NON-REDUCIBILITY OF ISOMORPHISM OF COUNTEREXAMPLES TO VAUGHT’S CONJECTURE TO THE ADMISSIBILITY EQUIVALENCE RELATION

WILLIAM CHAN

ABSTRACT. Let $T$ be a counterexample to Vaught’s conjecture. Let $E_T$ denote the isomorphism equivalence relation of $T$. Let $F_{\omega_1}$ be the countable admissible ordinal equivalence relation defined on $\omega_2$ by $x F_{\omega_1} y$ if and only if $\omega^x_1 = \omega^y_1$. This note will show that in $L$ and set-generic extensions of $L$, $\neg(E_T \leq_{\Delta^1_1} F_{\omega_1})$. Using ZFC, if $T$ is a non-minimal counterexample to Vaught’s conjecture, then $\neg(E_T \leq_{\Delta^1_1} F_{\omega_1})$.

1. Introduction

An equivalence relation $E$ on a Polish space $X$ is thin if and only if it does not have a perfect set of $E$-inequivalent elements.

Burgess showed that $\Sigma^1_1$ equivalence relations can either have countably many classes, $\aleph_1$-many but not perfectly many classes, or perfectly many classes. Equivalence relations of the second kind are the thin equivalence relations with uncountably many classes. There are a few notable examples of such equivalence relations.

Let $E_{\omega_1}$ be defined in $\omega^2$ be

$$x E_{\omega_1} y \iff (x,y \notin WO) \lor (x \equiv y)$$

where WO is the $\Pi^1_1$ set of reals coding well-orderings and $\equiv$ is order-isomorphism. $E_{\omega_1}$ is a thin equivalence relation with uncountably many classes and has a single $\Sigma^1_1$ and not $\Delta^1_1$ class consisting of the non-well-orderings.

The countable admissible ordinal equivalence relation $F_{\omega_1}$ is defined on $\omega^2$ by

$$x F_{\omega_1} y \iff \omega^x_1 = \omega^y_1$$

where $\omega^x_1$ is the least ordinal that can not be coded by a real recursive in $x$. $F_{\omega_1}$ is a $\Sigma^1_1$ thin equivalence relation with uncountably many classes which are all $\Delta^1_1$.

If $L$ is a recursive language and $T \subseteq L_{\omega_1,\omega}$ is a countable theory, then $E_T$ denotes isomorphism of models of $T$ with a single class consisting of the structures that do not model $T$. $E_T$ has all $\Delta^1_1$-classes as a consequence of Scott’s isomorphism theorem.

A notable class of thin $\Sigma^1_1$ equivalence relations that may or may not exist are the isomorphism relations of counterexamples to Vaught’s conjecture: $T \subseteq L_{\omega_1,\omega}$ is a counterexample to Vaught’s conjecture if $E_T$ is a thin equivalence relation with uncountably many classes.

The natural task is to compare these thin $\Sigma^1_1$ equivalence relations. A robust means of comparison is through $\Delta^1_1$ reductions: Suppose $E$ and $F$ are two equivalence relations on Polish spaces $X$ and $Y$, respectively. $E \leq_{\Delta^1_1} F$ if and only if there is a $\Delta^1_1$ function $\Phi : X \to Y$ so that for all $a,b \in X$, $\Phi(a) \sim F \Phi(b)$.

First, since $E_{\omega_1}$ has a single $\Sigma^1_1$ but not $\Delta^1_1$ class and both $F_{\omega_1}$ and $E_T$ have all $\Delta^1_1$ classes, it is impossible that $E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1}$ and $E_{\omega_1} \leq_{\Delta^1_1} E_T$. Secondly, by the boundedness theorem, $F_{\omega_1} \leq_{\Delta^1_1} E_{\omega_1}$ and $E_T \leq_{\Delta^1_1} E_{\omega_1}$, where $T$ is a counterexample to Vaught’s conjecture are also not possible.

The second failure is due to a significant global fact. The first failure seems local due to one $E_{\omega_1}$ class being non-$\Delta^1_1$ and this $E_{\omega_1}$ class even seems artificially added to ensure $E_{\omega_1}$ is defined on all of $\omega^2$.

To compare $E_{\omega_1}$ with $F_{\omega_1}$ and $E_T$ in a meaningful way which disregards this triviality, Zapletal defined the almost-$\Delta^1_1$ reducibility: $E \leq_{a\Delta^1_1} F$ if only if there is a countable $A \subseteq X$ and a $\Delta^1_1$ function $\Phi$ so that

March 9, 2017
for all \( x, y \notin [A]_E \), \( x E y \) if and only if \( \Phi(x) F \Phi(y) \). That is, an almost \( \Delta^1_1 \) reduction is a reduction that ignores at most a countable set of problematic classes.

Zapletal [7] showed that if there is a measurable cardinal and \( E \) is an \( \Sigma^1_1 \) equivalence relation with infinite pinned cardinal, then \( E_{\omega_1} \leq_{\Delta^1_1} E \). \( F_{\omega_1} \) has infinite pinned cardinals. If \( T \) is a counterexample to Vaught’s conjecture then \( E_T \) has infinite pinned cardinals. Hence if there is a measurable cardinal, then \( E_{\omega_1} \leq_{\Delta^1_1} E_T \) and \( E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1}. \) In fact, if \( 0^F \) exists, \( E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1}. \)

The author in [2] showed that in Gödel constructible universe \( \mathcal{L} \) and set generic extensions of \( L \), \( \neg(E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1}). \) This shows that \( E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1} \) is not provable in \( ZFC \) alone.

It remains to study the relationship between \( E_T \) and \( F_{\omega_1} \) when \( T \) is a counterexample to Vaught’s conjecture. Sy-David Friedman asked the first such question: If \( T \) is a counterexample to Vaught’s conjecture, can \( E_T \leq_{\Delta^1_1} F_{\omega_1} \)?

The author in [2] showed that in \( L \) and set-generic extensions of \( L \), \( \neg(E_T \leq_{\Delta^1_1} F_{\omega_1}) \) whenever \( T \) is a counterexample to Vaught’s conjecture. There the author put considerable effort into treating isomorphisms of counterexamples to Vaught’s conjecture as merely thin equivalence relations and used only one model-theoretic fact about the Vaught’s conjecture ([2] Fact 6.5).

This note will embrace admissible model theory as presented in [6] to give a less technical and more self-contained proof that in \( L \) and set-generic extensions of \( L \), \( \neg(E_T \leq_{\Delta^1_1} F_{\omega_1}) \) whenever \( T \) is a counterexample to Vaught’s conjecture. It will also be shown using just \( ZFC \) that if \( T \) is a non-minimal counterexample to Vaught’s conjecture, then \( \neg(E_T \leq_{\Delta^1_1} F_{\omega_1}) \).

Assume \( 0^F \) exists, one has that \( E_{\omega_1} \leq_{\Delta^1_1} F_{\omega_1}. \) Hence it is a natural question whether it is consistent, using perhaps some large cardinal axioms, that \( E_T \leq_{\Delta^1_1} F_{\omega_1}. \) There is some remarks on this at the end of the note, but this question remains open.

## 2. Infinitary Logic and Admissibility

**Definition 2.1.** Let \( \mathcal{L} \) be a countable recursive first order language. Let \( S(\mathcal{L}) \) denote the set of \( \mathcal{L} \)-structures with domain \( \omega \).

Under some fixed recursive coding of \( \mathcal{L} \)-structures by elements of \( \omega^2 \), \( S(\mathcal{L}) = \omega^2 \) but the former notation is used to clearly indicate reals are being considered as \( \mathcal{L} \)-structures on \( \omega \).

**Definition 2.2.** Let \( \mathcal{L} \) be a recursive language. Let \( T \subseteq \mathcal{L}_{\omega_1, \omega} \) be a countable theory (set of sentences). Define the equivalence relation \( E_T \) on \( S(\mathcal{L}) \) by

\[
x E_T y \iff (x \not\models T \land y \not\models T) \lor (x \equiv y)
\]

where \( \models \) is the \( \mathcal{L} \)-satisfaction relation and \( \equiv \) is the \( \mathcal{L} \)-isomorphism relation.

Let \( z \) be any real so that \( T \) belong to any admissible set containing \( z \). \( \models T \) is a \( \Delta^1_1(z) \) relation. \( \equiv \) is a \( \Sigma^1_1 \) relation. \( E_p \) is \( \Sigma^1_1(z) \). \( E_T \) has all classes \( \Delta^1_1 \).

**Definition 2.3.** Let \( E \) be an equivalence relation on a Polish space \( \omega^2 \). \( E \) is a thin equivalence relation if and only if for all perfect subsets \( P \subseteq \omega^2 \), there exist some \( x, y \in P \) so that \( x E y \).

**Definition 2.4.** Let \( \mathcal{L} \) be a recursive language. A countable theory \( T \subseteq \mathcal{L}_{\omega_1, \omega} \) is a counterexample to Vaught’s conjecture if and only if \( E_T \) is a thin equivalence relation with uncountably many classes.

**Definition 2.5.** Let \( A \) be a transitive set. \( A \) is an admissible set if and only if \( (A, \in) \models \text{KP} \). (\( \text{KP} \) is Kripke-Platek set theory.)

Let \( x \) be a set. \( \alpha \) is an \( x \)-admissible ordinal if and only if there is some admissible set \( A \) with \( x \in A \) and \( \alpha = A \cap \text{ON} \). Let \( A(x) \) denote the set of \( x \)-admissible ordinals.

Suppose \( \alpha \) is an admissible ordinal. Let \( \alpha^+ \) denote the least admissible ordinal above \( \alpha \).

If \( x \in \omega^2 \), then let \( \omega^x_1 \) be the least \( x \)-admissible ordinal.

**Definition 2.6.** Let \( F_{\omega_1} \) be the equivalence relation on \( \omega^2 \) defined by \( x F_{\omega_1} y \) if and only if \( \omega^x_1 = \omega^y_1 \).

\( F_{\omega_1} \) is a \( \Sigma^1_1 \) equivalence relation with all classes \( \Delta^1_1 \) and uncountably many classes.

**Definition 2.7.** Let \( \mathcal{L} \) be a recursive language. A fragment of \( \mathcal{L}_{\omega_1, \omega} \) is a collection \( \mathcal{F} \) of formulas of \( \mathcal{L}_{\omega_1, \omega} \) with some closure properties (see [1] Definition III.2.1).

Let \( A \) be a transitive set. Let \( \mathcal{L}_A = \mathcal{L}_{\omega_1, \omega} \cap A \).
\[ \text{Fact 2.10. Let } F \subseteq L_{\omega_1, \omega} \text{ be a fragment if and only if } F = L_A \text{ for some transitive set } A \text{ such that} \]
\[ (1) A \text{ is closed under pairing, union, and cartesian product. For all } a \in A, \text{ ON} \cap \text{tc}(a) \in A, \text{ where tc}(a) \text{ is the transitive closure of } a. \]
\[ (2) \text{If } \varphi(x_1, \ldots, x_n) \in L_{\omega_1, \omega} \cap A \text{ and } t_1, \ldots, t_n \in A \text{ are } L\text{-terms, then } \varphi(t_1, \ldots, t_n) \in A. \]
\[ \text{If } A \text{ is countable, then } L_A \text{ is called a countable fragment. If } A \text{ is admissible, then } L_A \text{ is called an admissible fragment.} \]

\[ \text{[6] showed how to enumerate the extensions of a scattered theory in various fragments and determined } \]
\[ \text{what admissible sets this enumeration belong to. These ideas are summarized below:} \]

\[ \text{Definition 2.8. Let } L \text{ be a recursive language. Let } F \text{ be a countable fragment of } L_{\omega_1, \omega}. \text{ Let } T \subseteq F \text{ be a } \]
\[ \text{theory in the fragment } F. \]
\[ T \text{ is finitarily consistent if not contradiction can be proved using the } T \text{ and the finitary inference rules.} \]
\[ T \text{ is } \omega\text{-complete in } F \text{ if and only for all sentences } \varphi \in F, \text{ either } \varphi \in T \text{ or } \neg \varphi \in T \text{ and whenever a sentence of the form } \bigwedge_{i \in \omega} \varphi_i \in T, \text{ then there is some } \varphi_i \in T. \]

\[ \text{Fact 2.9. If } T \text{ is finitarily consistent and } \omega\text{-complete in some fragment } F \subseteq L_{\omega_1, \omega}, \text{ then } T \text{ has a model.} \]
\[ \text{Proof. See [6] Proposition 4.1.} \]

\[ \text{Fact 2.10. Let } L \text{ be a recursive language. Let } \gamma \text{ be an ordinal. Suppose } (F_\alpha : \alpha < \gamma) \text{ and } (T_\alpha : \alpha < \gamma) \text{ are } \]
\[ \text{sequences of fragments and theories so that } T_\alpha \text{ is a finitary consistent and } \omega\text{-complete theory in the fragment } F_\alpha \text{ and whenever } \alpha < \beta < \gamma, \text{ } F_\alpha \subseteq F_\beta \text{ and } T_\alpha \subseteq T_\beta. \text{ Then } \bigcup_{\alpha < \gamma} T_\alpha \text{ is a finitary consistent and } \omega\text{-complete theory in the fragment } \bigcup_{\alpha < \gamma} F_\alpha. \]
\[ \text{Proof. See [6] Proposition 4.2.} \]

\[ \text{Definition 2.11. Let } L \text{ be a recursive language. Let } F \text{ be a countable fragment of } L_{\omega_1, \omega} \text{ and } T \text{ be theory in the fragment } F. \text{ } \]
\[ T \text{ is scattered if and only for all countable fragments } F' \supseteq F, \text{ there are only countably many finitarily consistent and } \omega\text{-complete theories } T' \subseteq F' \text{ which extend } T. \]
\[ \text{Fact 2.12. Let } T \text{ be a theory in some countable fragment. } T \text{ is a counterexample to Vaught's conjecture if and only if } T \text{ is scattered and has uncountably many countable models up to isomorphism.} \]

\[ \text{Definition 2.13. Let } L \text{ be a recursive language. Let } T \text{ be a countable theory in } L_{\omega_1, \omega}. \text{ The Morley tree of } T, \text{ denoted } MT(T), \text{ is defined as follows:} \]
\[ \text{Level 0 of } MT(T): \text{ Let } F_0 \text{ be the smallest countable fragment of } T. \text{ The nodes on level 0 are those } \]
\[ \text{finitary consistent and } \omega\text{-complete theories } T' \text{ in } F_0. \text{ For each such } T', \text{ let } F_{T'} = F_0. \]
\[ \text{Level } \alpha + 1: \text{ The nodes of level } \alpha \text{ have been defined and for each } U \text{ on level } \alpha, \text{ the fragment } F_U \text{ have been defined.} \]
\[ \text{Let } U \text{ be a node on level } \alpha. \text{ If } U \text{ is } \omega\text{-categorical, then } U \text{ will not extend to level } \alpha + 1. \text{ Suppose } U \text{ is not } \omega\text{-categorical.} \]
\[ U \text{ has a non-isolated } n\text{-type for some } n \in \omega. \text{ Let } F_U' \text{ be the smallest countable fragment containing } \bigwedge p \text{ for each non-isolated types } p \text{ in } S_n(U). \text{ Every finitarily consistent and } \omega\text{-complete theory } U' \text{ extending } U \text{ in the fragment } F_U' \text{ is a node on level } \alpha + 1. \text{ Let } F_{T'} = F_U'. \text{ The nodes of level } \alpha + 1 \text{ are exactly the objects obtained in this way.} \]
\[ \text{Level } \alpha \text{ where } \alpha \text{ is a limit: Suppose levels less than } \alpha \text{ have been defined and for each node } U \text{ in such level, } \]
\[ F_U \text{ have been defined. For each sequence } (T_\beta : \beta < \alpha) \text{ of nodes with } T_\beta \text{ on level } \beta \text{ and for all } \gamma < \beta < \alpha, \]
\[ T_\gamma \subseteq T_\beta, \text{ then put } T' = \bigcup_{\beta < \alpha} T_\beta \text{ and let } F_{T'} = \bigcup_{\beta < \alpha} F_\beta. \text{ (Note that } T' \text{ is an } \omega\text{-complete and finitary consistent theory in the fragment } F_{T'}. \text{ The nodes on level } \alpha \text{ are exactly those obtained in this way.} \]

\[ \text{Remark 2.14. In the successor case, note that if } U \text{ is some theory on level } \alpha \text{ of } MT(T) \text{ in the countable fragment } F_U \text{ which is not } \omega\text{-categorical, then there must be a non-isolated type in } U. \text{ To see this: Suppose } \]
\[ M \text{ and } N \text{ are two nonisomorphic models. Let } \bar{a} \text{ in } M \text{ and } \bar{b} \text{ in } N \text{ so that } tp^M(\bar{a}) = tp^N(\bar{b}) \text{ (here the type is taken in the fragment } F). \text{ Thus there is a partial isometry of } M \text{ to } N \text{ taking } \bar{a} \text{ to } \bar{b}. \text{ Let } a \in M.} \]
tp^M(\bar{a}) is isolated by some formula \psi(\bar{x}, x). M \models (\exists x)(\psi(\bar{a}, x)). So N \models (\exists x)(\psi(\bar{b}, x)). Let b \in N so that N \models \psi(b, b). Then bb realizes tp^M(\bar{a}). Hence the map \bar{a} to bb is a partial isometry of M to N. Using this and a back-and-forth argument, one can show that M and N are isomorphic.

Let p be any nonisolated type of U. Pick any \varphi(x) \in p. There exists some q \neq p so that \varphi \in q. So there is some \psi(x) so that \psi(\bar{x}) \in p and \neg \psi(\bar{x}) \in q. Hence (\exists x)(\bigwedge p) and (\exists x)(\bigwedge q) are consistent sentences in the fragment \mathcal{F}'_U. Hence U has at least two incompatible extension on level \alpha + 1.

**Fact 2.15.** For \alpha < \omega_1, level \alpha of \mathcal{M}T(T) is countable.

The following fact collects some definability properties of the Morley tree of a scattered theory:

**Fact 2.16.** Let \mathcal{L} be a recursive language. Let T be a scattered theory in \mathcal{L}. Suppose \alpha so that L_\alpha(T) is admissible. For all \beta < \alpha, \mathcal{M}T(T) \models \beta \in L_\alpha(T). Moreover, there is a \Sigma_1 relation \Xi(x, \gamma, z) (with no parameters) so that whenever T is scattered, L_\alpha(T) is admissible, b \in L_\alpha(T), and \beta \in ON \cap L_\alpha(T)

\[(\mathcal{M}T(T) \models \beta = b) \iff L_\alpha(T) \models \Xi(T, \beta, b).
\]

**Proof:** See [6] Proposition 4.4.

**Fact 2.17.** Let \mathcal{L} be a recursive language. The statement “T is a counterexample to Vaught’s conjecture” is \Sigma_1 definable in L_{\omega(T)} = (H_{\omega_1}(T)). By Schoenfield absoluteness, this statement is absolute to inner models containing T with the same \omega_1.

**Definition 2.18.** Let \alpha be a limit ordinal. Let T' be a node of \mathcal{M}T(T) in some level \gamma < \alpha. T' is unbounded below \alpha if and only if for all \beta with \gamma < \beta < \alpha, there is an extension of T' on level \beta.

A node T' \in \mathcal{M}T(T) on level \gamma is uniquely-unbounded below \alpha if and only if for all \beta such that \gamma < \beta < \alpha, there is a unique unbounded below \alpha node on level \beta that extends T'.

**Fact 2.19.** Let \mathcal{L} be a recursive language. Let T be a counterexample to Vaught’s conjecture. Let \alpha < \omega_1 be such that L_\alpha(T) is admissible. Let T' \in \mathcal{M}T(T) be unbounded below \alpha. Then there is some extension of T'' of T' on some level below \alpha which is uniquely-unbounded below \alpha.

**Proof:** The following argument appears in the proof of [6] Proposition 4.7:

Let T' belong to level \beta_0 of \mathcal{M}T(T). Note that for each \beta with \beta_0 < \beta < \alpha, T' has an extension on level \beta which is unbounded below \alpha. Otherwise, consider the relation

R(U, \gamma) \iff (\exists b)(\Xi(T, \gamma, b) \land (\forall L \in b)(\neg(U \subseteq L)))

\Xi is the \Sigma_1 relation of Fact 2.16. This is a \Sigma_1 relation in L_\alpha(T). Informally, this states that no nodes on level \gamma of \mathcal{M}T(T) is an extension of U. (\mathcal{M}T(T) \models \beta + 1 \in L_\alpha(T)) by Fact 2.16. Let X = \{U \in \mathcal{M}T(T) \mid \beta + 1 : U \supseteq T'\}. X \subseteq L_\alpha(T) by \Delta_1-separation. By the failure of the claim, one has that (\forall U \in X)(\exists \gamma < \delta)(R(U, \gamma)). By \Sigma_1-replacement, there exists some \delta < \alpha so that (\forall U \in X)(\exists \gamma < \delta)(R(U, \gamma)). This then implies T' is not unbounded below \alpha. Contradiction.

Now suppose there is no node \rangle U \rangle extending T' which is a uniquely-unbounded node below \alpha. Externally in the real universe, define the following map \Phi : \omega_2 \rightarrow \mathcal{M}T(T) \upharpoonright \alpha as follows with the property that for all s, \Phi(s) \in \mathcal{M}T(T) is an unbounded extension of T' and if s \subseteq t then \Phi(t) extends \Phi(s):

Let \Phi(\emptyset) = T'.

Suppose \Phi(s) has been defined for all s \in \omega_2. Since T' has no uniquely-unbounded below \alpha extension, neither does \Phi(s). Therefore, find some U_0 and U_1 extending \Phi(s), are both unbounded below \alpha, and are incompatible. Let \Phi(s'0) = U_0 and \Phi(s'1) = U_1.

For each x \in \omega_2, let U_x = \bigcup_{n \in \omega} \Phi(x \upharpoonright n). By Fact 2.10, U_x is a node on level \alpha of \mathcal{M}T(T). If x \neq y, then U_x and U_y is incompatible and hence different nodes on that level. Hence level \alpha of \mathcal{M}T(T) is not countable. Contradiction.

**Definition 2.20.** Let \mathcal{L} be a recursive language. A counterexample to Vaught’s conjecture T is a minimal counterexample to Vaught’s conjecture if and only if for all \varphi \in L_{\omega_1}, either \bar{T} \cup \{\varphi\} or \bar{T} \cup \{\neg \varphi\} has only countably many models.

**Corollary 2.21.** Let \mathcal{L} be a recursive language. Suppose T is a counterexample to Vaught’s conjecture. Then there is an extension T' of T which is a minimal counterexample to Vaught’s conjecture.
Fact 2.23. (Jensen’s Model Existence Theorem) Let bounded. Let an uniquely-unbounded node. Let \( L \) be a counterexample to Vaught’s conjecture.

Proof. This is well known result of Harnik and Makkai. The following is a proof using the above ideas. See \[1\], Lemma II.8.4. See \[3\] Section 4, Lemma 11.

Proof. See \[1\], Lemma II.8.4.

Fact 2.24. (Truncation Lemma) Suppose \( B = (B, \in) \) is a model of KP. Then WF\( (B) \) is also a model of KP.

Proof. See \[1\], Lemma II.8.4.

Fact 2.25. (Sacks’ Theorem) Let \( z \in \omega^2 \). Let \( \alpha \in \Lambda(z) \cap \omega_1 \). Then there exists some \( y \in \omega^2 \) with \( z \leq_T y \) and \( \omega^y_1 = \alpha \).

Proof. See \[5\]. See \[2\] Theorem 2.16 for a proof of this result using Fact 2.23.

Fact 2.26. Let \( \alpha \) be a countable admissible ordinal and \( z \in \omega^2 \) be such that \( \alpha < \omega^z_1 \). Then the set \( \{ y \in \omega^2 : \omega^y_1 = \alpha \} \) is \( \Delta^1_1(z) \).

Proof. See \[2\] Proposition 2.38 for the computations.

3. Nonreducibility Results

Fact 3.1. Let \( \mathcal{L} \) be a recursive language. Let \( T \subseteq \mathcal{L}_{\omega_1^\omega} \) be a counterexample to Vaught’s conjectures. Let \( y \in \omega^2 \) so that \( T \in L_{\omega^y_1}(y) \). Suppose \( U \) is a terminal node of \( \mathcal{M}(T) \) on some level above \( \omega^y_1 + 1 \). Then the \( E_T \) equivalence class \( \{ x \in S(\mathcal{L}) : x \models U \} \) is not \( \Sigma^1_1(y) \).

Proof. This result is analogous to \[2\] Theorem 6.4. The proof is as follows:

Let \( B = \{ x \in S(\mathcal{L}) : x \models U \} \). Suppose \( B \) is \( \Sigma^1_1(y) \). Let \( R \) be a \( y \)-recursive tree on \( 2 \times \omega \) so that \( x \in B \iff (\exists f)(f \in [R^\omega]) \).

Let \( \mathcal{J} \) be a language which is \( \Delta^1_1 \) in \( L_{\omega^y_1}(y) \) consisting of:

(i) A binary relation symbol \( \bar{\epsilon} \).
(ii) For each \( a \in L_{\omega^y_1}(y) \), a new constant symbol \( \bar{a} \).
(iii) Two distinct constant symbols, \( \dot{c} \) and \( \dot{d} \).

Let \( H \) be a theory in the countable admissible fragment \( J_{\omega_1}(\omega) \) which is \( \Sigma_1 \) definable in \( L_{\omega_1}(\omega) \) defined consisting of the following sentences:

(I) \( ZFC \models P \).

(II) For each \( a \in L_{\omega_1}(\omega) \), \( \forall v \in a \) \( \forall z \in v = \dot{z} \).

(III) \( \dot{c} \subseteq \omega, \dot{d} : \omega \rightarrow \omega \).

(IV) \( \dot{d} \in [\mathbb{R}]^x_1 \).

\( H \) is consistent: Let \( x \) be any model of \( U \). Then \( x \in B \) so there is some \( f : \omega \rightarrow \omega \) so that \( f \in [\mathbb{R}]^x_1 \). Let \( B \) be the \( \mathcal{J} \)-structure defined by: The domain is \( B = H_{R_1} \). \( \dot{c}^B = \dot{c}^x \). Let \( \dot{c}^x = x \). Let \( \dot{d}^B = f \). Then \( B \models H \).

By Fact 2.23 let \( C \models H \) so that \( L_{\omega_1}(\omega) \subseteq C \), \( WF(C) \) is transitive, and \( WF(C) \cap ON = \omega_1^C \). Let \( c = \dot{c}^C \) and \( d = \dot{d}^C \). By (III), \( c \subseteq \omega \) and \( d : \omega \rightarrow \omega \). Hence \( c, d \in WF(C) \). By \( \Delta_1 \)- absoluteness from \( C \) down to \( WF(C) \) and then up to the real universe \( V \), one has that \( d \in [\mathbb{R}]^x_1 \). Hence \( c \models U \). By Fact 2.24 \( WF(B) \) is an admissible set. \( c \) then belongs to an admissible set of ordinal height \( \omega_1^B \). By a result of Nadel (see [6] Section 2), \( c \) is a model of a terminal node of \( MT(T) \) (i.e. \( \omega \)-categorical theory) on level no higher than \( \omega_1^B + 1 \). But \( c \models U \) and \( U \) is an \( \omega \)-categorical theory on level above \( \omega_1^B + 1 \). Contradiction.

\[ \square \]

**Theorem 3.2.** Let \( \mathcal{L} \) be a recursive language. Let \( T \subseteq \mathcal{L}_{\omega_1, \omega} \) be a counterexample to Vaught’s conjecture. Suppose there is a \( \Delta_1^1 \) function \( \Phi : S(\mathcal{L}) \rightarrow \omega \) 2 which witnesses \( E_T \leq \Delta_1^1 F_{\omega_1} \). Then there is some real \( z \) so that for all \( \alpha \in \Lambda(z) \), \( \alpha^+ \notin \Lambda(z) \).

**Proof.** This is [3] Theorem 6.9.

Let \( z \in \omega^2 \) so that \( \Phi \) is \( \Delta_1^1(z) \) and \( T \subseteq \mathcal{L}_{\omega_1}(z) \).

First, this result will be shown for any \( \alpha \in \Lambda(z) \cap \omega_1 \). By Schoenfeld absoluteness, \( \Phi \) continues to witness \( E_T \leq \Delta_1^1 F_{\omega_1} \). Hence to conclude this theorem for all \( \alpha \in \Lambda(z) \), one should apply the countable case to a Coll(\( \omega_1 \), \( \alpha \))-extension.

So suppose there is some \( \alpha \in \Lambda(z) \cap \omega_1 \) so that \( \alpha^+ \in \Lambda(z) \). Use Fact 2.22 to find three distinct terminal nodes \( A, B, \) and \( C \) on \( MT(T) \) on levels above \( \alpha + 1 \) and below \( \alpha^+ \). Let \( M, N, \) and \( P \) be some countable models of \( A, B, \) and \( C \), respectively. Since \( \Phi \) is a reduction of \( E_T \) to \( F_{\omega_1} \), there is at most one \( X \in \{ M, N, P \} \) so that \( \omega_1^X = \alpha \). If such an \( X \) exists, then without loss of generality suppose \( X = P \).

Claim: \( \omega_1^{\Phi(M)} \geq \alpha^+ \) and \( \omega_1^{\Phi(N)} \geq \alpha^+ \).

To prove the claim: Suppose \( \omega_1^{\Phi(M)} < \alpha^+ \). \( M \) and \( P \) are not isomorphic since \( A \) and \( C \) are distinct terminal nodes on the \( MT(T) \). Since \( \Phi \) is a reduction and \( \omega_1^{\Phi(P)} = \alpha \), \( \omega_1^{\Phi(M)} \neq \alpha \). Since there are no admissible ordinals between \( \alpha \) and \( \alpha^+ \), \( \omega_1^{\Phi(M)} < \alpha \). Since \( A \) is a terminal node, for all \( X \in S(\mathcal{L}) \),

\[ X \models A \iff X \models M \iff \Phi(X) \in [\Phi(M)]_{F_{\omega_1}} \]

Using Fact 2.25 find some \( y \in \omega^2 \) so that \( z \leq_T y \) and \( \omega_1^y = \alpha \). Then \( [\Phi(M)]_{F_{\omega_1}} \) is \( \Delta_1^1(y) \) by Fact 2.26. This observation and the above, shows that \( \{ X \in S(\mathcal{L}) : X \models A \} \) is \( \Sigma_1^1(y, z) = \Sigma_1^1(y) \). However, this contradicts Fact 3.1. This proves the claim.

By Fact 2.16 for all \( \beta < \alpha^+ \), \( MT(T) \mid \beta \in L_{\alpha^+}(z) \). Hence \( A, B \in L_{\alpha^+}(z) \). Since \( \Phi \) is \( \Delta_1^1(z) \), let \( \tilde{R} \) be a \( z \)-recursive tree on \( 2 \times 2 \times \omega \) so that

\[ \Phi(x) = y \iff (\exists f)(f \in [R(x,y)]) \]

Let \( J \) be a language which is \( \Delta_1 \) in \( L_{\alpha^+}(z) \) consisting of:

(i) A binary relation symbol \( \dot{\xi} \).

(ii) For each \( \alpha \in L_{\alpha^+}(z) \), a new constant symbol \( \dot{a} \).

(iii) Six distinct constant symbols \( \dot{M}, \dot{N}, \dot{c}, \dot{d}, \dot{f}, \) and \( \dot{g} \).

Let \( H \) be the theory in the countable admissible fragment \( J_{\alpha^+}(z) \) which is \( \Sigma_1 \) definable in \( L_{\alpha^+}(z) \) consisting of the following sentences:

(I) \( ZFC \models P \).

(II) For each \( \alpha \in L_{\alpha^+}(z) \), \( \forall v \in a \) \( v \models \dot{a} \).

(III) \( \dot{M}, \dot{N}, \dot{c}, \dot{d} \subseteq \dot{\omega} \). \( \dot{f}, \dot{g} : \dot{\omega} \rightarrow \dot{\omega} \).
(IV) \( f \in [R^{(M, c)}] \) and \( \hat{y} \in [R^{(N, d)}] \).
(V) \( M \models A \) and \( \hat{N} \models B \).
(VI) For each \( \beta < \alpha^+ \), \( \beta \) is not \( \check{\mathcal{C}} \)-admissible and \( \beta \) is not \( \check{d} \)-admissible.

\( H \) is consistent: Let \( f, g \in \omega^\omega \) so that \( f \in [R^{(M, \Phi(M))}] \) and \( g \in [R^{(N, \Phi(N))}] \). Let \( \mathcal{B} \) be the \( \mathcal{J} \)-structure defined by: The domain is \( B = H_{R_\mathcal{B}} \). \( \mathcal{C} \subseteq B \). \( M^B = M, N^B = N, \check{d}^B = \Phi(M), \check{d}^B = \Phi(N) \). \( \hat{f}^B = f, \hat{g}^B = g \). Hence no real satisfying the condition of Theorem 3.2 can exists.

By Fact 2.23 let \( c \models H \) so that \( L_{\check{c}}(z) \subseteq C \), WF(C) is transitive, and WF(C) \( \cap \mathbb{ON} = \alpha^+ \). Let \( M' = \hat{M}^C \), \( N' = \hat{N}^C, c = \check{c}, d = \check{d}, f' = \check{f}, g' = \check{g} \). Since all the objects above are either subsets of \( \omega \) or functions from \( \omega \) to \( \omega \), they all belong to WF(C). By Fact 2.24 WF(C) is an admissible set of ordinal height \( \alpha^+ \). Hence \( \omega^\omega_1, \omega^\omega_\check{\mathcal{C}} \leq \alpha^+ \). By \( \check{\Delta}_1 \)-absoluteness, each \( \beta < \alpha^+ \) is not \( \check{\mathcal{C}} \)-admissible or \( \check{d} \)-admissible. Hence \( \omega^\omega_1 = \omega^\omega_\check{\mathcal{C}} = \alpha^+ \). By \( \check{\Delta}_1 \)-absoluteness, \( f' \in [R^{(\check{M}', \check{\mathcal{C}})}], g' \in [R^{(\check{N}', \check{\mathcal{C}})}], M' \models \check{A}, \check{N}' \models \check{B} \). In particular \( c = \Phi(M') \) and \( d = \Phi(N') \). Thus \( \omega^\mathcal{C}_\Phi(M') = \omega^\mathcal{C}_\Phi(N') = \alpha^+ \). Therefore \( \Phi(M') \models E_{\omega_1}(\Phi(N')) \). \( \Phi(M') \models A \) and \( \Phi(N') \models B \) implies that \( \neg(M' \equiv N') \). Thus \( \neg(\Phi(M'), E_{\omega_1}(\Phi(N'))) \). \( \Phi \) is not a reduction. Contradiction.

**Corollary 3.3.** In \( L \) and set-generic extensions of \( L \), \( \neg(E_T \leq \Delta^*_1, E_{\omega_1}) \) whenever \( T \) is a countereexample to Vaught’s conjecture.

**Proof.** For any real \( z \in \omega^2 \) for some \( \beta < \omega_1 \). Above \( \beta \), every admissible ordinal is \( \omega \)-admissible. Hence no real satisfying the condition of Theorem 3.2 can exists.

Given that \( E_{\omega_1} \leq \Delta^*_1, E_{\omega_1} \) holds if \( \check{\mathcal{C}} \) exists and \( \neg(E_{\omega_1} \leq \Delta^*_1, E_{\omega_1}) \) in set-generic extensions of \( L \), a natural question is whether it is consistent (possible relative to some large cardinal axiom) that \( E_T \leq \Delta^*_1, E_{\omega_1} \) where \( T \) is a countereexample to Vaught’s conjecture. The following result is provable in ZFC:

**Theorem 3.4.** Let \( \mathcal{L} \) be a recursive language. Let \( T \subseteq L_{\omega_1, \omega} \) be a countereexample to Vaught’s conjecture. Suppose there is a \( z \in \omega^2 \) so that there is a countable \( \omega \)-recursively inaccessible ordinal \( \alpha \) with \( T \models L_{\omega_1}(z) \) and \( MT(T) \) has two nodes on level \( \alpha \). Then there is no \( \Delta^*_1(z) \) function \( \Phi : S(\mathcal{L}) \rightarrow \omega^2 \) witnessing \( E_T \leq \Delta^*_1, E_{\omega_1} \).

**Proof.** Let \( A \) and \( B \) be two distinct tree on level \( \alpha \) of \( MT(T) \). For \( \gamma < \alpha \), let \( A_\gamma \) and \( B_\gamma \) be the unique node on level \( \gamma \) which \( A \) and \( B \) extends, respectively. For some \( \beta < \alpha, A_\beta \neq B_\beta \). By Fact 2.16 for each \( \gamma < \alpha, MT(T) \vdash \gamma \in L_{\omega_1}(z) \). So \( A_\beta, B_\beta \in L_{\omega_1}(T) \). Let \( M, N \in S(\mathcal{L}) \) be so that \( M \models A \) and \( N \models B \). In particular, \( M \models A_\beta \) and \( N \models B_\beta \).

Since \( \Phi \models \Delta^*_1(z) \), let \( \check{R} \) be a \( \omega \)-recursive tree on \( 2 \times 2 \times \omega \) so that \( \Phi(x) = y \Leftrightarrow (\exists f)(f \in [R^{(x, y)}]) \).

Let \( \mathcal{J} \) be a language which is \( \Delta_1 \) in \( L_{\omega_1}(z) \) consisting of:

(i) A binary relation symbol \( \check{\mathcal{E}}. \)
(ii) For each \( a \in L_{\omega_1}(z) \), a new constant symbol \( \hat{a} \).
(iii) Six distinct constant symbols \( \check{M}, \check{N}, \check{c}, \check{d}, \hat{f} \) and \( \hat{g} \).

Let \( H \) be the theory in the countable admissible fragment \( \mathcal{J}_{L_{\omega_1}(z)} \) which is \( \Sigma_1 \)-definable in \( L_{\omega_1}(z) \) consisting of the following sentences:

(I) \( ZFC - P \)
(II) For each \( a \in L_{\omega_1}(z), (\forall v)(v \check{\mathcal{E}} \hat{a} \Leftrightarrow \bigvee_{b \in a} v = \check{b}) \).
(III) \( \check{M}, \check{N}, \check{c}, \check{d} \subseteq \check{\omega}. \check{f}, \check{g} : \check{\omega} \rightarrow \check{\omega}. \)
(IV) \( f \in [R^{M, \check{\mathcal{C}}}], \check{g} \in [R^{N, \check{\mathcal{C}}}]. \)
(V) \( M \models A_\beta \) and \( N \models B_\beta \).
(VI) For each \( \gamma < \alpha \), \( \gamma \) is not \( \check{\mathcal{C}} \)-admissible and \( \gamma \) is not \( \check{d} \)-admissible.

\( H \) is shown to be consistent like in Theorem 3.2. In the model used to show the consistency of \( H, \check{c} \) and \( \check{d} \) will be interpreted as \( M \) and \( N \). Only the argument to show of (VI) is somewhat different than the situation in Theorem 3.2. (VI) would follow from that fact that \( \omega_1^\Phi(M) \geq \alpha \) and \( \omega_1^\Phi(N) \geq \alpha \). Suppose not, then \( \omega_1^\Phi(M) < \alpha \). Since \( \alpha \) is \( \omega \)-recursively inaccessible, there is some \( \omega \)-admissible ordinal \( \gamma \) so that \( \omega_1^\Phi(M) < \gamma < \alpha \). Using Fact 2.25 find some \( y \in \omega^2 \) so that \( z \leq_T y \) and \( \omega^\gamma_\check{\mathcal{C}} \gamma \). Let \( A_M \) be the terminal
node extending $A$ for which $M$ is the unique model of $M$ up to isomorphism. $A_M$ is on level $\alpha$ or higher of $\mathcal{MT}(T)$. Since $\Phi$ is a reduction, one has for all $X \in S(\mathcal{Z})$

$$X \models A_M \iff X E_T M \iff \Phi(X) \in [\Phi(M)]_{F_{\omega_1}}.$$ 

Since $\omega_1^{\Phi(M)} < \gamma = \omega_1^n$, Fact 2.26 implies that $[\Phi(M)]_{F_{\omega_1}}$ is $\Delta_1^1(y)$. Hence $[M]_{E_T}$ is $\Delta_1^1(z, y) = \Delta_1^1(y)$. This contradicts Fact 3.1.

Let $C \models H, \ M'$, and $N'$ be as in the proof of Theorem 3.2. As in that proof, $\omega_1^{\Phi(M')} = \omega_1^{\Phi(N')} = \alpha$. So $\Phi(M') F_{\omega_1} \Phi(N')$. However, $M' \models A_\beta$ and $N' \models B_\beta$. Hence $M'$ and $N'$ cannot be isomorphic. So $\neg(M' E_T N')$. This contradicts $\Phi$ being a reduction. □

**Corollary 3.5.** Suppose $T$ is a non-minimal counterexample to Vaught’s conjecture. Then $\neg(E_T \leq \Delta_1^1 F_{\omega_1}).$

**Proof.** Suppose $\Phi$ witnesses $E_T \leq F_{\omega_1}$. If $T$ is a non-minimal counterexample to Vaught’s conjecture. Let $z \in \omega^2$ be so that $T \in \mathcal{L}_{\omega_1}(z)$ and $\Phi$ is $\Delta_1^1(z)$. $\mathcal{MT}(T)$ has two distinct nodes $U$ and $V$ on some level $\gamma$ which are unbounded below $\omega_1$. Let $\alpha$ be any $z$-recursive ordinal greater than $\gamma$. Since $U$ and $V$ are both unbounded below $\omega_1$, there must be $U'$ and $V'$ on level $\alpha$ which extends $U$ and $V$. Then this contradicts Theorem 3.3. □

A natural question is whether it is consistent assuming large cardinals and there exist counterexamples to Vaught conjecture $T$ and some real $z$ so that all the $z$-recursively inaccessible levels of $\mathcal{MT}(T)$ has only one node. Montalbán informed the author that this is possible using Turing determinacy. The following is an argument using sharps.

**Fact 3.6.** Suppose $T$ is a minimal counterexample to Vaught’s conjecture. Suppose $T \in \mathcal{L}_{\gamma}(z)$, for some countable ordinal $\gamma$, and $z^2$ exists. Then $\mathcal{MT}(T)$ has a unique node on each $z^2$-admissible level above $\gamma$.

**Proof.** Let $\Psi(\alpha)$ be the statement

$$\text{I}_{\text{Coll}(\omega, \alpha)} \models \text{Coll}(\omega, \alpha) \ 	ilde{T} \text{ is uniquely-unbounded below } \alpha.$$ 

Since $T$ is a minimal counterexample to Vaught’s conjecture, $\psi(\omega_1)$ holds. Since $\mathcal{MT}(T) \subseteq \mathcal{L}(z)$ by Fact 2.16 and by absoluteness, $L(z) \models \psi(\omega_1^T)$. However $\omega_1^T$ is a Silver’s indiscernible for $L(z)$. Therefore, for any Silver indiscernible $\alpha$ for $L(z)$ above $\gamma$, $L(z) \models \psi(\alpha)$. All $z^2$-admissible ordinals are Silver’s indiscernibles for $L(z)$. Hence for all $z^2$-admissible ordinals above $\gamma$, $L(z) \models \psi(\alpha)$. Then by absoluteness, $T$ is uniquely-unbounded below $\alpha$ for each countable $z^2$-admissible ordinal above $\gamma$. This implies for all $z^2$-admissible ordinal above $\gamma$, $\mathcal{MT}(T)$ has a unique node on level $\alpha$. □

This result seems to suggests that perhaps if $T$ is a minimal counterexample in $L$ and $0^z$ exists, then $E_T \leq \Delta_1^1 F_{\omega_1}$ may be possible.

**Question 3.7.** If $T$ is a minimal counterexample to Vaught’s conjecture in $L$ and $0^z$ exists, then does $E_T \leq \Delta_1^1 F_{\omega_1}$ hold?

**References**


DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125
E-mail address: wcchan@caltech.edu