} = \min R_i$, $r_{max} = \max r_j$ and $r_{min} = \min r_j$. We say the secondary network adopts power assignment scheme $\mathcal{A}(C)$ if for $i \in T^{(s)}$, $P_i = C r_i^2 P$.

**Theorem 1.** Under power assignment $\mathcal{A}(C_1)$ and hybrid protocol model, if $\Delta_{ps} > \Delta_s$, then for any active primary link $(X_i, X_{R_{RX(i)}})$, the interference suffered by $X_{R_{RX(i)}}$ from the secondary network is upper bounded by $C_2 R_i^2 \gamma$, for some $C_2 = \Theta(C_1)$.

**Proof:** Let $B(X, r)$ be the disk centered at $X$ with radius $r$. Then all $B(Y_j, \Delta_{sp} r_j/2), j \in T^{(s)}$ should be mutually disjoint according to Corollary 1. As well, $B(Y_j, \Delta_{ps} r_j/2), j \in T^{(s)}$ are disjoint with $B(X_{R_{RX(i)}}, \Delta_{sp} R_i/2)$. Since $\Delta_{ps} > \Delta_s$, then all $B(Y_j, \Delta_{sp} r_j/2), j \in T^{(s)}$, $B(X_{R_{RX(i)}}, \Delta_{sp} R_i/2)$ are
pairwise disjoint. Denote $D_{ij} = B(X_{\text{Rx}(i)}, |X_{\text{Rx}(i)} - Y_j|) \cap B(Y_j, \Delta_s r_j/2)$, it is clear that all $D_{ij}$ are disjoint (see Figure 1). Denote by $E, F$ the two points where $B(X_{\text{Rx}(i)}, |X_{\text{Rx}(i)} - Y_j|)$ intersects $B(Y_j, \Delta_s r_j/2)$. It is clear that $\vartriangle EY_jX_{\text{Rx}(i)} = \vartriangle EY_jX_{\text{Rx}(i)} \geq \pi/3$ because $|X_{\text{Rx}(i)} - Y_j| > \Delta_s r_j/2$. So the area of $D_{ij}$ is at least one third of $B(Y_j, \Delta_s r_j/2)$. Let $I_{sp}(i)$ denote the interference at receiver $X_{\text{Rx}(i)}$ from secondary network and $dA$ be the area element,

$$I_{sp}(i) = \sum_{j \in T^{(i)}} \frac{P_j}{|Y_j - X_{\text{Rx}(i)}|^\gamma} = \sum_{j \in T^{(i)}} \frac{4C_1 P}{\pi \Delta_s^2} \int_{B(Y_j, \Delta_s r_j/2)} \frac{dA}{|Y_j - X_{\text{Rx}(i)}|^\gamma} \leq \sum_{j \in T^{(i)}} \frac{12C_1 P}{\pi \Delta_s^2} \int_{D_{ij}} \frac{dA}{|Y_j - X_{\text{Rx}(i)}|^\gamma} \leq \frac{12C_1 P}{\pi \Delta_s^2} \int_{\cup_{j \in T^{(i)}} D_{ij}} \frac{dA}{|X - X_{\text{Rx}(i)}|^\gamma}$$

Since $(\cup_{j \in T^{(i)}} D_{ij}) \cap B(X_{\text{Rx}(i)}, \Delta_s r_i/2) = 0$, we have,

$$I_{sp}(i) \leq \frac{12C_1 P}{\pi \Delta_s^2} \int_{|X - X_{\text{Rx}(i)}| \geq \Delta_s r_i/2} \frac{dA}{|X - X_{\text{Rx}(i)}|^\gamma} = \frac{24C_1 P}{\Delta_s^2 (\gamma - 2)} \left( \frac{2}{\Delta_s r_i} \right)^{\gamma - 2} = C_2 P R_i^{2-\gamma}$$

**C. Interference at Secondary Nodes**

Now we focus on the interference at secondary nodes. The main challenge is to bound the uncontrollable interference from the primary network.

**Theorem 2.** Under power assignment $A(C_1)$ and hybrid protocol model, for any active link $(Y_i, Y_{\text{Rx}(i)})$, the interference at $Y_{\text{Rx}(i)}$ from the primary network is upper bounded by $c_3 P R_{\text{min}}^{-\gamma}$, for some constant $c_3$.

**Proof:** Denote by $I_{ps}(i) = \sum_{j \in T^{(i)}} \frac{P_j}{|Y_{\text{Rx}(i)} - Y_j|^\gamma}$ the interference at $Y_{\text{Rx}(i)}$ from primary network. Pick $X'$ as the interfering primary transmitter closest to $Y_{\text{Rx}(i)}$. From Corollary 1, distance between any primary transmitter and $Y_{\text{Rx}(i)}$ should be larger than $\Delta_p R_{\text{min}}/2 + \Delta_{ps} r_i/2$; distance between any two primary transmitter is larger than $\Delta_p R_{\text{min}}/2$.

Now consider the case that $\Delta_p < \Delta_p$. (Note that if $\Delta_{sp} > \Delta_p$, $I_{ps}(i)$ will be smaller and the upper bound still holds.) Then all $X, j \in T^{(i)}$ is at least $\Delta_p R_{\text{min}}/2$ away from $Y_{\text{Rx}(i)}$. First consider the interference contributed by $X'$, $\frac{P}{|X' - Y_{\text{Rx}(i)}|^\gamma} \leq \int_{B_{ij}} \frac{P}{|X' - Y_{\text{Rx}(i)}|^\gamma} \frac{dA}{|X' - Y_{\text{Rx}(i)}|^\gamma}$

Next consider the interference from some other primary transmitter $X_j$. Let $B_{ij} = B(X_j, \Delta_p R_{\text{min}}/2) \cap B(Y_{\text{Rx}(i)}, |X_j - Y_{\text{Rx}(i)}|)^c$, as shown in Figure 1, then,

$$\int_{B_{ij}} \frac{P}{|X' - Y_{\text{Rx}(i)}|^\gamma} \frac{dA}{|X' - Y_{\text{Rx}(i)}|^\gamma} \leq \int_{B_{ij}} \frac{P}{|X_j - Y_{\text{Rx}(i)}|^\gamma} \frac{dA}{|X_j - Y_{\text{Rx}(i)}|^\gamma} = \frac{2^{\gamma + 2} P}{\pi \Delta_p^2 R_{\text{min}}^2} \int_{B_{ij}} \frac{dA}{|X - Y_{\text{Rx}(i)}|^\gamma}$$

In sum, let $I'_{ps}(i) = \sum_{j \in T^{(i)}} \frac{P_j}{|X_j - Y_{\text{Rx}(i)}|^\gamma}$,

$$I'_{ps}(i) \leq \frac{2^{\gamma + 2} P}{\pi \Delta_p^2 R_{\text{min}}^2} \sum_{j \in T^{(i)} \setminus \{X_i\}} \int_{B_{ij}} \frac{dA}{|X - Y_{\text{Rx}(i)}|^\gamma} = \frac{2^{\gamma + 2} P}{\pi \Delta_p^2 R_{\text{min}}^2} \int_{x - Y_{\text{Rx}(i)} > \Delta_p R_{\text{min}}} \frac{dA}{|X - Y_{\text{Rx}(i)}|^\gamma} \leq \frac{2^{\gamma + 2} P}{\pi \Delta_p^2 R_{\text{min}}^2} \int_{x - Y_{\text{Rx}(i)} > \Delta_p R_{\text{min}}} \frac{dA}{|X - Y_{\text{Rx}(i)}|^\gamma}$$

Combining the contribution from $X'$, $I_{ps}(i) \leq \left( \frac{2^\gamma}{\Delta_{sp}} + \frac{2^{\gamma + 2}}{(\gamma - 2) \Delta_p} \right) C_4 P R_i^{2-\gamma}$

We should also take into account the interference between secondary links. Power assignment scheme $A$ is well designed so that it not only restricts the interference from the secondary network to the primary network, but also that between secondary links, as shown by the next theorem. Its proof is similar to Theorem 1 and is omitted for space concern.

**Theorem 3.** Under power assignment scheme $A(C_1)$ and hybrid protocol model, for any active secondary link $(Y_i, Y_{\text{Rx}(i)})$, the interference at $Y_{\text{Rx}(i)}$ from all other simultaneously active secondary links is upper bounded by $I_{ss}(i) \leq C_4 P R_i^{2-\gamma}$, where $C_4 = \frac{2^{4\gamma - 2}}{(\gamma - 2) \Delta_p} C_1$. 

---

**Fig. 1:** Analyzing the interference. Left plot shows an example for $D_{ij}$ and right plot for $B_{ij}$. 

---
D. Physical Feasibility of the Hybrid Protocol Model

Last we show under appropriate conditions, hybrid protocol model is indeed physical feasible. We begin with a lemma.

Lemma 3. Given $A, B, C, a, b > 0$, if $\frac{C^2}{A} \leq \frac{b}{a+b}$ and $\frac{C}{A} \geq a + b$, then $\frac{C}{A} \geq a$.

Then first consider the primary network.

Lemma 4. If $S^{(p)} \in \mathcal{D}(\alpha + \epsilon)$, $(X_i, X_{Rt(x)}) \in S^{(p)}$, and $I_{sp}(i) \leq \frac{PR_{i}^{-\gamma}}{C_0} \geq \frac{1}{C_0 R_{i}^{2}} \geq \alpha (\alpha + \epsilon)$, then $(X_i, X_{Rt(x)})$ is feasible under physical model $\mathcal{P}(\alpha, \beta)$. That is to say, $N + T_{i} + I_{pp}(i) \geq \alpha$, where $I_{pp}(i)$ is interference at $X_{Rt(x)}$ from other simultaneously active primary links.

Proof: Let $A = PR_{i}^{-\gamma}, B = N + I_{pp}(i), C = I_{sp}(i), a = \alpha, b = \epsilon$. Note that $S^{(p)} \in \mathcal{D}(\alpha + \epsilon)$ implies $A/B \geq a + b$, then the result holds from Lemma 3.

Lemma 5. If $\Delta_{ps} > \Delta_{x}, S \in \mathcal{H}$ and $S^{(p)} \in \mathcal{D}(\alpha + \epsilon)$, then under power assignment $A(C_1)$ such that $C_2 \leq \frac{1}{\alpha (\alpha + \epsilon) R_{max}^{2}}$, all primary links are feasible under physical model $\mathcal{P}(\alpha, \beta)$.

Proof: For any $(X_i, X_{Rt(x)}) \in S^{(p)}$, from Theorem 1, $I_{sp}(i) \leq C_0 R_{i}^{2} \gamma P$, thus the SIR at $X_{Rt(x)}$ satisfies,

$$\frac{PR_{i}^{-\gamma}}{I_{sp}(i)} \geq \frac{C_0 R_{i}^{2} \gamma P}{I_{sp}(i)} \geq \frac{1}{C_0 R_{i}^{2}} \geq \alpha (\alpha + \epsilon)$$

Then from Lemma 4 we have the assertion.

Now turn to the secondary network. It can be easily shown that Lemma 6 follows from Theorem 2 and Lemma 7 from Theorem 3.

Lemma 6. Under power assignment $A(C_1)$ with $C_1 \geq \frac{c_5 c_6 R_{min}}{R_{max}^{2}}$, if $S^{(p)} \in \mathcal{H}$, then for any $(Y_i, Y_{Rt(x)}), i \in T^{(s)}$, it holds:

$$\frac{C_1 P_{i} R_{i}^{2} \gamma}{I_{ps}(i)} \geq c_5$$

Lemma 7. Under the condition of Lemma 6, if $\Delta_{x} = \left(48 \frac{\gamma - 2}{\gamma - 1} c_6\right)^{\frac{1}{2}}$, for any $(Y_i, Y_{Rt(x)}), i \in T^{(s)}$, follows:

$$\frac{C_1 P_{i} R_{i}^{2} \gamma}{N + I_{ss}(i)} \geq c_6$$

Last we are ready to prove the final result.

Theorem 4. If $r_{\gamma - 2}^{\max} = o\left(\frac{R_{min}}{R_{max}^{2}}\right)$, $\Delta_{p} \leq (\alpha + \epsilon)^{\frac{1}{\gamma}} - 1$, and $\Delta_{ps} \leq \Delta_{x} \geq \left(24 \frac{\gamma - 2}{\gamma - 1} \beta\right)^{\frac{1}{\gamma}}$, then there exists power assignment $A(C_1)$, such that for any $S^{(p)} \in \mathcal{D}(\alpha + \epsilon)$, holds $S^{(p)} \in \mathcal{H}(\Delta_{p}, \Delta_{ps}, \Delta_{sp}, \Delta_{x})$. And if we schedule secondary network transmisions in the way such that $S^{(p)} \cup S^{(s)} \in \mathcal{H},$ holds $S^{(p)} \cup S^{(s)} \in \mathcal{P}(\alpha, \beta)$.

Proof: The first claim follows from Lemma 1. To prove the second claim, first notice every active primary link is physical feasible if the condition of Lemma 5 is verified, i.e., if

$$C_2 = \Theta(C_1) = o\left(\frac{1}{R_{max}^{2}}\right)$$

On the other hand, consider the secondary network, if we can ensure

$$C_1 = \omega\left(\frac{R_{min}^{\gamma}}{R_{i}^{2}}\right)$$

then according to Lemma 6, (4) holds for any positive constant $c_5$. In combination with Lemma 7, it is clear that SINR at any secondary receiver is greater than $\frac{1}{\alpha (\min(c_5, c_6))}$ = $\beta$. Since $r_{\gamma - 2}^{\max} = o\left(\frac{R_{min}^{\gamma}}{R_{max}^{2}}\right)$, we can indeed find $C_1$ and $C_2$, such that (5) and (6) hold, proving the theorem.

The condition $r_{\gamma - 2}^{\max} = o(R_{min}^{\gamma} / R_{max}^{2})$ characterizes the regime that primary links are homogeneous in range. In other words, if this condition fails, it implies that the scheduling of the primary network is somewhat “chaotic” and simple decision model like $\mathcal{H}$ do not suffice to identify transmission opportunities. Fortunately, this condition usually holds because $R_{max}$ and $R_{min}$ typically do not differ much in order and we tend to employ a small $r$.

IV. AVAILABILITY OF TRANSMISSION OPPORTUNITIES

Section III addresses the problem of how to identify transmission chances for secondary networks: given a set $S^{(p)}$ of simultaneously active primary links, we allow a set $S^{(s)}$ of simultaneously active secondary links according to hybrid protocol model $\mathcal{H}$. This section, on the other hand, considers the problem that for those secondary links which desire to transmit, how frequently do these chances occur. Of particular interest is to compare this result with an identical standalone network. Standalone networks provide trivial performance upper bounds since cognitive secondary networks will suffer from additional transmission constraints imposed by primary networks. To alleviate the performance loss due to such constraints, it is intuitive that one should reduce the range of secondary links, and this fact is indeed verified by hybrid protocol model and Theorem 4. This section will further show if $r_{\max} = o(R_{min})$, then for quite general cases, such performance loss is insignificant and has no impact in order sense. In other words, all secondary links have a constant ratio of time to be unconstrained as if they were in a standalone network. Besides, note it is well known that to achieve better scalability, a smaller range is also favorable. This coincidence implies that secondary networks can reach the optimal scaling performance of a standalone network. Now we formally introduce the concept of unconstraint, and analyze the unconstrained time in the following subsections.

Definition 5. Given arbitrary $S_{s,a}^{(a)} \in \mathcal{L}_{s}^{a}(\Delta_{a})$ and arbitrary $S^{(p)} \in \mathcal{L}_{p}(\Delta_{p})$, there exists an unique maximal $S^{(s)} \subset S_{s,a}^{(a)}$ such that $S^{(p)} \cup S^{(s)} \in \mathcal{H}(\Delta_{p}, \Delta_{ps}, \Delta_{sp}, \Delta_{p})$. We say a link $(Y_i, Y_{Rt(x)}) \in S_{s,a}^{(a)}$ is unconstrained if $(Y_i, Y_{Rt(x)}) \in S^{(s)}$.

Note the fraction of time that the link is constrained characterizes the performance loss relative to the corresponding standalone network.
A. Cell Partitioning Round-Robin Mode

We start with the case that primary networks operate according to a common scheduling paradigm: the cell partitioning round-robin active scheme. It first spatially tessellates the network into cells, then assigns color to each cell, such that cells with the same color, if limit their transmissions to neighbors, will not interfere with each other. Then we allow cells with the same color to transmit simultaneously, and let different colors take turns to be active. A simple TDMA scheme will suffice. This very widely employed scheme \[4\], \[10\], \[3\], \[8\] features a high degree of spatial concurrency and thus frequency reuse. It is deterministic and therefore simple.

To the best of our knowledge, all previous works on asymptotic analysis of cognitive networks focused on some particular variants of such TDMA scheme. We now show for a generic primary scheduling policy which operates in the round-robin fashion, the unconstrained time fraction for any short range secondary link is constant, as a simple consequence of the hybrid protocol model.

Definition 6. A network tessellation is a set of disjoint cells \( \{V_i \subseteq \mathcal{O}\} \). A round-robin TDMA scheme is a scheduling scheme that i) tessellates the network into cells such that every cell is contained in a disk of radius \( \rho(n) \), ii) allows non-interfering cells to be simultaneously active and transmit to neighbor cells, where two cells \( V_i, V_j \) are non-interfering if \( \sup\{|E - F| : E \in V_i, F \in V_j\} \geq (2 + \Delta_p)4\rho(n) \), and iii) activates different groups of cells in a round-robin TDMA fashion, and guarantees every cell can be active for at least \( c_T \) fraction of time in one round, for some constant \( c_T > 0 \).

The existence of round-robin TDMA schemes is a consequence of the well-known theorem about vertex coloring of graphs. The next theorem shows such scheme is favorable to secondary networks because it deterministically ensures transmission opportunities not only for every primary cell, but also for every secondary link.

Theorem 5. If the primary network operates according to a round-robin TDMA scheme and \( \Delta_p > 2 \), \( \Delta_{sp} \leq \frac{\Delta_p - 2}{2} \), then every secondary link with range \( r = o(R_{min}) \) has at least \( c_T \) fraction of time to be unconstrained in one round.

Proof: Consider a generic link \( (Y_i, Y_{R_k(i)}) \), pick a point \( X \) such that \( |X - Y_i| = (4 + 2\Delta_p)\rho(n) \), and denote by \( V \) the cell \( X \) belongs to, we claim whenever \( V \) is scheduled to be active, \( (Y_i, Y_{R_k(i)}) \) is unconstrained. To that end, we first verify transmitter \( Y_i \) will not upset transmissions in \( V \). Indeed, any point \( E \) belongs to \( V \) should lie within distance \( 2\rho(n) \) from \( X \), thus any point \( F \) belongs to a neighbor cell of \( V \) should lie within distance \( 4\rho(n) \) from \( X \), then

\[
|Y_i - F| \geq (4 + 2\Delta_p)\rho(n) - 4\rho(n) = 2\Delta_p\rho(n) \geq (1 + \Delta_{sp})4\rho(n) \geq (1 + \Delta_{sp})|E - F|
\]

Now consider another simultaneous active cell \( V' \), it is clear that any point \( X' \in V' \) is at least \( (2 + \Delta_p)4\rho(n) \) away from \( X \), then \( |X' - Y_i| \geq (4 + 2\Delta_p)\rho(n) = |X - Y_i| \). Together with (7), condition (2) is verified. Besides, since \( r = o(R_{min}) \), condition (3) is obvious. This completes the proof.

We observe that hybrid protocol model \( H \), \( \Delta_p > 2 \) is critical to guarantee transmission opportunities for secondary nodes, as shown in Theorem 5. Equivalently, it implies \( \alpha + \epsilon \geq 2' \). This is an assumption about primary networks and we assume it always holds from now on. However, we conjecture this assumption is not fundamental and can be relaxed by introducing a criterion with more flexible form, i.e., allowing \( \Delta_{sp} \) and \( \Delta_{ps} \) to be dependent on \( n \). Such decision models may have a better capability of digging into the potential of available gaps, at the cost of complexity. We leave for future work a more in-depth analysis of such cases.

B. Independent Relay Mode

Theorem 5 suffices to provide rich scaling results on cognitive networks, for the scenario it considers, i.e., the round-robin TDMA scheme, covers most centralized control networks. However, some other cases are also of interest such as networks which employ distributed CSMA protocol [9]. Exceptions also exist in centralized control networks, such as the protocol proposed in [5], which schedules the network in a more aggressive way. In words, Theorem 5 relies on the scheduling of primary networks, but sometimes we may want to relax this requirement. In the following it is shown that some general assumptions on the routing protocol of primary networks are sufficient to reach similar result.

Intuitively, according to the hybrid protocol model, on one hand primary transmissions will not be too dense spatially, thus leave gaps for the secondary network. On the other hand, they also mute nearby secondary links. We shall show every primary link can create some gap and mute some area. More formally, given link \( (X_i, X_{R_k(i)}) \) and \( (Y_j, Y_{R_k(j)}) \), we say the former triggers the latter if \( (Y_j, Y_{R_k(j)}) \) is unconstrained as long as \( (X_i, X_{R_k(i)}) \) is active, and shades the latter per contra. Because nodes are i.i.d. distributed, whether a primary link can create some gap and mute some area is “independently” distributed (relayed) to primary links, according to the law of large numbers, if traffic is somewhat “independently” distributed (relayed) to primary links, according to the law of large numbers, if a secondary link is shaded for a long time w.h.p., i.e., the primary traffic nearby is intense, it will also be triggered for considerable time.

Lemma 8. Consider link \( (X_i, X_{R_k(i)}) \) and \( (Y_j, Y_{R_k(j)}) \), if \( \Delta_p > 2 \), \( \Delta_{sp} \leq \frac{\Delta_{ps} - 2}{2} \) and \( r_j = o(R_i) \), then a sufficient condition that \( (X_i, X_{R_k(i)}) \) triggers \( (Y_j, Y_{R_k(j)}) \) is \( Y_j \) lies in the ring of points with distance to line segment \( X_iX_{R_k(i)} \) larger than \( (1 + \Delta_{sp})R_i \) and less than \( \Delta_p R_i / 2 \). A necessary condition that \( (X_i, X_{R_k(i)}) \) shades \( (Y_j, Y_{R_k(j)}) \) is \( Y_j \) lies within the \( (1 + \Delta_{sp})R_i \) neighborhood of line segment \( X_iX_{R_k(i)} \) (Figure 2).

Proof: The necessary condition is obvious. As to the sufficient condition, (3) holds for \( r_j = o(R_i) \). It is also clear that \( |Y_j - X_{R_k(j)}| \geq (1 + \Delta_{sp})R_i \). For any other active primary link \( (X_k, X_{R_k(k)}) \), \( k \neq i \), its receiver is at least \( \Delta_p R_i / 2 \) away from \( Y_j \) due to Corollary 1. Therefore (2) also holds, proving
Fig. 2: An active primary link \((X_i, X_{Rx(i)})\) can shade some area (the dark region) and trigger some area (the outside ring).

the lemma.

As a consequence we can term \((X_i, X_{Rx(i)})\) triggers \((Y_j, Y_{Rx(j)})\) and \((X_i, X_{Rx(i)})\) triggers \(Y_j\) interchangeably.

**Definition 7.** Consider a regular network tessellation of square cells. We assume every source route traffic to its destination along these cells in multi-hop fashion, such that at every hop a packet is transmitted to a relay node in a neighbor cell. We say the network, and is in accord with the design principles of relaying implies there are no special designated nodes in cells. We assume every source route traffic to its destination.

**Lemma 9.** For an independent relay protocol, to ensure asymptotic connectivity of the overall network, the side length \(L\) of square cells is at least \(\Theta(\sqrt{\log n/n})\).

**Lemma 10.** For an independent relay protocol, there exist positive constants \(c_8\) and \(c_9\), such that w.h.p. every cell contains more than \(c_8nL^2\) and less than \(c_9nL^2\) primary nodes.

**Lemma 11.** Consider arbitrary neighboring cells \(V_1, V_2\) and link \((Y_j, Y_{Rx(j)})\), let \(X_i\) and \(X_{Rx(i)}\) be independently and uniformly distributed in \(V_1\) and \(V_2\), respectively. Denote by \(p\) the probability that \((X_i, X_{Rx(i)})\) triggers \((Y_j, Y_{Rx(j)})\) and \(q\) the probability of shading. Then \(\forall\) constant \(\delta_1, \delta_2 > 0\), among all primary links from \(V_1\) to \(V_2\), w.h.p., there are at least \(p(1 - \delta_1)(c_8nL^2)^2\) links that trigger \((Y_j, Y_{Rx(j)})\), and at most \(q(1 + \delta_2)(c_9nL^2)^2\) links that shade it.

**Proof:** We only prove the first part of the lemma. From Lemma 10 there are at least \((c_8nL^2)^2\) nodes in each cell, denoted by \(X_{uk}\) and \(X_{vl}\), \(k, l = 1, \ldots, c_9nL^2\). Thus we have at least \((c_8nL^2)^2\) candidate links from \(V_1\) to \(V_2\). Consider this subset of links, define \(I_{k,l}\) as

\[
I_{k,l} = \begin{cases} 
1 & \text{if } (X_{uk}, X_{vl}) \text{ triggers } (Y_j, Y_{Rx(j)}) \\
0 & \text{otherwise}
\end{cases}
\]

Because nodes are i.i.d., \(\{I_{k,l}\}\) are identically distributed and with probability \(p\) equals to 1. Besides, \(I_{a,b}\) and \(I_{k,l}\) are independent if \(a \neq k\) and \(b \neq l\). Construct \(I_k\) as

\[
I_k = \sum_{i=1}^{\sum_{j=1}^{2\cdot c_8nL^2 - 1}} I_{i+k, i+k+1} \quad \text{if } k \text{ odd}
\]

\[
I_k = \sum_{i=1}^{\sum_{j=1}^{2\cdot c_8nL^2 - 1}} I_{i+k, i+k+1} \quad \text{if } k \text{ even}
\]

Then \(I_k\) is sum of i.i.d. random variables. Applying the law of large numbers, one can easily show that \(\forall\delta_3 < 2, k < \delta_3c_8nL^2, \forall\delta_1 > 0\), the following holds w.h.p.

\[
I_k > \begin{cases} 
p(1 - \delta_1)(c_8nL^2 - \frac{k}{2}) & \text{if } k \text{ odd} \\
p(1 - \delta_1)(c_8nL^2 - \frac{k}{2}) & \text{if } k \text{ even}
\end{cases}
\]

Last, note the relationship of summing \(\{I_{k,l}\}\) and \(\{I_k\}\),

\[
\sum_{k} \sum_{l} I_{k,l} = \sum_{k=1}^{\sum_{j=1}^{2\cdot c_8nL^2 - 1}} I_k \geq \sum_{k=1}^{\sum_{j=1}^{2\cdot c_8nL^2 - 1}} I_k 
\]

\[
\geq (p(1 - \delta_1) - (2 - \delta_3))(c_8nL^2)^2 \quad (w.h.p.)
\]

Making \(\delta_3\) arbitrarily close to 2, we have the claim. \(\blacksquare\)

In the next step we characterize the relation between \(p\) and \(q\). The main idea is to couple the triggering and shading events through a continuous transformation in \(\mathbb{R}^4\). We first cite a property of Lebesgue measure [15].

**Lemma 12.** (Integration by change of variable) Let \(S \subset \mathbb{R}^n\) be an open set and \(L\) be a Lebesgue measure on \(S\). Let \(T(x) = (y_1(x), \ldots, y_n(x))\), \(x = (x_1, \ldots, x_n) \in S\) be a given homeomorphism \(T: S \to \mathbb{R}^n\) with the continuous derivatives \(\frac{\partial y_i}{\partial x_j}\), \(i, j = 1, \ldots, n\) on \(S\) and we note with \(\tau(T, x) = \left(\frac{\partial y_i}{\partial x_j}\right)\) the nondegenerate Jacobian matrix for all \(x \in S\). Then for any non-negative borelian function \(f\) defined on the open set \(T(S)\) we have

\[
\int_{T(S)} f(y)dy = \int_{S} f(T(x))|\tau(T, x)|dx
\]

where \(dx, dy\) denote the integration with respect to \(L\).

**Theorem 6.** Define \(p, q\) as in Lemma 11 and under the condition of Lemma 8, there exists constant \(c_{10} > 0\), such that \(p > c_{10}q\).

**Proof:** Without loss of generality, let \(V_1 = [-1, 0] \times [0, 1]\), \(V_2 = [0, 1] \times [0, 1]\), and \((\Omega, \mathcal{F}, L)\) be the probability space of interest, where \(\Omega = V_1 \times V_2\) and \(L\) is the Lebesgue measure restricted on \(\Omega\). Given \(\omega = (x_1, y_1, x_2, y_2) \in \Omega\), define \(T_\theta: \Omega \to \Omega\) as \(T_\theta(\omega) = \omega' = (x'_1, y'_1, x'_2, y'_2)\):

\[
\begin{align*}
x'_1 &= -x_1 \\
y'_1 &= y_1 + (1 - \theta)\frac{x_1(y_2 - y_1)}{x_1 - x_2} \\
x'_2 &= \theta x_2 \\
y'_2 &= \theta y_2 + (1 - \theta)\left( y_1 + \frac{x_1(y_2 - y_1)}{x_1 - x_2} \right)
\end{align*}
\]

3With abuse of notation, we use \(\Omega\) to denote a set and \(\omega\) an element instead of order when no confusion is caused.
Intuitively, let $E = (x_1, y_1)$ and $F = (x_2, y_2)$, then $T_\theta$ linearly shrinks line segment $EF$ to $E'F'$ with $\theta < 1$, preserving its geometric topology. See Figure 3 for an example.

Let $d(\omega)$ be the distance from $Y_j$ to line segment $EF$ and $S_{\text{shd}} = \{\omega \in \Omega : Y_j$ is triggered\}, $S_{\text{trs}} = \{\omega \in \Omega : Y_j$ is triggered\}. Assume $S_{\text{shd}}$ is not empty and consider any $\omega_0 \in S_{\text{shd}}$ and the set $H(\omega_0) = \{\omega' \in \Omega : \omega' = T_\theta(\omega_0), \theta > 0\}$. Define $\theta_{\text{cr}}$ and $\theta_{\text{min}}$ such that,

$$
d(T_{\theta_{\text{cr}}}(\omega_0)) = \theta_{\text{cr}}|EF| = |EF|_{\text{cr}},
$$
$$
2d(T_{\theta_{\text{min}}}(\omega_0)) = \theta_{\text{min}}|EF| = |EF|_{\text{min}},
$$

According to Lemma 8, it is clear that $\theta_{\text{cr}}$ and $\theta_{\text{min}}$ uniquely exist and $\theta_{\text{cr}} > \theta_{\text{min}}$. Besides, $\forall \omega' \in H(\omega_0), \omega' \in S_{\text{shd}} \Rightarrow |EF'| > |EF|_{\text{cr}}, |EF|_{\text{min}} \leq |EF'| \leq |EF|_{\text{cr}} \Rightarrow \omega' \in S_{\text{trs}}$. Therefore we can introduce a mapping from the set of line segments with length $(|EF|_{\text{cr}}, \sqrt{5})$ to those with length $(|EF|_{\text{min}}, |EF|_{\text{cr}})$, where $\sqrt{5}$ is an upper bound of $|EF'|$, i.e.,

$$
\theta(\omega_0) = \frac{|EF|_{\text{min}} + |EF|_{\text{cr}} - |EF|_{\text{min}} - |EF|_{\text{cr}}}{\sqrt{5}}
$$

Then $\forall \omega \in S_{\text{shd}}, T_{\theta}(\omega) \in S_{\text{trs}}$, i.e., we construct a mapping between $S_{\text{shd}}$ and $S_{\text{trs}}$, and by invoking Lemma 12 the relation between $p$ and $q$ can be established. To that end, we first simply $\theta(\omega)$. It is obvious on a little thought that $\forall S \subset S_{\text{shd}}, \theta \leq \min_{\omega \in S} \frac{|EF|_{\text{cr}} - |EF|_{\text{cr}} - |EF|_{\text{cr}}}{\sqrt{5}}$, then $\mathcal{L}(T_{\theta}(\omega)(S)) \geq \mathcal{L}(T_{\theta}(\omega)(S))$.

And,

$$
\frac{|EF|_{\text{cr}} - |EF|_{\text{min}}}{\sqrt{5}} \geq \frac{d_{\text{cr}}}{\sqrt{5}(1 + \Delta_p)} - \frac{2d_{\text{min}}(\omega_0)}{\Delta_p} \geq \frac{d_{\text{cr}}}{\sqrt{5}(1 + \Delta_p)} \geq \frac{\Delta_p + \Delta_p - 2}{2\sqrt{5}\Delta_p} d_{\text{cr}}
$$

Last, observe in the worst case that $Y_j$ lies in $V_1$ or $V_2$, there are some $S'_{\text{shd}} \subset S_{\text{shd}}, \mathcal{L}(S'_{\text{shd}}) > 1/8\mathcal{L}(S_{\text{shd}})$, such that $S'_{\text{shd}} \supset \cup_{\omega \in S'_{\text{shd}}} H(\omega)$ and $\forall \omega \in S_{\text{shd}}, d(\omega) > 1/4$. Therefore $d_{\text{cr}} > 1/4$. Define $\hat{\theta} \triangleq \frac{\Delta_p + \Delta_p - 2}{8\sqrt{5}\Delta_p} \frac{1}{\hat{\theta}^3}$, follows,

$$
q = \Pr\{Y_j$ is shadowed\} = $\mathcal{L}(S_{\text{shd}}) \leq 8\mathcal{L}(S'_{\text{shd}})$
$$
$$
\leq 8\mathcal{L}(T_{\hat{\theta}}(S'_{\text{shd}})) / \hat{\theta}^3
$$

Let $c_{12} = \hat{\theta}^3 / 8$, we complete the proof.

**Theorem 7.** If primary network employs an independent relay protocol and $\Delta_p > 2, \Delta_p \leq \frac{\Delta_p}{2}$, then every secondary link with range $r = a(R_{\text{min}})$ has at least on average $c_{11}$ fraction of time to be unconstrained, where constant $c_{11} > 0$.

**Proof:** Without loss of generality, consider a time interval of unit length and a particular secondary link $(Y_j, Y_{\text{Rx}}(j))$, we only discuss the case that $Y_j$ is shadowed by transmissions from some cell $V_1$ to $V_2$ for a least some constant fraction of time, otherwise the proof is trivial. This implies that the shading probability $q$ is lower bounded by $q_1(\theta) = 1 - \frac{C_0}{\theta}$, and $C_0 \lambda_p = \Theta(1)$, where $C_0$ is the number of flows that choose this route, and $\lambda_p = O(1/\sqrt{n})$ is the per-node throughput of primary network. Then from Theorem 6 we have the triggering probability $p > c_{10}q_1 = \Theta(1)$.

According to Lemma 10 and Lemma 11, the fraction of candidate links that trigger $Y_j$ is at least $p_1 = \frac{c_2}{2c_0}$. Let $I_k$ be the logical indicator function, and define $J = \sum_{i=1}^{C_0} I_{\text{flow i chooses a link that triggers Y_j}}$, then $J$ is sum of i.i.d. Bernoulli random variables with mean $p_2 > p_1$. Denote $E$ as expectation, by applying Chernoff bounds we get:

$$
\Pr\left\{ J < \frac{1}{2}E[J] \right\} \leq \frac{1}{2}C_0p_2 < e^{-C_0p_2/8}
$$

(8) indicates $J$ will be triggered for a constant fraction of time. And we need to show this fact holds uniformly for all secondary links. To that end, we tessellate the network into $C_0/n$ subquares for some $C_0 = \omega(1)$, then it is clear that all secondary transmitters within a same subquare share the same status of being unconstrained or not. Denote $J_k = \sum_{i=1}^{C_0} I_{\text{flow i triggers subquare k}}$, then by the sub-additivity of probability measure:

$$
\Pr\left\{ \bigcap_{k} J_k > \frac{1}{2}E[J_k] \right\} \geq 1 - \sum_{k} \Pr\left\{ J_k < \frac{1}{2}E[J_k] \right\}
$$

where the last limit holds for any $C_0 = n^\mu, \mu \in \mathbb{R}$ due to $C_0p_2 = \Omega(1/\lambda_p) = \Omega(1/\sqrt{n})$. Therefore w.h.p. every secondary link is triggered for at least $\frac{1}{2}C_0p_2\lambda_p = \Theta(1)$ seconds.

**V. OPTIMAL PERFORMANCE SCALING**

In this section we present results on throughput and delay scaling of general cognitive networks as well as a number of corollaries under various specific settings.
Theorem 8. If the primary network operates in the round-robin TDMA or the independent relay fashion, for any protocol interference model based scheme that schedules and routes the secondary network such that $r_{\text{max}} = o(R_{\text{min}})$, $r_{\text{max}}^2 = o(R_{\text{min}}/R_{\text{max}}^2)$ w.h.p., and achieves per-node throughput $\lambda_s$ and delay $D_s$ in the case that secondary network is standalone, there exists a corresponding scheme which can achieve per-node throughput $\Theta(\lambda_s)$ and delay $\Theta(D_s)$ when primary network is standalone, and denote by $\mathcal{D}_s$ the set of standalone ad hoc networks to cognitive networks. Their cooperation between primary and secondary nodes, which is straightforward from [4], [10].

**Corollary 2.** The optimal throughput delay tradeoff is $D_p = \Theta(n\lambda_p)$, $\lambda_p \leq \Theta(1/\sqrt{m})$ for primary network and $D_s = \Theta(m\lambda_s)$, $\Theta(n\lambda_p/m) < \lambda_s \leq \Theta(1/\sqrt{m})$ for secondary network, if $\Theta(1/\sqrt{m}) > \Theta(n\lambda_p/m)$.

**Corollary 3.** If primary nodes move according to random walk model, then the optimal throughput delay tradeoff for primary network is $D_p = \Theta(n\lambda_p)$ if $\lambda_p \leq \Theta(1/\sqrt{n})$, $D_p = \Theta(n\log n)$ if $\Theta(1/\sqrt{n}) < \lambda_p \leq \Theta(1)$. And the optimal throughput delay tradeoff for secondary network is $D_s = \Theta(m\lambda_s)$, $\Theta(n\min(1/\sqrt{n}, \lambda_p)/m) < \lambda_s \leq \Theta(1/\sqrt{m})$, if $\Theta(1/\sqrt{m}) > \Theta(n\min(1/\sqrt{n}, \lambda_p)/m)$.

We can extend the theorem to other variations of ad hoc networks, such as hybrid networks [8].

**Corollary 4.** If the primary network is equipped with $k = \Omega(\sqrt{n})$ base stations, the capacity of it is $\lambda_p = \Theta(k/n)$, and the optimal throughput delay tradeoff for the secondary network is $D_s = \Theta(m\lambda_s)$, $\Theta(\sqrt{n}\lambda_p/m) < \lambda_s \leq \Theta(1/\sqrt{m})$, if $\Theta(1/\sqrt{m}) > \Theta(\sqrt{\lambda_s}n/m)$.

The above corollaries are consequences of centralized TDMA scheduling. In the following we consider two examples of independently relaying. An interesting case is that primary networks make use of distributed random access protocols such as carrier-sensing multi-access (CSMA) [9].

**Corollary 5.** If the primary network employs independent relay protocol and CSMA protocol. The capacity of primary network is $\Theta(1/\sqrt{n}\log n)$. The optimal throughput delay tradeoff for secondary network is $D_s = \Theta(m\lambda_s)$, $\Theta(\sqrt{n}/m\log n) < \lambda_s \leq \Theta(1/\sqrt{m})$, if $m = \Omega(n^M)$ for some constant $M > 1$.

Now we consider a primary network with general mobility [5]. The next result follows from the mobile version of Theorem 7, which is analogous to the static one.

**Corollary 6.** If the mobility of primary nodes can be characterized by a stationary spatial distribution function with support of diameter $f(n) = \omega(1/\sqrt{n})$, then the capacity of primary network is $\lambda_p = \Theta(f(n))$. The optimal throughput delay tradeoff for secondary network is $D_s = \Theta(m\lambda_s)$, $\Theta(\sqrt{n}/m) < \lambda_s \leq \Theta(1/\sqrt{m})$, if $m = \omega(n)$.

Last, the results we can obtain is not limited to the cases listed above. Since our framework only relies on a few general conditions, it is flexible and is able to accommodate various cognitive networks with different specific forms. For instance, one can otherwise let both the networks or only the secondary network be mobile.

### A. Gaussian Channel Model

The physical interference model is a fixed rate on-off channel model. An alternative, i.e., the Gaussian channel model, generalizes data rate to be continuous in SINR, based on Shannon’s capacity formula for the additive Gaussian noise channel. We now briefly extend our results to this

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3Refer to [5] for a rigorous definition.
model. Specifically, the communication rate of a primary link \((X_i, X_j)\) is given by

\[
W = \log \left( 1 + \frac{P|X_i - X_j|^{-\gamma}}{N + I_{pp} + I_{sp}} \right)
\]

where \(I_{pp}\) and \(I_{sp}\) are interference from primary network and secondary network, respectively (cf. Section 3). Similarly we define the channel model for secondary links. We limit our interest to the case that \(W\) is bounded between two positive constants, i.e., \(W_1 < W < W_2\). We note this is also the most realistic case and suffice to generalize the results of prior works [12], [11]. Let \(\lambda_p\) be the per-node throughput of the primary network in the absence of the secondary network.

**Operation Rule 3.** The secondary scheduling should ensure that

\[
\frac{\lambda_p}{\lambda_p^{s.a.}} \geq 1 - \delta_{\text{loss}}.
\]

**Theorem 9.** Theorem 8 also holds under Gaussian channel model if we substitute Operation Rule 3 for Operation Rules 1 and 2, with \(\delta_{\text{loss}} \in (0, 1)\).

**Proof:** We claim that Operation Rules 1 and 2 with appropriate parameters are sufficient conditions for Operation Rule 3. Indeed, set \(\alpha \leq e^{W_1} - 1\) and \(\epsilon \leq ((1 + \alpha) - (1 + \alpha)^{1-\delta_{\text{loss}}} + 1/\delta_{\text{loss}}\), it is easy to show that

\[
\log \left( 1 + \frac{P|X_i - X_{R(i)}|^{-\gamma}}{N + I_{pp} + I_{sp}} \right) \geq \log(1 + \alpha + \epsilon) \geq 1 - \delta_{\text{loss}}
\]

for any active link \((X_i, X_{R(i)})\). On the other hand, to ensure the scheduling of secondary network under the Gaussian channel model is feasible under \(\mathcal{P}\), we set \(\beta \leq e^{W_1} - 1\). The rest part of the theorem is obvious.

**VI. Conclusions**

This paper studies the throughput and delay scaling of general cognitive networks and characterizes the conditions for them to achieve the same throughput and delay scaling as standalone networks. We propose a hybrid protocol model for secondary nodes to identify transmission opportunities and show that based on it communication schemes of standalone networks can be easily extended to secondary networks, without harming the performance of primary networks.

In particular, we show that secondary networks can obtain the same optimal performance as standalone networks when primary networks are classic static networks, networks with random walk mobility, hybrid networks, CSMA networks or networks with general mobility. Our work provides fundamental insight on the understanding and design of cognitive networks.

**References**


