

LOSSY DATA COMPRESSION:
NONASYMPTOTIC FUNDAMENTAL LIMITS

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Abstract

The basic tradeoff in lossy compression is that between the compression ratio (rate) and the fidelity of reproduction of the object that is compressed. Traditional (asymptotic) information theory seeks to describe the optimum tradeoff between rate and fidelity achievable in the limit of infinite length of the source block to be compressed. A perennial question in information theory is how relevant the asymptotic fundamental limits are when the communication system is forced to operate at a given fixed blocklength. The finite blocklength (delay) constraint is inherent to all communication scenarios. In fact, in many systems of current interest, such as real-time multimedia communication, delays are strictly constrained, while in packetized data communication, packets are frequently on the order of 1000 bits.

Motivated by critical practical interest in non-asymptotic information-theoretic limits, this thesis studies the optimum rate-fidelity tradeoffs in lossy source coding and joint source-channel coding at a given fixed blocklength.

While computable formulas for the asymptotic fundamental limits are available for a wide class of channels and sources, the luxury of being able to compute exactly (in polynomial time) the non-asymptotic fundamental limit of interest is rarely affordable. One can at most hope to obtain bounds and approximations to the information-theoretic non-asymptotic fundamental limits. The main findings of this thesis include tight bounds to the non-asymptotic fundamental limits in lossy data compression and transmission, valid for general sources without any assumptions on ergodicity or memorylessness. Moreover, in the stationary memoryless case this thesis shows a simple formula approximating the nonasymptotic fundamental limits which involves only two parameters of the source.

This thesis considers scenarios where one must put aside traditional asymptotic thinking. A striking observation made by Shannon states that separate design of source and channel codes achieves the asymptotic fundamental limit of joint source-channel coding. At finite blocklengths, however, joint source-channel code design brings considerable performance advantage over a separate one. Furthermore, in some cases uncoded transmission is the best known strategy in the non-asymptotic regime, even if it is suboptimal asymptotically. This thesis also treats the lossy compression problem in which the compressor observes the source through a noisy channel, which is asymptotically equivalent to a certain conventional lossy source coding problem but whose nonasymptotic fidelity-rate tradeoff is quite different.

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Chapter 1

Introduction

1.1 Nonasymptotic information theory

This thesis drops the key simplifying assumption of the traditional (asymptotic) information theory that allows the blocklength to grow indefinitely and focuses on the best tradeoffs achievable in the finite blocklength regime in lossy data compression and joint source-channel coding. Since in many systems of current interest, such as real-time multimedia communication, delays are strictly constrained, while in packetized data communication, packets are frequently on the order of 1000 bits, non-asymptotic information-theoretic limits bear critical practical interest. Not only the knowledge of such theoretical limits allows us to assess what is achievable at a given fixed blocklength, but it also sheds light on the characteristics of a good code. For example, as we will show in Section 2.9, a Gaussian source can be represented with small mean-square error even nonasymptotically when all codewords lie on the surface of an n -dimensional sphere. The nonasymptotic analysis motivates the re-evaluation of existing codes and design principles based on asymptotic insights (for example, in Chapter 3 we will see that separate design of source and channel codes is suboptimal in the finite blocklength regime) and provides theoretical guidance for new code design (sometimes no coding at all performs best, see Section 3.8).

The challenge in attaining the ambitious goal of assessing the optimal nonasymptotically achievable tradeoffs is that information-theoretic nonasymptotic fundamental limits in general do not admit exact computable (in polynomial time) solutions. In fact, many researchers used to believe that such a nonasymptotic information theory could be nothing more than a collection of brute-force computations tailored to a specific problem. As Shannon admitted in 1953 [1, p.188], “A finite

delay theory of information... would indeed be of great practical importance, but the mathematical difficulties are formidable.” Despite those challenges, we are able to show very general tight bounds and simple approximations to the non-asymptotic fundamental limits of lossy data compression and transmission.

In doing so, we follow an approach to nonasymptotic information theory pioneered by Polyanskiy and Verdú [2–4] in the context of channel coding. A roadmap propounded in [3,4] to finite blocklength analysis of communications systems can be outlined as follows.

- 1) Identify the key random variable that governs the information-theoretic limits of a given communication system.
- 2) Derive novel bounds in terms of that random variable that tightly sandwich the nonasymptotic fundamental limit.
- 3) Perform a refined analysis of the new bounds leading to a compact approximation of the nonasymptotic fundamental limit.
- 4) Demonstrate that the approximation is accurate in the regime of practically relevant blocklengths.

While refinements to Shannon’s asymptotic channel coding theorem [5] have been studied previously using a central limit theorem approximation (e.g. [6]) and a large deviations approximation (e.g. [7]), the issue of how relevant these approximations are at a given fixed blocklength had not been addressed until the work of Polyanskiy and Verdú [3].

To put the contribution of this thesis in perspective, we now proceed to review the recent advancements in accessing the nonasymptotic fundamental limits of channel coding and almost lossless data compression.

1.2 Channel coding

The basic task of channel coding is to transmit M equiprobable messages over a noisy channel so that they can be distinguished reliably at the receiver end (see Fig. 1.2). The information-theoretic fundamental limit of channel coding is the maximum (over all encoding and decoding functions, regardless of their complexity) number of messages compatible with a given probability of error ϵ . In the standard block setting, the encoder maps the message to a sequence of channel input symbols

of length n , while the decoder observes that sequence contaminated by random noise and attempts to recover the original message. The *channel coding rate* is given by $\frac{\log M}{n}$.

The maximum channel coding rate compatible with vanishing error probability achievable in the limit of large blocklength is called the channel capacity and is denoted by C . Shannon's ground-breaking result [5] states that if the channel is stationary and memoryless, the channel capacity is given by the maximum (over all channel input distributions) mutual information between the channel input and its output. The mutual information is the expectation of the random variable called the *information density*

$$i_{X;Y}(x; y) = \log \frac{dP_{Y|X=x}}{dP_Y}(y) \quad (1.1)$$

where $P_X \rightarrow P_{Y|X} \rightarrow P_Y$ ¹, and, as demonstrated in [3, 4], it is precisely that random variable that determines the finite blocklength coding rate. In particular, the Gaussian approximation of $R(n, \epsilon)$, the maximum achievable coding rate at blocklength n and error probability ϵ , is given by, for finite alphabet stationary memoryless channels [3, 4, 6]

$$nR(n, \epsilon) = nC - \sqrt{nV}Q^{-1}(\epsilon) + O(\log n) \quad (1.2)$$

where V is the *channel dispersion* given by the variance of the channel information density evaluated with the capacity-achieving input distribution, and $Q^{-1}(\cdot)$ is the inverse of the standard Gaussian complementary cdf. As demonstrated in [3, 4], the approximation in (1.2) is not only beautifully concise but is also extremely tight for the practically relevant blocklengths of order 1000.



Figure 1.1: Channel coding setup.

1.3 Lossless data compression

In the basic setup of (almost) lossless compression, depicted in Fig. 1.2, a discrete source S is compressed so that encoder and decoder agree with probability at least $1 - \epsilon$, i.e. $\mathbb{P}[S \neq Z] \leq \epsilon$. The nonasymptotic fundamental limit is given by the minimum number $M^*(\epsilon)$ of distinct source

¹We write $P_X \rightarrow P_{Y|X} \rightarrow P_Y$ to indicate that P_Y is the marginal of $P_X P_{Y|X}$, i.e. $P_Y(y) = \sum_x P_{Y|X}(y|x)P_X(x)$.

encoder outputs compatible with a given error probability ϵ .

As pointed out by Verdú [8], the nonasymptotic fundamental limits of fixed-length almost lossless and variable-length strictly lossless data compression without prefix constraints are identical. In the latter case, ϵ corresponds to the probability of exceeding a target encoded length.

Lossless data compression is an exception in the world of information theory because the optimal encoding and decoding functions are known, permitting an exact computation of the nonasymptotic fundamental limit $M^*(\epsilon)$. The optimal encoder indexes M largest probability source outcomes and discards the rest, thereby maximizing the probability of correct reconstruction. A parametric expression for $M^*(\epsilon)$ is provided in [9]. The random variable which corresponds (in a sense that can be formalized) to the number of bits required to compress a given source outcome is the *information* in s , defined by

$$\iota_S(s) = \log \frac{1}{P_S(s)} \quad (1.3)$$

For example, the information in k i.i.d. coin flips with bias p is given by

$$\iota_{S^k}(s^k) = j \log \frac{1}{p} + (k - j) \log \frac{1}{1 - p} \quad (1.4)$$

where j is the number of ones in s^k . If we let the blocklength increase indefinitely, by the law of large numbers the information in most source outcomes would be near its mean, which is equal to $kh(p)$, where $h(\cdot)$ is the binary entropy function

$$h(p) = p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p} \quad (1.5)$$

At finite blocklengths, however, the stochastic variability of $\iota_{S^k}(S^k)$ plays an important role. In fact, $R(k, \epsilon)$, the minimum achievable source coding rate at blocklength k and error probability ϵ , is tightly bounded in terms of the cdf of that random variable [9]. Moreover, the minimum achievable compression rate of k i.i.d. random variables with common distribution P_S can be expanded as [6]²

$$kR(k, \epsilon) = kH(S) + \sqrt{kV(S)}Q^{-1}(\epsilon) - \frac{1}{2} \log k + O(1) \quad (1.6)$$

where $H(S)$ and $V(S)$ are the entropy and the varentropy of the source, given by the mean and the variance of the information in S , respectively. The idea behind this approximation is that by the central-limit theorem the distribution of $\iota_{S^k}(S^k) = \sum_{i=1}^k \iota_S(S_i)$ is close to $\mathcal{N}(kH(S), kV(S))$, and

²See also [9] where the completeness of Strassen's proof is disputed.

since the nonasymptotic fundamental limit is bounded in terms of the cdf of that random variable we have (1.6).

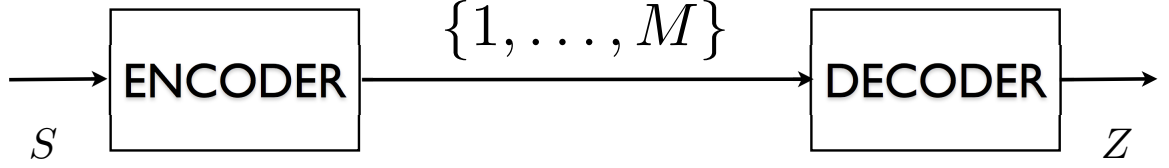


Figure 1.2: Source coding setup.

1.4 Lossy data compression

The fundamental problem of lossy compression is to represent an object with the fidelity of reproduction possible while satisfying a constraint on the compression ratio (rate). In the basic setup of lossy compression, depicted in Fig. 1.2, we are given a source alphabet \mathcal{M} , a reproduction alphabet $\widehat{\mathcal{M}}$, a *distortion measure* $d: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]$ to assess the fidelity of reproduction, and a probability distribution of the object S to be compressed down to M distinct values.

Unlike the channel coding setup of Section 1.2 as well as the (almost) lossless data compression setup of Section 1.3 in both of which the figure of merit is the block error rate, the lossy compression framework permits a study of reproduction of digital data under a symbol error rate constraint. Moreover, the lossy compression formalism encompasses compression of continuous and mixed sources, such as photographic images, audio and video.

Following the information-theoretic philosophy put forth by Claude Shannon in [5], this thesis is concerned with the fundamental rate-fidelity tradeoff achievable, regardless of coding complexity, rather than with designing efficient encoding and decoding functions, which is the main focus of coding theory.

In the conventional fixed-to-fixed (or block) setting \mathcal{M} and $\widehat{\mathcal{M}}$ are the k -fold Cartesian products of the letter alphabets \mathcal{S} and $\hat{\mathcal{S}}$, and the object to be compressed becomes a vector S^k of length k , and the *compression rate* is given by $\frac{\log M}{k}$. The remarkable insight made by Claude Shannon [10] reveals that it pays to describe the entire vector S^k at once, rather than each entry individually, even if the entries are independent. Indeed, in the basic task to digitize i.i.d. Gaussian random variables, the optimum one bit quantizer achieves the mean square error of $\frac{\pi-2}{\pi} \approx 0.36$, while the optimal block code in the limit achieves the mean square error of $\frac{1}{4}$. In another basic example where the goal is to compress 1000 fair coin flips down to 500, the optimum bit-by-bit strategy stores half

of the bits and guesses the rest, achieving the average BER of $1/4$, while the optimum rate- $\frac{1}{2}$ block code achieves the average BER of 0.11 , in the limit of infinite blocklength [10]. Note however that achieving these asymptotic limits requires unbounded blocklength, while the best estimate of the BER of the optimum rate- $\frac{1}{2}$ code operating on blocks of length 1000 provided by the classical theory is that the BER is between $1/4$ and 0.11 . To narrow down the gap between the upper and lower bounds to the optimal finite blocklength coding rate is one of the main goals of this thesis.

The minimum rate achievable in the limit of infinite length of the source block compatible with a given average distortion d is called the *rate-distortion* function and is denoted $R(d)$. Shannon's beautiful result [10,11] states that the rate-distortion function of a stationary memoryless source with separable (i.e. additive, or per-letter) distortion measure is given by the minimal mutual information between the source and the reproduction subject to an average distortion constraint. For example, for the i.i.d. binary source with bias p , the rate-distortion function is given by

$$R(d) = h(p) - h(d) \quad (1.7)$$

where $d \leq p < \frac{1}{2}$ is the tolerable bit error rate.

As we will see in Chapter 2, the key random variable governing non-asymptotic fundamental limits in lossy data compression is the so-called \mathbf{d} -tilted information, $j_S(s, d)$ (Section 2.2), which essentially quantifies the number of bits required to represent source outcome s of the source S with distortion measure \mathbf{d} within distortion d . For example, the BER-tilted information in k i.i.d. coin flips with bias p is given by

$$j_{S^k}(s^k, d) = j \log \frac{1}{p} + (k - j) \log \frac{1}{1 - p} - kh(d) \quad (1.8)$$

where j is the number of ones in s^k . If $d = 0$, the \mathbf{d} -tilted information is given by the information in s^k in (1.4). As in Sections 1.2 and 1.3, the mean of the key random variable, $j_{S^k}(S^k, d)$, is equal to the asymptotic fundamental limit, $kR(d)$, which agrees with the intuition that long sequences concentrate near their mean. At finite blocklength, however, the whole distribution of $j_{S^k}(S^k, d)$ matters. Indeed, tight upper and lower bounds to the nonasymptotic fundamental limit that we develop in Section 2.5 connect the probability that the distortion of the best code with M representation points exceeds level d (operational quantity) to the probability that the \mathbf{d} -tilted information exceeds $\log M$ (information-theoretic quantity). Moreover, as we show in Section 2.6, unless the blocklength is extremely short, the minimum finite blocklength coding rate for stationary

memoryless sources with separable distortion is well approximated by

$$kR(k, d, \epsilon) = kR(d) + \sqrt{k\mathcal{V}(d)}Q^{-1}(\epsilon) + O(\log k), \quad (1.9)$$

where ϵ is the probability that the distortion incurred by the reproduction exceeds d , $\mathcal{V}(d)$ is the *rate-dispersion function* which equals the variance of the d -tilted information. For our coin flip example,

$$\mathcal{V}(d) = p(1-p) \log^2 \frac{1-p}{p} \quad (1.10)$$

Notice that in this example the rate-dispersion function does not depend on d , and if the coin is fair, then $\mathcal{V}(d) = 0$, which implies that in that case the rate-distortion function is approached not at $O\left(\frac{1}{\sqrt{k}}\right)$ speed but much faster (in fact, as $\frac{1}{2k} \log k$, as we show in Section 2.7).

1.5 Lossy joint source-channel coding

In Chapter 3, we consider the lossy joint source-channel coding (JSCC) setup depicted in Fig. 1.3. The goal is to reproduce the source output under a distortion constraint as in Chapter 2, but the coding task is complicated by the presence of a noisy channel between the data source and its receiver. Now, not only one needs to partition the source space and choose a representative point for each region carefully so as to minimize the incurred distortion (the source coding task), but also to place message points in the channel input space intelligently so that after these points are contaminated by the channel noise, they still can be distinguished reliably at the receiver end (the channel coding task).



Figure 1.3: Joint source-channel coding setup.

In the standard block coding setup, encoder input and decoder output become k -vectors S^k and Z^k , and the channel input and output become n -vectors X^n and Y^n , and the JSCC coding rate is given by $\frac{k}{n}$. For a large class of sources and channels, in the limit of large blocklength, the maximum achievable JSCC rate compatible with vanishing excess distortion probability is characterized by the ratio $\frac{C}{R(d)}$ [11], where C is the *channel capacity*, i.e. the maximum asymptotically achievable channel

coding rate. The striking observation made by Shannon is that the asymptotic fundamental limit can be achieved by concatenating the optimum source and channel codes. This means that designing these codes separately does not incur any loss of performance, provided that the blocklength is allowed to grow without limit.

We will see in Chapter 3 that the situation is completely different at finite blocklengths, where joint design can bring significant gains, both in terms of the rate achieved and implementation complexity. This is one of those instances in which nonasymptotic information theory forces us to unlearn the lessons instilled by traditional asymptotic thinking.

More specifically, consider the Gaussian approximation of $R(n, \epsilon)$, the maximum achievable coding rate at blocklength n and error probability ϵ , which is given by (1.2) for finite alphabet stationary memoryless channels.

Concatenating the channel code in (1.2) and the source code in (1.9), we obtain the following achievable rate region compatible with probability ϵ of exceeding d :

$$nC - kR(d) \leq \min_{\eta + \zeta \leq \epsilon} \left\{ \sqrt{nV}Q^{-1}(\eta) + \sqrt{k\mathcal{V}(d)}Q^{-1}(\zeta) \right\} + O(\log n) \quad (1.11)$$

However in Section 3.5 we will see that the maximum number of source symbols transmissible using a given channel blocklength n satisfies

$$nC - kR(d) = \sqrt{nV + k\mathcal{V}(d)}Q^{-1}(\epsilon) + O(\log n) \quad (1.12)$$

which in general is strictly better than (1.11). The intuition is that separated scheme produces an error when either the channel realization is too noisy (small channel information density) or the source realization is too atypical (large d-tilted information), whereas in the joint scheme, a particularly good source realization may compensate for a particularly bad channel realization and vice versa.

In addition to deriving new general achievability and converse bounds for JSCC and performing their Gaussian approximation analysis, in Chapter 3 we revisit the dilemma of whether one should or should not code when operating under delay constraints. Symbol-by-symbol (uncoded) transmission is known to achieve the Shannon limit when the source and channel satisfy a certain probabilistic matching condition [12]. In Section 3.8 we show that even when this condition is not satisfied, symbol-by-symbol transmission, though asymptotically suboptimal, in some cases constitutes the best known strategy in the non-asymptotic regime.

1.6 Noisy lossy compression

Chapter 4 considers a lossy compression setup in which the encoder has access only to a noise-corrupted version X of a source S , and as before, we are interested in minimizing (in some stochastic sense) the distortion $d(S, Z)$ between the true source S and its rate-constrained representation Z (see Fig. 1.4). This problem arises if the object to be compressed is the result of an uncoded transmission over a noisy channel, or if it is observed data subject to errors inherent to the measurement system. Some examples include speech in a noisy environment, or photographic images corrupted by noise introduced by the image sensor and circuitry. Since we are concerned with preserving the original information in the source rather than preserving the noise, the distortion measure is defined with respect to the source.

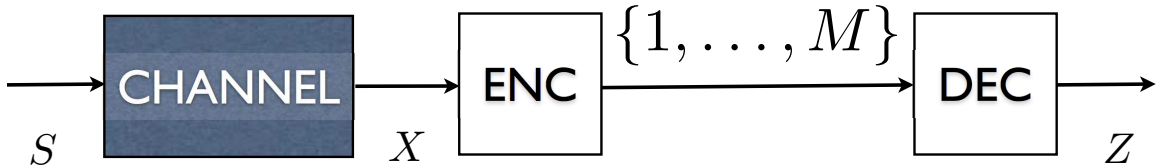


Figure 1.4: Noisy source coding setup.

What is intriguing about the noisy source coding problem is that, asymptotically, it is known to be equivalent to the traditional noiseless rate-distortion problem with a modified distortion function, but nonasymptotically, there is a sizable gap between the two minimum achievable rates, as we demonstrate in Chapter 4 by giving new nonasymptotic achievability and converse bounds to the achievable noisy source coding rate and performing their Gaussian approximation analysis. In particular, we show that the achievable noisy source coding rate of a discrete stationary memoryless source over a discrete stationary memoryless channel under a separable distortion measure can be approximated by (1.9) where the rate-distortion function $R(d)$ is that of the asymptotically equivalent noiseless rate-distortion function, and the rate-dispersion function $\mathcal{V}(d)$ is replaced by the *noisy rate-dispersion function* $\tilde{\mathcal{V}}(d)$ that satisfies

$$\tilde{\mathcal{V}}(d) > \mathcal{V}(d) \tag{1.13}$$

The gap between the two rate-dispersion functions, explicitly identified in Chapter 4, is due to the stochastic variability of the channel from S to X , which nonasymptotically cannot be neglected. That additional randomness introduced by the channel slows down the rate of approach to the asymptotic fundamental limit.

1.7 Channel coding under cost constraints

Chapter 5 studies the channel problem depicted in Fig. 1.1 where channel inputs have cost associated to them. Such constraints arise for example due to limited transmit power. Despite the obvious differences between lossy compression under distortion constraint and channel coding under cost constraint, there are certain mathematical parallels between the two problems that permitted us to apply an approach similar to that used in Chapter 2 to solve the former problem to characterize the nonasymptotic fundamental limit in the latter. More precisely, in Chapter 5 we define the \mathbf{b} -tilted information in a given channel input x , $J_{X,Y}^*(x, \beta)$, which parallels the notion of the \mathbf{d} -tilted information in lossy compression. We show that the maximum achievable channel coding rate under a cost constraint can be bounded in terms of the distribution of that random variable.

1.8 Remarks

We conclude the introductory chapter with a few remarks about our general approach.

1.8.1 Nonasymptotic converse and achievability bounds

Since nonasymptotic fundamental limits cannot in general be computed exactly, nonasymptotic information theory must rely on bounds. Although non-asymptotic bounds can be distilled from classical proofs of coding theorems, these bounds usually leave much tightness to be desired in the non-asymptotic regime [3, 4]. For example, observe the sizable gap between the tightest known achievability and converse bounds in Fig 1.5. Because these bounds were derived with the asymptotics in mind, asymptotically they converge to the correct limit, but in the displayed region of blocklengths they are frustratingly loose. For another example, recall that the classical achievability scheme in JSCC uses separate source/channel coding, which is rather suboptimal non-asymptotically. An accurate finite blocklength analysis therefore calls for novel upper and lower bounds that sandwich tightly the non-asymptotic fundamental limit. This thesis shows such bounds, which hold in full generality, without any assumptions on the source alphabet, stationarity or memorylessness, for each of the discussed lossy compression problems.

1.8.2 Average distortion vs. excess distortion probability

While the classical Shannon paper [11] as well as many subsequent installments in rate-distortion theory focused on the average distortion between the source and its representation by the decoder,

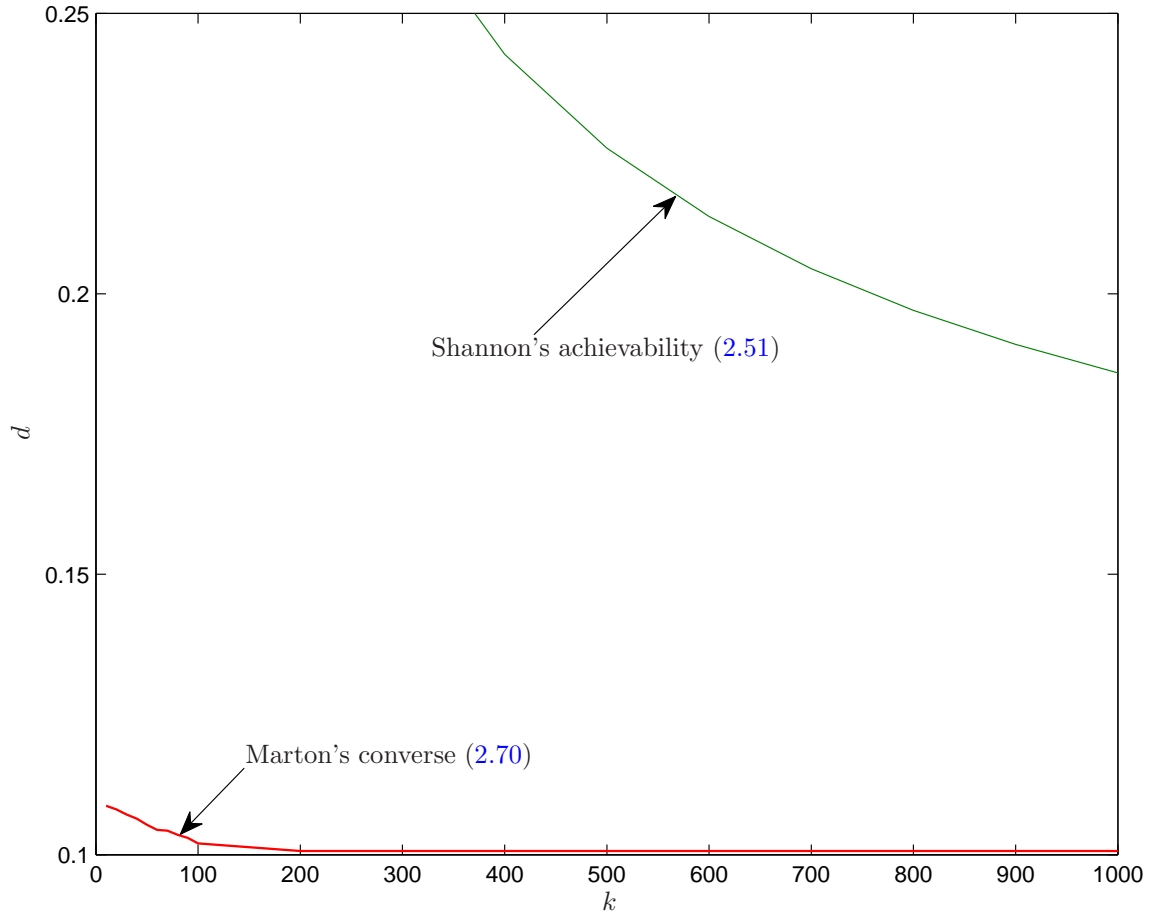


Figure 1.5: Known bounds to the nonasymptotic fundamental limit for the binary memoryless source with bias $p = \frac{2}{5}$ compressed at rate $\frac{1}{2}$ under the constraint that the probability that the fraction of erroneous bits exceeds d is no larger than $\epsilon = 10^{-4}$.

the figure of merit in this thesis is the excess distortion [13–18], i.e. our fidelity criterion is the probability of exceeding a certain distortion threshold. The excess distortion criterion is relevant in applications where, if more than a given fraction of bits is erroneous, the entire packet must be discarded. Moreover, for a given code, the excess distortion constraint is, in a way, more fundamental than the average distortion constraint, because varying d over its entire range and evaluating the probability of exceeding d gives full information about the distribution (and not just its mean) of the distortion incurred at the decoder output. This is not overly crucial if the blocklength is allowed to increase indefinitely, because due to the ergodic principle, most distortions incurred at the decoder output will eventually be close to their average. However, at a given fixed blocklength, the spread of those distortions around their average value is non-negligible, so focusing on just the average fails to provide a complete description of system performance. Excess distortion is thus a natural way to look at lossy compression problems at finite blocklengths.

1.8.3 Large deviations vs. central limit theorem

While the law of large numbers leads to asymptotic fundamental limits, the evaluation of the speed of convergence to those asymptotic limits calls for more sophisticated tools. There are two complementary approaches to such finer asymptotic analysis: the large deviations analysis which leads to the reliability function [7, 13], and the Gaussian approximation analysis which leads to dispersion [3, 6]. The reliability function approximation and the Gaussian approximation to the non-asymptotic fundamental limit are tight in different operational regimes. In the former, a rate which is strictly suboptimal with respect to the asymptotic fundamental limit is fixed, and the reliability function measures the exponential decay of the excess distortion probability to 0 as the blocklength increases. The error exponent approximation is tight if the error probability a system can tolerate is extremely small. However, as observed in [3] in the context of channel coding, already for probability of error as low as 10^{-6} to 10^{-1} , which is the operational regime for many high data rate applications, the Gaussian approximation, which gives the optimal rate achievable at a given error probability as a function of blocklength, is tight. Note that Marton’s converse bound in Fig. 1.5 is tight in terms of the large deviations asymptotics but is very loose at finite blocklengths, even for the quite low excess distortion probability of 10^{-4} in Fig. 1.5. This thesis follows the philosophy of [3] and performs the Gaussian approximation analysis of the new bounds.

Chapter 2

Lossy data compression

2.1 Introduction

In this chapter we present new achievability and converse bounds to the minimum sustainable lossy source coding rate as a function of blocklength and excess probability, valid for general sources and general distortion measures, and, in the case of stationary memoryless sources with separable distortion, their Gaussian approximation analysis. The material in this chapter was presented in part in [19–22].

Section 2.2 presents the basic notation and properties of the \mathbf{d} -tilted information. Section 2.3 introduces the definitions of the fundamental finite blocklengths limits. Section 2.4 reviews the few existing finite blocklength achievability and converse bounds for lossy compression, as well as various relevant asymptotic refinements of Shannon’s lossy source coding theorem. Section 2.5 shows the new general upper and lower bounds to the minimum rate at a given blocklength. Section 2.6 studies the asymptotic behavior of the bounds using Gaussian approximation analysis. The evaluation of the new bounds and a numerical comparison to the Gaussian approximation is detailed for:

- stationary binary memoryless source (BMS) with bit error rate distortion¹ (Section 2.7);
- stationary discrete memoryless source (DMS) with symbol error rate distortion (Section 2.8);
- stationary Gaussian memoryless source (GMS) with mean-square error distortion (Section 2.9).

¹Although the results in Section 2.7 are a special case of those in Section 2.8, it is enlightening to specialize our results to the simplest possible setting.

2.2 Tilted information

The information density between realizations of two random variables with joint distribution $P_S P_{Z|S}$ is denoted by

$$\iota_{S;Z}(s; z) \triangleq \log \frac{dP_{Z|S=s}}{dP_Z}(z) \quad (2.1)$$

Further, for a discrete random variable S , the information in outcome s is denoted by

$$\iota_S(s) \triangleq \log \frac{1}{P_S(s)} \quad (2.2)$$

When $\mathcal{M} = \mathcal{S}^k$, under appropriate conditions, the number of bits that it takes to represent s divided by $\iota_S(s)$ converges to 1 as k increases. Note that if S is discrete, then $\iota_{S;S}(s; s) = \iota_S(s)$.

For given P_S and distortion measure, denote

$$\mathbb{R}_S(d) \triangleq \inf_{\substack{P_{Z|S}: \\ \mathbb{E}[\mathbf{d}(S, Z)] \leq d}} I(S; Z) \quad (2.3)$$

$$\mathbb{D}_S(R) \triangleq \inf_{\substack{P_{Z|S}: \\ I(S; Z) \leq R}} \mathbb{E}[\mathbf{d}(S, Z)] \quad (2.4)$$

We impose the following basic restrictions on the source and the distortion measure.

(a) $\mathbb{R}_S(d)$ is finite for some d , i.e. $d_{\min} < \infty$, where

$$d_{\min} \triangleq \inf \{d: \mathbb{R}_S(d) < \infty\} \quad (2.5)$$

(b) The infimum in (2.3) is achieved by a unique $P_{Z|S}^*$ such that $\mathbb{E}[\mathbf{d}(S, Z^*)] = d$.

The uniqueness of the rate-distortion function achieving transition probability kernel in restriction (b) is imposed merely for clarity of presentation (see Remark 2.9 in Section 2.6). Moreover, as we will see in Appendix B.4, even the requirement that the infimum in (2.3) is actually achieved can be relaxed.

The counterpart of (2.2) in lossy data compression, which roughly corresponds to the number of bits one needs to spend to encode s within distortion d , is the following.

Definition 2.1 (d-tilted information). *For $d > d_{\min}$, the \mathbf{d} -tilted information in s is defined as²*

$$j_S(s, d) \triangleq \log \frac{1}{\mathbb{E}[\exp \{\lambda^* d - \lambda^* \mathbf{d}(s, Z^*)\}]} \quad (2.6)$$

²Unless stated otherwise, all log's and exp's in the thesis are arbitrary common base.

where the expectation is with respect to the unconditional distribution³ of Z^* , and

$$\lambda^* \triangleq -\mathbb{R}'_S(d) \quad (2.7)$$

It can be shown that (b) guarantees differentiability of $\mathbb{R}_S(d)$, thus (2.6) is well defined.

The following two theorems summarize crucial properties of \mathbf{d} -tilted information.

Theorem 2.1. Fix $d > d_{\min}$. For P_{Z^*} -almost every z , it holds that

$$j_S(s, d) = \imath_{S; Z^*}(s; z) + \lambda^* \mathbf{d}(s, z) - \lambda^* d \quad (2.8)$$

where λ^* is that in (2.7), and $P_{SZ^*} = P_S P_{Z^*|S}$. Moreover,

$$\mathbb{R}_S(d) = \min_{P_{Z|S}} \mathbb{E} [\imath_{S; Z}(S; Z) + \lambda^* \mathbf{d}(S, Z)] - \lambda^* d \quad (2.9)$$

$$= \min_{P_{Z|S}} \mathbb{E} [\imath_{S; Z^*}(S; Z) + \lambda^* \mathbf{d}(S, Z)] - \lambda^* d \quad (2.10)$$

$$= \mathbb{E} [j_S(S, d)] \quad (2.11)$$

and for all $z \in \widehat{\mathcal{M}}$

$$\mathbb{E} [\exp \{ \lambda^* d - \lambda^* \mathbf{d}(S, z) + j_S(S, d) \}] \leq 1 \quad (2.12)$$

with equality for P_{Z^*} -almost every z .

If the source alphabet \mathcal{M} is a finite set, for $s \in \mathcal{M}$, denote the partial derivatives

$$\dot{\mathbb{R}}_S(s, d) \triangleq \frac{\partial}{\partial P_{\bar{S}}(s)} \mathbb{R}_{\bar{S}}(d) \mid_{P_{\bar{S}}=P_S} \quad (2.13)$$

Theorem 2.2. Assume that the source alphabet \mathcal{M} is a finite set. Suppose that for all $P_{\bar{S}}$ in some Euclidean neighborhood of P_S , $\text{supp}(P_{\bar{Z}^*}) = \text{supp}(P_{Z^*})$, where $\mathbb{R}_{\bar{S}}(d) = I(\bar{S}; \bar{Z}^*)$. Then

$$\frac{\partial}{\partial P_{\bar{S}}(s)} \mathbb{E} [j_{\bar{S}}(S, d)] \mid_{P_{\bar{S}}=P_S} = \frac{\partial}{\partial P_{\bar{S}}(s)} \mathbb{E} [\imath_{\bar{S}}(S)] \mid_{P_{\bar{S}}=P_S} \quad (2.14)$$

$$= -\log e, \quad (2.15)$$

$$\dot{\mathbb{R}}_S(s, d) = j_S(s, d) - \log e, \quad (2.16)$$

$$\text{Var} [\dot{\mathbb{R}}_S(S, d)] = \text{Var} [j_S(S, d)]. \quad (2.17)$$

³Henceforth, Z^* denotes the rate-distortion-achieving reproduction random variable at distortion d , i.e. $P_S \rightarrow P_{Z^*|S} \rightarrow P_{Z^*}$ where $P_{Z^*|S}$ achieves the infimum in (2.3).

A measure-theoretic proof of (2.8) and (2.12) can be found in [23, Lemma 1.4]. The equality in (2.10) is shown in [24]. The equality in (2.17) was first observed in [25]. For completeness, proofs of Theorems 2.1 and 2.2 are included in Appendix B.1.

Remark 2.1. While Definition 2.1 does not cover the case $d = d_{\min}$, for discrete random variables with $\mathbf{d}(s, z) = 1 \{s \neq z\}$ it is natural to define 0-tilted information as

$$J_S(s, 0) \triangleq \iota_S(s) \quad (2.18)$$

Example 2.1. For the BMS with bias $p \leq \frac{1}{2}$ and bit error rate distortion,

$$J_{S^k}(s^k, d) = \iota_{S^k}(s^k) - kh(d) \quad (2.19)$$

if $0 \leq d < p$, and 0 if $d \geq p$.

Example 2.2. For the GMS with variance σ^2 and mean-square error distortion,⁴

$$J_{S^k}(s^k, d) = \frac{k}{2} \log \frac{\sigma^2}{d} + \left(\frac{|s^k|^2}{\sigma^2} - k \right) \frac{\log e}{2} \quad (2.20)$$

if $0 < d < \sigma^2$, and 0 if $d \geq \sigma^2$.

2.2.1 Generalized tilted information

For two random variables Z and \bar{Z} defined on the same space, denote the relative information by

$$\iota_{Z \parallel \bar{Z}}(z) \triangleq \log \frac{dP_Z}{dP_{\bar{Z}}}(z) \quad (2.21)$$

If Z is distributed according to $P_{Z|S=s}$, we abbreviate the notation as

$$\iota_{Z|S \parallel \bar{Z}}(s; z) \triangleq \log \frac{dP_{Z|S=s}}{dP_{\bar{Z}}}(z) \quad (2.22)$$

in lieu of $\iota_{Z|S=s \parallel \bar{Z}}(z)$. The familiar information density in (2.1) between realizations of two random variables with joint distribution $P_S P_{Z|S}$ follows by particularizing (2.22) to $\{P_{Z|S}, P_Z\}$, where $P_S \rightarrow P_{Z|S} \rightarrow P_Z$.

⁴We denote the Euclidean norm by $|\cdot|$, i.e. $|s^k|^2 = s_1^2 + \dots + s_k^2$.

For a given $P_{\bar{Z}}$, s , d , λ , and the following particular choice of $P_{Z|S} = P_{Z^*|S}$

$$dP_{\bar{Z}^*|S=s}(z) \triangleq \frac{dP_{\bar{Z}}(z) \exp(-\lambda d(s, z))}{\mathbb{E}[\exp(-\lambda d(s, \bar{Z}))]} \quad (2.23)$$

it holds that (for $P_{\bar{Z}}$ -a.e. z)

$$J_{\bar{Z}}(s, \lambda) \triangleq \log \frac{1}{\mathbb{E}[\exp(-\lambda d(s, \bar{Z}))]} \quad (2.24)$$

$$= \imath_{\bar{Z}^*|S} \bar{Z}(s; z) + \lambda d(s, z) \quad (2.25)$$

where in (2.24), as in (2.6), the expectation is with respect to unconditional distribution of \bar{Z} . We refer to the function

$$J_{\bar{Z}}(s, \lambda) - \lambda d \quad (2.26)$$

as the *generalized d-tilted information* in s . The d -tilted information in s is obtained by particularizing the generalized d -tilted information to $P_{\bar{Z}} = P_{Z^*}$, $\lambda = \lambda^*$:

$$J_S(s, d) = J_{Z^*}(s, \lambda^*) - \lambda^* d \quad (2.27)$$

Generalized d -tilted information is convenient because it exhibits the same beautiful symmetries as the regular d -tilted information (see (2.25)) while allowing the freedom in the choice of the output distribution $P_{\bar{Z}}$ as well as λ , which leads to tighter non-asymptotic bounds.

Generalized d -tilted information is closely linked to the following optimization problem [26].

$$\mathbb{R}_{S, \bar{Z}}(d) \triangleq \min_{\substack{P_{Z|S}: \\ \mathbb{E}[d(S; Z)] \leq d}} D(P_{Z|S} \| P_{\bar{Z}} | P_S) \quad (2.28)$$

Note that

$$\mathbb{R}_S(d) = \min_{P_{\bar{Z}}} \mathbb{R}_{S, \bar{Z}}(d) \quad (2.29)$$

As long as $d > d_{\min|S, \bar{Z}}$, where

$$d_{\min|S, \bar{Z}} \triangleq \inf \{d: \mathbb{R}_{S, \bar{Z}}(d) < \infty\} \quad (2.30)$$

the minimum in (2.28) is always achieved [23] (unlike that in (2.3)) by $P_{\bar{Z}^*|S}$ defined in (2.23) with $\lambda = \lambda_{S, \bar{Z}}^*$, where

$$\lambda_{S, \bar{Z}}^* \triangleq -\mathbb{R}'_{S, \bar{Z}}(d) \quad (2.31)$$

and the function $J_{\bar{Z}}(s, \lambda_{S, \bar{Z}}^*) - \lambda_{S, \bar{Z}}^* d$ possesses properties analogous to those of the function $J_S(s, d)$ listed in (2.9)–(2.11). In particular (cf. (2.11))

$$\mathbb{R}_{S, \bar{Z}}(d) = \mathbb{E} \left[J_{\bar{Z}}(S, \lambda_{S, \bar{Z}}^*) \right] - \lambda_{S, \bar{Z}}^* d \quad (2.32)$$

2.2.2 Tilted information and distortion d -balls

The distortion d -ball around s is denoted by

$$B_d(s) \triangleq \{z \in \widehat{\mathcal{M}}: \mathbf{d}(s, z) \leq d\} \quad (2.33)$$

As the following result shows, generalized tilted information is closely related to the (unconditional) probability that \bar{Z} falls within distortion d from S .

Lemma 2.3. *Fix a distribution $P_{\bar{Z}}$ on the output alphabet $\widehat{\mathcal{M}}$. It holds that*

$$\begin{aligned} & \sup_{\delta > 0, \lambda > 0} \left\{ \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d - \lambda \delta) \mathbb{P} [d - \delta < \mathbf{d}(s; \bar{Z}^*) \leq d | S = s] \right\} \\ & \leq P_{\bar{Z}}(B_d(s)) \end{aligned} \quad (2.34)$$

$$\leq \inf_{\lambda > 0} \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d) \mathbb{P} [\mathbf{d}(s, \bar{Z}^*) \leq d | S = s] \quad (2.35)$$

where the conditional probability in the left side of (2.34) and the right side of (2.35) is with respect to $P_{\bar{Z}^*|S}$ defined in (2.23).

Proof. To show (2.35), write

$$P_{\bar{Z}}(B_d(s)) = \mathbb{E} \left[\exp(-\iota_{\bar{Z}^*|S=s||\bar{Z}}(\bar{Z}^*)) \mathbf{1} \{ \mathbf{d}(s, \bar{Z}^*) \leq d \} | S = s \right] \quad (2.36)$$

$$\leq \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d) \mathbb{P} [\mathbf{d}(s, \bar{Z}^*) \leq d | S = s] \quad (2.37)$$

where

- (2.36) applies the usual change of measure argument;
- (2.37) uses (2.25) and

$$\mathbf{1} \{ \mathbf{d}(s, \bar{Z}^*) \leq d \} \leq \exp(\lambda d - \lambda \mathbf{d}(s, \bar{Z}^*)) \quad (2.38)$$

The lower bound in (2.34) is shown by similarly leveraging (2.25) as follows.

$$P_{\bar{Z}}(B_d(s)) = \mathbb{E} \left[\exp \left(-\iota_{\bar{Z}^*|S=s||\bar{Z}}(\bar{Z}^*) \right) 1 \{d(s, \bar{Z}^*) \leq d\} \mid S = s \right] \quad (2.39)$$

$$\geq \mathbb{E} \left[\exp \left(-\iota_{\bar{Z}^*|S=s||\bar{Z}}(\bar{Z}^*) \right) 1 \{d - \delta \leq d(s, \bar{Z}^*) \leq d\} \mid S = s \right] \quad (2.40)$$

$$\geq \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d - \lambda \delta) \mathbb{P} [d - \delta \leq d(s, \bar{Z}^*) \leq d \mid S = s] \quad (2.41)$$

□

If $P_{\bar{Z}} = P_{Z^*}$, we may weaken (2.35) by choosing $\lambda = \lambda^*$ to conclude

$$P_{Z^*}(B_d(s)) \leq \exp(-J_S(s, d)) \quad (2.42)$$

As we will see in Theorem 2.11, under certain regularity conditions equality in (2.42) can be closely approached.

2.3 Operational definitions

In fixed-length lossy compression, the output of a general source with alphabet \mathcal{M} and source distribution P_S is mapped to one of the M codewords from the reproduction alphabet $\widehat{\mathcal{M}}$. A lossy code is a (possibly randomized) pair of mappings $\mathbf{f}: \mathcal{M} \mapsto \{1, \dots, M\}$ and $\mathbf{c}: \{1, \dots, M\} \mapsto \widehat{\mathcal{M}}$. A distortion measure $\mathbf{d}: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]$ is used to quantify the performance of a lossy code. Given decoder \mathbf{c} , the best encoder simply maps the source output to the closest (in the sense of the distortion measure) codeword, i.e. $\mathbf{f}(s) = \arg \min_m \mathbf{d}(s, \mathbf{c}(m))$. The average distortion over the source statistics is a popular performance criterion. A stronger criterion is also used, namely, the probability of exceeding a given distortion level (called *excess-distortion probability*). The following definitions abide by the excess distortion criterion.

Definition 2.2. *An (M, d, ϵ) code for $\{\mathcal{M}, \widehat{\mathcal{M}}, P_S, \mathbf{d}: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]\}$ is a code with $|\mathbf{f}| = M$ such that*

$$\mathbb{P} [\mathbf{d}(S, \mathbf{c}(\mathbf{f}(S))) > d] \leq \epsilon \quad (2.43)$$

The minimum achievable code size at excess-distortion probability ϵ and distortion d is defined by

$$M^*(d, \epsilon) \triangleq \min \{M: \exists (M, d, \epsilon) \text{ code}\} \quad (2.44)$$

Note that the special case $d = 0$ and $\mathbf{d}(s, z) = 1 \{s \neq z\}$ corresponds to almost-lossless compression.

Definition 2.3. In the conventional fixed-to-fixed (or block) setting in which \mathcal{M} and $\widehat{\mathcal{M}}$ are the k -fold Cartesian products of alphabets \mathcal{S} and $\hat{\mathcal{S}}$, an (M, d, ϵ) code for $\{\mathcal{S}^k, \hat{\mathcal{S}}^k, P_{\mathcal{S}^k}, d^k: \mathcal{S}^k \times \hat{\mathcal{S}}^k \mapsto [0, +\infty]\}$ is called a (k, M, d, ϵ) code.

Fix ϵ , d and blocklength k . The minimum achievable code size and the finite blocklength rate-distortion function (excess distortion) are defined by, respectively

$$M^*(k, d, \epsilon) \triangleq \min \{M: \exists (k, M, d, \epsilon) \text{ code}\} \quad (2.45)$$

$$R(k, d, \epsilon) \triangleq \frac{1}{k} \log M^*(k, d, \epsilon) \quad (2.46)$$

Alternatively, using an average distortion criterion, we employ the following notations.

Definition 2.4. An $\langle M, d \rangle$ code for $\{\mathcal{M}, \widehat{\mathcal{M}}, P_{\mathcal{S}}, \mathbf{d}: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]\}$ is a code with $|\mathbf{f}| = M$ such that $\mathbb{E}[\mathbf{d}(S, \mathbf{c}(\mathbf{f}(S)))] \leq d$. The minimum achievable code size at average distortion d is defined by

$$M^*(d) \triangleq \min \{M: \exists \langle M, d \rangle \text{ code}\} \quad (2.47)$$

Definition 2.5. If \mathcal{M} and $\widehat{\mathcal{M}}$ are the k -fold Cartesian products of alphabets \mathcal{S} and $\hat{\mathcal{S}}$, an $\langle M, d \rangle$ code for $\{\mathcal{S}^k, \hat{\mathcal{S}}^k, P_{\mathcal{S}^k}, \mathbf{d}^k: \mathcal{S}^k \times \hat{\mathcal{S}}^k \mapsto [0, +\infty]\}$ is called an $\langle k, M, d \rangle$ code.

Fix d and blocklength k . The minimum achievable code size and the finite blocklength rate-distortion function (average distortion) are defined by, respectively

$$M^*(k, d) \triangleq \min \{M: \exists \langle k, M, d \rangle \text{ code}\} \quad (2.48)$$

$$R(k, d) \triangleq \frac{1}{k} \log M^*(k, d) \quad (2.49)$$

In the limit of long blocklengths, the minimum achievable rate is characterized by the rate-distortion function [5, 11].

Definition 2.6. The rate-distortion function is defined as

$$R(d) \triangleq \limsup_{k \rightarrow \infty} R(k, d) \quad (2.50)$$

Fixing the rate rather than the distortion, we define the distortion-rate functions in a similar manner:

- $D(k, R, \epsilon)$: the minimum distortion threshold achievable at block length k , rate R and excess probability ϵ .
- $D(k, R)$: the minimum average distortion achievable at blocklength k and rate R .
- $D(R)$: the minimum average distortion achievable in the limit of large blocklength at rate R .

In the simplest setting of a stationary memoryless source with separable distortion measure, i.e. when $P_{S^k} = P_S \times \dots \times P_S$, $\mathbf{d}(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k \mathbf{d}(s_i, z_i)$, we will write P_S , P_Z , $\mathbb{R}_S(d)$, $J_S(s, d)$, $J_Z(s, \lambda)$ to denote single-letter distributions on \mathcal{S} , $\hat{\mathcal{S}}$ and the functions in (2.3), (2.28), (2.6), (2.26), respectively, evaluated with those single-letter distributions and a single-letter distortion measure.

In the review of prior work in Section 2.4 we will use the following concepts related to variable-length coding. A variable-length code is a pair of mappings $\mathbf{f}: \mathcal{M} \mapsto \{0, 1\}^*$ and $\mathbf{c}: \{0, 1\}^* \mapsto \hat{\mathcal{M}}$, where $\{0, 1\}^*$ is the set of all possibly empty binary strings. It is said to operate at distortion level d if $\mathbb{P}[\mathbf{d}(S, \mathbf{c}(\mathbf{f}(S))) \leq d] = 1$. For a given code (\mathbf{f}, \mathbf{c}) operating at distortion d , the length of the binary codeword assigned to $s \in \mathcal{M}$ is denoted by $\ell_d(s) = \text{length of } \mathbf{f}(s)$.

2.4 Prior work

In this section, we summarize the main available bounds on the fixed-blocklength fundamental limits of lossy compression and we review the main relevant asymptotic refinements to Shannon's lossy source coding theorem.

2.4.1 Achievability bounds

Returning to the general setup of Definition 2.2, the basic general achievability result can be distilled [2] from Shannon's coding theorem for memoryless sources:

Theorem 2.4 (Achievability [2, 11]). *Fix P_S , a positive integer M and $d \geq d_{\min}$. There exists an (M, d, ϵ) code such that*

$$\epsilon \leq \inf_{P_{Z|S}} \left\{ \mathbb{P}[\mathbf{d}(S, Z) > d] + \inf_{\gamma > 0} \left\{ \mathbb{P}[\iota_{S;Z}(S; Z) > \log M - \gamma] + e^{-\exp(\gamma)} \right\} \right\} \quad (2.51)$$

Theorem 2.4 was obtained by independent random selection of the codewords. It is the most general existing achievability result (i.e. existence result of a code with a guaranteed upper bound on error probability). In particular, it allows us to deduce that for stationary memoryless sources

with separable distortion measure, i.e. when $P_{S^k} = P_S \times \dots \times P_S$, $d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k d(s_i, z_i)$, it holds that

$$\limsup_{k \rightarrow \infty} R(k, d) \leq \mathbb{R}_S(d) \quad (2.52)$$

$$\limsup_{k \rightarrow \infty} R(k, d, \epsilon) \leq \mathbb{R}_S(d) \quad (2.53)$$

where $\mathbb{R}_S(d)$ is defined in (2.3), and $0 < \epsilon < 1$.

For three particular setups of i.i.d. sources with separable distortion measure, we can cite the achievability bounds of Goblick [27] (fixed-rate compression of a finite alphabet source), Pinkston [28] (variable-rate compression of a finite-alphabet source) and Sakrison [29] (variable-rate compression of a Gaussian source with mean-square error distortion).

The bounds of Goblick [27] and Pinkston [28] were obtained using random coding and a type counting argument.

Theorem 2.5 (Achievability [27, Theorem 2.1]). *For a finite alphabet i.i.d. source with single-letter distribution P_S and separable distortion measure, the minimum average distortion achievable at blocklength k and rate R satisfies*

$$\begin{aligned} D(k, R) \leq \min_{P_Z, \lambda > 0} & \left\{ \mathbb{E} [J'_Z(S, \lambda)] + \mathbb{E} [J'_Z(S, \lambda)|_{\lambda=0}] \left[\frac{3 \log k}{\sqrt{k}} \left(\frac{T_\lambda}{V_\lambda^{3/2}} + \frac{2T_0}{V_0^{3/2}} \right) + \frac{V_\lambda e^{-\frac{\sqrt{k}}{2V_\lambda^2}}}{\sqrt{2\pi} \sqrt[4]{k}} + \frac{2V_0 e^{-\frac{\sqrt{k}}{2V_0^2}}}{\sqrt{2\pi} \sqrt[4]{k}} \right. \right. \\ & \left. \left. + \exp \left(-(kR - 1) F(k) \exp \left(-k \left[\mathbb{E} [J_Z(S, \lambda)] - \lambda \mathbb{E} [J'_Z(S, \lambda)] + \frac{1 + \lambda}{\sqrt[4]{k}} \right] \right) \right) \right] \right\} + \frac{1}{\sqrt[4]{k}} \end{aligned} \quad (2.54)$$

where $(\cdot)'$ denotes differentiation with respect to λ , J_Z is defined by (2.25) with P_Z and λ , V_λ and V_0 are the variances of $J'_Z(S, \lambda)$ and $J'_Z(S, \lambda)|_{\lambda=0}$, respectively, while T_λ and T_0 are their third absolute moments, and

$$F(k) = \frac{1}{(2\pi k)^{\frac{1}{2}} |\mathcal{S}|(|\hat{\mathcal{S}}|-1)} \exp \left(-\frac{1}{2} |\mathcal{S}| |\hat{\mathcal{S}}| - \lambda \max_{(s, z) \in \mathcal{S} \times \hat{\mathcal{S}}} d(s, z) - \sum_{(s, z) \in \mathcal{S} \times \hat{\mathcal{S}}} \frac{1}{P_{Z^*|S}(z|s)} \right) \quad (2.55)$$

Note that

$$\mathbb{E} [J'_Z(S, \lambda)] = \mathbb{E} [d(S, \bar{Z}^*)] \quad (2.56)$$

$$\mathbb{E} [J'_Z(S, \lambda)|_{\lambda=0}] = \mathbb{E} [d(S, \bar{Z})] \quad (2.57)$$

where $P_{\bar{Z}^*|S}$ is that in (2.23) and $P_{\bar{Z}|S} = P_{\bar{Z}}$.

Theorem 2.6 (Achievability [28, (6)]). *For a finite alphabet i.i.d. source with single-letter distribution P_S and separable distortion measure, there exists a variable-length code such that the distortion between S^k and its reproduction almost surely does not exceed d , and*

$$\mathbb{E}[\ell(S^k)] \leq \min_{P_{\bar{Z}}, \lambda > 0} \{k(\mathbb{E}[J_{\bar{Z}}(S, \lambda)] - \lambda \mathbb{E}[J'_{\bar{Z}}(S, \lambda)]) - \log G(k)\} + |S| \log(k+1) + 1 \quad (2.58)$$

where

$$G(k) = \frac{\exp\left(-\alpha_k \lambda \sqrt{k \max_{s \in S} |J''_{\bar{Z}}(s, \lambda)|} - \frac{1}{2} \alpha_k^2\right)}{\sqrt{2\pi} \left(\alpha_k + \lambda \sqrt{k \max_{s \in S} |J''_{\bar{Z}}(s, \lambda)|}\right)} \left(1 - \frac{1}{\left(\alpha_k + \lambda \sqrt{k \min_{s \in S} |J''_{\bar{Z}}(s, \lambda)|}\right)^2}\right) \quad (2.59)$$

$$\alpha_k = Q^{-1} \left(\frac{1}{2} - \frac{33}{4\sqrt{k}} \max_{s \in S} \frac{T_s}{\sqrt{V_s}} \right) \quad (2.60)$$

and V_s and T_s are the variance and the third absolute moment of $\mathbb{E}[d(S, \bar{Z})|S = s]$, respectively.

Sakrison's achievability bound, stated next, relies heavily on the geometric symmetries of the Gaussian setting.

Theorem 2.7 (Achievability [29]). *Fix blocklength k , and let S^k be a Gaussian vector with independent components of variance σ^2 . There exists a variable-length code achieving average mean-square error d such that*

$$\mathbb{E}[\ell(S^k)] \leq -\frac{k-1}{2} \log \left(\frac{d}{\sigma^2} - \frac{1}{1.2k} \right) + \frac{1}{2} \log k + \log 4\pi + \frac{2}{3} \log e + \frac{5 \log e}{12(k+1)} \quad (2.61)$$

2.4.2 Converse bounds

The basic converse used in conjunction with (2.51) to prove the rate-distortion fundamental limit with average distortion is the following simple result, which follows immediately from the data processing lemma for mutual information:

Theorem 2.8 (Converse [11]). *Fix P_S , integer M and $d \geq d_{\min}$. Any $\langle M, d \rangle$ code must satisfy*

$$\mathbb{R}_S(d) \leq \log M \quad (2.62)$$

where $\mathbb{R}_S(d)$ is defined in (2.3).

Shannon [11] showed that in the case of stationary memoryless sources with separable distortion,

$\mathbb{R}_{S^k}(d) = k\mathbb{R}_S(d)$. Using Theorem 2.8, it follows that for such sources

$$\mathbb{R}_S(d) \leq R(k, d) \quad (2.63)$$

for any blocklength k and any $d > d_{\min}$, which together with (2.52) gives

$$R(d) = \mathbb{R}_S(d) \quad (2.64)$$

The strong converse for lossy source coding [30] states that if the compression rate R is fixed and $R < \mathbb{R}_S(d)$, then $\epsilon \rightarrow 1$ as $k \rightarrow \infty$, which together with (2.53) yields that for i.i.d. sources with separable distortion and any $0 < \epsilon < 1$

$$\limsup_{k \rightarrow \infty} R(k, d, \epsilon) = \mathbb{R}_S(d) = R(d) \quad (2.65)$$

More generally, the strong converse holds provided that the source is stationary ergodic and the distortion measure is subadditive [31].

For prefix-free variable-length lossy compression, the key non-asymptotic converse was obtained by Kontoyiannis [32] (see also [33] for a lossless compression counterpart).

Theorem 2.9 (Converse [32]). *If a prefix-free variable-length code for P_S operates at distortion level d , then for any $\gamma > 0$*

$$\mathbb{P}[\ell_d(S) \leq J_S(S, d) - \gamma] \leq 2^{-\gamma} \quad (2.66)$$

The following nonasymptotic converse can be distilled from Marton's fixed-rate lossy compression error exponent [13].

Theorem 2.10 (Converse [13]). *Assume that $d_{\max} = \max_{s,z} d(s, z) < +\infty$. Fix $d_{\min} < d < d_{\max}$ and an arbitrary $(\exp(R), d, \epsilon)$ code.*

- *If*

$$R < \mathbb{R}_S(d), \quad (2.67)$$

then the excess-distortion probability is bounded away from zero:

$$\epsilon \geq \frac{\mathbb{D}_S(R) - d}{d_{\max} - d} \quad (2.68)$$

- If R satisfies

$$\mathbb{R}_S(d) < R < \max_{P_{\bar{S}}} \mathbb{R}_S(d), \quad (2.69)$$

where the maximization is over the set of all probability distributions on \mathcal{M} , then

$$\epsilon \geq \sup_{\delta > 0, P_{\bar{S}}} \left(\frac{\mathbb{D}_{\bar{S}}(R) - d}{d_{\max} - d} - P_{\bar{S}}(G_{\delta}^c) \right) \exp(-D(\bar{S} \| S) - \delta), \quad (2.70)$$

where the supremization is over all probability distributions on \mathcal{M} satisfying $R_{\bar{S}}(d) > R$, and

$$G_{\delta} = \{s \in \mathcal{M} : \iota_{\bar{S} \| S}(s) \leq D(\bar{S} \| S) + \delta\} \quad (2.71)$$

Proof. Inequality in (2.68) is that in [13, (7)]. To show (2.70), fix an arbitrary code (f, c) whose rate satisfies (2.69), fix an auxiliary $P_{\bar{S}}$ satisfying $R_{\bar{S}}(d) > R$, and write

$$\mathbb{P}[\mathbf{d}(S, c(f(S))) > d] \geq \mathbb{P}[\mathbf{d}(S, c(f(S))) > d, S \in G_{\delta}] \quad (2.72)$$

$$= \mathbb{E}[\exp(-\iota_{\bar{S} \| S}(\bar{S})) \mathbf{1}\{\mathbf{d}(\bar{S}, c(f(\bar{S}))) > d, \bar{S} \in G_{\delta}\}] \quad (2.73)$$

$$\geq \mathbb{P}[\mathbf{d}(\bar{S}, c(f(\bar{S}))) > d, \bar{S} \in G_{\delta}] \exp(-D(\bar{S} \| S) - \delta) \quad (2.74)$$

$$\geq (\mathbb{P}[\mathbf{d}(\bar{S}, c(f(\bar{S}))) > d] - P_{\bar{S}}(G_{\delta}^c)) \exp(-D(\bar{S} \| S) - \delta) \quad (2.75)$$

$$\geq \left(\frac{\mathbb{D}_{\bar{S}}(R) - d}{d_{\max} - d} - P_{\bar{S}}(G_{\delta}^c) \right) \exp(-D(\bar{S} \| S) - \delta) \quad (2.76)$$

where

- (2.73) is by the standard change of measure argument,
- (2.75) is by the union bound,
- (2.76) is due to (2.68).

□

It turns out that the converse in Theorem 2.10 results in rather loose lower bounds on $R(k, d, \epsilon)$ unless k is very large, in which case the rate-distortion function already gives a tight lower bound. Generalizations of the error exponent results in [13] (see also [34, 35] for independent developments) are found in [14–18].

2.4.3 Gaussian Asymptotic Approximation

The “lossy asymptotic equipartition property (AEP)” [36], which leads to strong achievability and converse bounds for variable-rate quantization, is concerned with the almost sure asymptotic behavior of the distortion d -balls. Second-order refinements of the “lossy AEP” were studied in [32, 37, 38].⁵

Theorem 2.11 (“Lossy AEP”). *For memoryless sources with separable distortion measure satisfying the regularity restrictions (i)–(iv) in Section 2.6,*

$$\log \frac{1}{P_{Z^{k*}}(B_d(S^k))} = \sum_{i=1}^k \mathcal{I}_S(S_i, d) + \frac{1}{2} \log k + O(\log \log k)$$

almost surely.

Remark 2.2. Note the different behavior of almost lossless data compression:

$$\log \frac{1}{P_{Z^{k*}}(B_0(S^k))} = \log \frac{1}{P_{S^k}(S^k)} = \sum_{i=1}^k \mathcal{I}_S(S_i) \quad (2.77)$$

Kontoyiannis [32] pioneered the second-order refinement of the variable-length rate-distortion function showing that for stationary memoryless sources with separable distortion measures the stochastically optimum prefix-free description length at distortion level d satisfies

$$\ell_d^*(S^k) = \sum_{i=1}^k \mathcal{I}_S(S_i, d) + O(\log k) \text{ a.s.} \quad (2.78)$$

2.4.4 Asymptotics of redundancy

Considerable attention has been paid to the asymptotic behavior of the redundancy, i.e. the difference between the average distortion $D(k, R)$ of the best k -dimensional quantizer and the distortion-rate function $D(R)$. For finite-alphabet i.i.d. sources, Pilc [40] strengthened the positive lossy source coding theorem by showing that

$$D(k, R) - D(R) \leq -\frac{\partial D(R)}{\partial R} \frac{\log k}{2k} + o\left(\frac{\log k}{k}\right) \quad (2.79)$$

Zhang, Zang and Wei [41] proved a converse to (2.79), thereby showing that for memoryless sources with finite alphabet,

$$D(k, R) - D(R) = -\frac{\partial D(R)}{\partial R} \frac{\log k}{2k} + o\left(\frac{\log k}{k}\right) \quad (2.80)$$

⁵The result of Theorem 2.11 was pointed out in [32, Proposition 3] as a simple corollary to the analyses in [37, 38]. See [39] for a generalization to α -mixing sources.

Using a geometric approach akin to that of Sakrison [29], Wyner [42] showed that (2.79) also holds for stationary Gaussian sources with mean-square error distortion, while Yang and Zhang [37] extended (2.79) to abstract alphabets. Note that as the average overhead over the distortion-rate function is dwarfed by its standard deviation, the analyses of [37, 40–42] are bound to be overly optimistic since they neglect the stochastic variability of the distortion.

2.5 New finite blocklength bounds

In this section we give achievability and converse results for any source and any distortion measure according to the setup of Section 2.3. When we apply these results in Sections 2.6 - 2.9, the source S becomes an k -tuple (S_1, \dots, S_k) .

2.5.1 Converse bounds

Our first result is a general converse bound, expressed in terms of \mathbf{d} -tilted information.

Theorem 2.12 (Converse, \mathbf{d} -tilted information). *Assume the basic conditions (a)–(b) in Section 2.3 are met. Fix $d > d_{\min}$. Any (M, d, ϵ) code must satisfy*

$$\epsilon \geq \sup_{\gamma \geq 0} \{\mathbb{P}[J_S(S, d) \geq \log M + \gamma] - \exp(-\gamma)\} \quad (2.81)$$

Proof. Let the encoder and decoder be the random transformations $P_{X|S}: \mathcal{M} \mapsto \{1, \dots, M\}$ and $P_{Z|X}: \{1, \dots, M\} \mapsto \widehat{\mathcal{M}}$. Let Q_X be equiprobable on $\{1, \dots, M\}$, and let Q_Z denote the marginal

of $P_{Z|X}Q_X$. We have⁶, for any $\gamma \geq 0$

$$\begin{aligned} & \mathbb{P}[j_S(S, d) \geq \log M + \gamma] \\ &= \mathbb{P}[j_S(S, d) \geq \log M + \gamma, \mathbf{d}(S, Z) > d] + \mathbb{P}[j_S(S, d) \geq \log M + \gamma, \mathbf{d}(S, Z) \leq d] \end{aligned} \quad (2.82)$$

$$\leq \epsilon + \sum_{s \in \mathcal{M}} P_S(s) \sum_{x=1}^M P_{X|S}(x|s) \sum_{z \in B_d(s)} P_{Z|X}(z|x) 1\{M \leq \exp(j_S(s, d) - \gamma)\} \quad (2.83)$$

$$\leq \epsilon + \exp(-\gamma) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d)) \sum_{x=1}^M \frac{1}{M} \sum_{z \in B_d(s)} P_{Z|X}(z|x) \quad (2.84)$$

$$= \epsilon + \exp(-\gamma) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d)) Q_Z(B_d(s)) \quad (2.85)$$

$$\leq \epsilon + \exp(-\gamma) \sum_{z \in \widehat{\mathcal{M}}} Q_Z(z) \sum_{s \in \mathcal{M}} P_S(s) \exp(\lambda^* d - \lambda^* \mathbf{d}(s, z) + j_S(s, d)) \quad (2.86)$$

$$\leq \epsilon + \exp(-\gamma) \quad (2.87)$$

where

- (2.84) follows by upper-bounding

$$P_{X|S}(x|s) 1\{M \leq \exp(j_S(s, d) - \gamma)\} \leq \frac{\exp(-\gamma)}{M} \exp(j_S(s, d)) \quad (2.88)$$

for every $(s, x) \in \mathcal{M} \times \{1, \dots, M\}$,

- (2.86) uses (2.42) particularized to Z distributed according to Q_Z , and
- (2.87) is due to (2.12).

□

Remark 2.3. Theorem 2.12 gives a pleasing generalization of the almost-lossless data compression converse bound [2], [43, Lemma 1.3.2]. In fact, skipping (2.86), the above proof applies to the case $d = 0$ and $\mathbf{d}(s, z) = 1\{s \neq z\}$, which corresponds to almost-lossless data compression.

Remark 2.4. As explained in Appendix B.4, condition (b) can be dropped from the assumptions of Theorem 2.12.

Our next converse result, which is tighter than the one in Theorem 2.12 in some cases, is based on binary hypothesis testing. The optimal performance achievable among all randomized tests

⁶We write summations over alphabets for simplicity. All our results in Sections 2.5 and 2.6 hold for arbitrary probability spaces.

$P_{W|S}: \mathcal{M} \rightarrow \{0, 1\}$ between probability distributions P and Q on \mathcal{M} is denoted by (1 indicates that the test chooses P):⁷

$$\beta_\alpha(P, Q) = \min_{\substack{P_{W|S}: \\ \mathbb{P}[W=1] \geq \alpha}} \mathbb{Q}[W=1] \quad (2.89)$$

In fact, Q need not be a probability measure, it just needs to be σ -finite in order for the Neyman-Pearson lemma and related results to hold.

Theorem 2.13 (Converse).⁸ *Let P_S be the source distribution defined on the alphabet \mathcal{M} . Any (M, d, ϵ) code must satisfy*

$$M \geq \sup_Q \inf_{z \in \widehat{\mathcal{M}}} \frac{\beta_{1-\epsilon}(P_S, Q)}{\mathbb{Q}[\mathbf{d}(S, z) \leq d]} \quad (2.90)$$

where the supremum is over all distributions on \mathcal{M} .

Proof. Let $(P_{X|S}, P_{Z|X})$ be an (M, d, ϵ) code. Fix a distribution Q_S on \mathcal{M} , and observe that $W = 1 \{\mathbf{d}(S, Z) \leq d\}$ defines a (not necessarily optimal) hypothesis test between P_S and Q_S with $\mathbb{P}[W=1] \geq 1 - \epsilon$. Thus,

$$\begin{aligned} \beta_{1-\epsilon}(P_S, Q_S) &\leq \sum_{s \in \mathcal{M}} Q_S(s) \sum_{m=1}^M P_{X|S}(m|s) \sum_{z \in \widehat{\mathcal{M}}} P_{Z|X}(z|m) 1\{\mathbf{d}(s, z) \leq d\} \\ &\leq \sum_{m=1}^M \sum_{z \in \widehat{\mathcal{M}}} P_{Z|X}(z|m) \sum_{s \in \mathcal{M}} Q_S(s) 1\{\mathbf{d}(s, z) \leq d\} \end{aligned} \quad (2.91)$$

$$\leq \sum_{m=1}^M \sum_{z \in \widehat{\mathcal{M}}} P_{Z|X}(z|m) \sup_{z' \in \widehat{\mathcal{M}}} \mathbb{Q}[\mathbf{d}(S, z') \leq d] \quad (2.92)$$

$$= M \sup_{z \in \widehat{\mathcal{M}}} \mathbb{Q}[\mathbf{d}(S, z) \leq d] \quad (2.93)$$

□

Suppose for a moment that S takes values on a finite alphabet, and let us further lower bound (2.90) by taking Q to be the equiprobable distribution on \mathcal{M} , $Q = U$. Consider the set $\Omega \subset \mathcal{M}$ that has total probability $1 - \epsilon$ and contains the most probable source outcomes, i.e. for any source outcome $s \in \Omega$, there is no element outside Ω having probability greater than $P_S(s)$. For any $s \in \Omega$, the optimum binary hypothesis test (with error probability ϵ) between P_S and U must choose P_S . Thus the numerator of (2.90) evaluated with $Q = U$ is proportional to the number of elements in Ω , while the denominator is proportional to the number of elements in a distortion ball of radius

⁷Throughout, P, Q denote distributions, whereas \mathbb{P}, \mathbb{Q} are used for the corresponding probabilities of events on the underlying probability space.

⁸Theorem 2.13 was suggested by Dr. Yury Polyanskiy.

d. Therefore (2.90) evaluated with $Q = U$ yields a lower bound to the minimum number of \mathbf{d} -balls required to cover Ω .

Remark 2.5. In general, the lower bound in Theorem 2.13 is not achievable due to overlaps between the distortion \mathbf{d} -balls that comprise the covering. One special case when it is in fact achievable is almost lossless data compression on a countable alphabet \mathcal{M} . To encompass that case, it is convenient to relax the restriction in (2.89) that requires Q to be a probability measure and allow it to be a σ -finite measure, so that $\beta_\alpha(P_S, Q)$ is no longer bounded by 1. Note that Theorem 2.13 would still hold. Letting U to be the counting measure on \mathcal{M} (i.e. U assigns unit weight to each letter), we have (Appendix B.2)

$$\beta_{1-\epsilon}(P_S, U) \leq M^*(0, \epsilon) \leq \beta_{1-\epsilon}(P_S, U) + 1 \quad (2.94)$$

The lower bound in (2.94) is satisfied with equality whenever $\beta_{1-\epsilon}(P_S, U)$ is achieved by a non-randomized test.

The last result in this section is a lossy compression counterpart of the lossless variable length compression bound in [9, Theorem 3], which, unlike the converse in [32] (Theorem 2.9), does not impose the prefix condition on the encoded sequences.

Theorem 2.14 (Converse, variable-length lossy compression). *For a nonnegative integer k , the encoded length of any variable-length code operating at distortion d must satisfy⁹*

$$\mathbb{P}[\ell_d(S) \geq k] \geq \max_{\gamma > 0} \{\mathbb{P}[J_S(S, d) \geq k + \gamma] - \exp(-\gamma)\} \quad (2.95)$$

Proof. Let the encoder and decoder be the random transformations $P_{X|S}$ and $P_{Z|X}$, where X takes values in $\{1, 2, \dots\}$. Let Q_S be equiprobable on $\{1, \dots, \exp(k) - 1\}$, and let Q_Z denote the marginal of $P_{Z|X}Q_S$.

⁹All log's and exp's involved in Theorem 2.14 are binary base.

$$\begin{aligned}
& \mathbb{P}[j_S(S, d) \geq k + \gamma] \\
&= \mathbb{P}[j_S(S, d) \geq k + \gamma, X > \exp(k) - 1] + \mathbb{P}[j_S(S, d) \geq k + \gamma, X \leq \exp(k) - 1] \tag{2.96}
\end{aligned}$$

$$\leq \epsilon + \sum_{s \in \mathcal{M}} P_S(s) \sum_{x=1}^{\exp(k)-1} P_{X|S}(x|s) \sum_{z \in B_d(s)} P_{Z|X}(z|x) 1\{\exp(k) \leq \exp(j_S(s, d) - \gamma)\} \tag{2.97}$$

$$\leq \epsilon + \exp(-\gamma) \tag{2.98}$$

where the proof of (2.98) repeats that of (2.87).

□

2.5.2 Achievability bounds

The following result gives an exact analysis of the excess probability of random coding, which holds in full generality.

Theorem 2.15 (Exact performance of random coding). *Denote by $\epsilon_d(c_1, \dots, c_M)$ the probability of exceeding distortion level d achieved by the optimum encoder with codebook (c_1, \dots, c_M) . Let Z_1, \dots, Z_M be independent, distributed according to an arbitrary distribution P_Z on the reproduction alphabet. Then*

$$\mathbb{E}[\epsilon_d(Z_1, \dots, Z_M)] = \mathbb{E}[(1 - P_Z(B_d(S)))^M] \tag{2.99}$$

Proof. Upon observing the source output s , the optimum encoder chooses arbitrarily among the members of the set

$$\arg \min_{i=1, \dots, M} d(s, c_i)$$

The indicator function of the event that the distortion exceeds d is

$$1\left\{\min_{i=1, \dots, M} d(s, c_i) > d\right\} = \prod_{i=1}^M 1\{d(s, c_i) > d\} \tag{2.100}$$

Averaging over both the input S and the choice of codewords chosen independently of S , we get

$$\mathbb{E} \left[\prod_{i=1}^M 1 \{d(S, Z_i) > d\} \right] = \mathbb{E} \left[\mathbb{E} \left[\prod_{i=1}^M 1 \{d(S, Z_i) > d\} \mid S \right] \right] \quad (2.101)$$

$$= \mathbb{E} \prod_{i=1}^M \mathbb{E} [1 \{d(S, Z_i) > d\} \mid S] \quad (2.102)$$

$$= \mathbb{E} \left[\left(\mathbb{P} [d(S, \bar{Z}) > d \mid S] \right)^M \right] \quad (2.103)$$

where in (2.102) we have used the fact that Z_1, \dots, Z_M are independent even when conditioned on S . \square

Invoking Shannon's random coding argument, the following achievability result follows immediately from Theorem 2.15.

Theorem 2.16 (Achievability). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{P_{\bar{Z}}} \mathbb{E} \left[\left(1 - P_{\bar{Z}}(B_d(S)) \right)^M \right] \quad (2.104)$$

where the infimization is over all random variables defined on $\widehat{\mathcal{M}}$, independent of S .

The bound in Theorem 2.15, based on the exact performance of random coding, is a major stepping stone in random coding achievability proofs for lossy source coding found in literature (e.g. [11, 27–29, 32, 37, 38, 41]), all of which loosen (2.104) using various tools with the goal to obtain expressions that are easier to analyze.

Applying $(1 - p)^M \leq e^{-Mp}$ to (2.104), one obtains the following more numerically stable bound.

Corollary 2.17 (Achievability). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{P_{\bar{Z}}} \mathbb{E} \left[e^{-MP_{\bar{Z}}(B_d(S))} \right] \quad (2.105)$$

where the infimization is over all random variables defined on $\widehat{\mathcal{M}}$, independent of S .

Shannon's bound in Theorem 2.4 can be obtained from Theorem 2.16 by using the nonasymptotic covering lemma:

Lemma 2.18 ([44, Lemma 5]). *Fix $P_{Z|S}$ and let $P_{S\bar{Z}} = P_S P_{\bar{Z}}$ where $P_S \rightarrow P_{Z|S} \rightarrow P_{\bar{Z}}$. It holds that*

$$\mathbb{E} \left[\left(\mathbb{P} [d(S, \bar{Z}) > d] \right)^M \right] \leq \inf_{\gamma > 0} \left\{ \mathbb{P} [i_{S;Z}(S; Z) > \log \gamma] + \mathbb{P} [d(S, Z) > d] + e^{-\frac{M}{\gamma}} \right\} \quad (2.106)$$

Proof. We give a derivation from [45]. Introducing an auxiliary parameter $\gamma > 0$ and observing the inequality [45]

$$(1 - p)^M \leq e^{-Mp} \quad (2.107)$$

$$\leq e^{-\frac{M}{\gamma} \min\{1, \gamma p\} + |1 - \gamma p|^+} \quad (2.108)$$

we upper-bound the left side of (2.106) as

$$\mathbb{E} \left[(1 - P_{\bar{Z}}(B_d(S)))^M \right] \leq e^{-\frac{M}{\gamma}} \mathbb{E} [\min(1, \gamma P_{\bar{Z}}(B_d(S)))] + \mathbb{E} [|1 - \gamma P_{\bar{Z}}(B_d(S))|^+] \quad (2.109)$$

$$\leq e^{-\frac{M}{\gamma}} + \mathbb{E} [|1 - \gamma P_{\bar{Z}}(B_d(S))|^+] \quad (2.110)$$

$$= e^{-\frac{M}{\gamma}} + \mathbb{E} [|1 - \gamma \mathbb{E} [\exp(-\iota_{S;Z}(S; Z)) \mathbf{1}\{\mathbf{d}(S, Z) \leq d\} | S]|^+] \quad (2.111)$$

$$\leq e^{-\frac{M}{\gamma}} + \mathbb{E} [|1 - \gamma \exp(-\iota_{S;Z}(S; Z)) \mathbf{1}\{\mathbf{d}(S, Z) \leq d\}|^+] \quad (2.112)$$

$$\leq e^{-\frac{M}{\gamma}} + \mathbb{P} [\iota_{S;Z}(S; Z) > \log \gamma] + \mathbb{P} [\iota_{S;Z}(S; Z) \leq \log \gamma, \mathbf{d}(S, Z) > d] \quad (2.113)$$

$$\leq e^{-\frac{M}{\gamma}} + \mathbb{P} [\iota_{S;Z}(S; Z) > \log \gamma] + \mathbb{P} [\mathbf{d}(S, Z) > d] \quad (2.114)$$

where

- (2.111) is by the usual change of measure argument;
- (2.112) is due to the convexity of $|\cdot|^+$;
- (2.113) is obtained by bounding

$$\gamma \exp(-\iota_{S;Z}(s; z)) \geq \begin{cases} 1 & \text{if } \iota_{S;Z}(s; z) \leq \log \gamma \\ 0 & \text{otherwise} \end{cases} \quad (2.115)$$

Optimizing (2.114) over the choice of $\gamma > 0$ and $P_{Z|S}$, we obtain the Shannon bound (2.51). \square

Relaxing (2.104) using (2.18), we obtain (2.51).

The following relaxation of (2.109) gives a tight achievability bound in terms of the generalized \mathbf{d} -tilted information (defined in (2.26)).

Theorem 2.19 (Achievability, generalized d-tilted information). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{\gamma, \beta, \delta, P_{\bar{Z}}} \mathbb{E} \left[\inf_{\lambda > 0} \left\{ 1 \{ J_{\bar{Z}}(S, \lambda) - \lambda d > \log \gamma - \log \beta - \lambda \delta \} + |1 - \beta \mathbb{P}[d - \delta \leq \mathbf{d}(S, \bar{Z}^*) \leq d | S]|^+ \right. \right. \\ \left. \left. + e^{-\frac{M}{\gamma}} \min(1, \gamma \exp(-J_{\bar{Z}}(S, \lambda) + \lambda d)) \right\} \right] \quad (2.116)$$

where $P_{\bar{Z}^*|S}$ is the transition probability kernel defined in (2.23).

Proof. Fix an arbitrary probability distribution $P_{\bar{Z}}$ defined on the output alphabet $\widehat{\mathcal{M}}$. The first term in the right side of (2.109) is upper-bounded using (2.35) as

$$\min \{1, \gamma P_{\bar{Z}}(B_d(s))\} \leq \min \{1, \gamma \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d)\} \quad (2.117)$$

The second term in (2.109) is upper-bounded using (2.34) as

$$|1 - \gamma P_{\bar{Z}}(B_d(s))|^+ \\ \leq |1 - \gamma \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d - \lambda \delta) \mathbb{P}[d - \delta \leq \mathbf{d}(S, \bar{Z}^*) \leq d | S = s]|^+ \quad (2.118)$$

$$\leq |1 - \beta \mathbb{P}[d - \delta \leq \mathbf{d}(S, \bar{Z}^*) \leq d | S = s]|^+ + 1 \{ J_{\bar{Z}}(s, \lambda) - \lambda d > \log \gamma - \log \beta \} \quad (2.119)$$

where (2.118) uses (2.34), and to obtain (2.119), we bounded

$$\gamma \exp(-J_{\bar{Z}}(s, \lambda) + \lambda d) \geq \begin{cases} \beta & \text{if } J_{\bar{Z}}(s, \lambda) + \lambda d \leq \log \gamma - \log \beta - \lambda \delta \\ 0 & \text{otherwise} \end{cases} \quad (2.120)$$

Assembling (2.117) and (2.119), optimizing with respect to the choice of λ and taking the expectation over S , we obtain (2.109). \square

In particular, relaxing the bound in (2.109) by fixing $\lambda = \lambda^*$ and $P_{\bar{Z}} = P_{Z^*}$, where $P_{Z^*|S}$ achieves $\mathbb{R}_S(d)$, we obtain the following bound.

Corollary 2.20 (Achievability, d-tilted information). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \min_{\gamma, \beta, \delta} \left\{ \mathbb{P}[J_S(S, d) > \log \gamma - \log \beta - \lambda^* \delta] \right. \\ \left. + \mathbb{E} \left[|1 - \beta \mathbb{P}[d - \delta \leq \mathbf{d}(S, Z^*) \leq d | S]|^+ \right] + e^{-\frac{M}{\gamma}} \mathbb{E}[\min(1, \gamma \exp(-J_S(S, d)))] \right\} \quad (2.121)$$

The following result, which relies on a converse for channel coding to show an achievability result

for source coding, is obtained leveraging the ideas in the proof of [46, Theorem 7.3] attributed to Wolfowitz [47].

Theorem 2.21 (Achievability). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{P_{Z|S}} \left\{ \mathbb{P}[\mathbf{d}(S, Z) > d] + \inf_{\gamma > 0} \left\{ \sup_{z \in \widehat{\mathcal{M}}} \mathbb{P}[\imath_{S;Z}(S; z) \geq \log M - \gamma] + \exp(-\gamma) \right\} \right\} \quad (2.122)$$

Proof. For a given source P_S , fix an arbitrary $P_{Z|S}$ and consider the following construction.

Codebook construction: Consider the Feinstein code [48] achieving maximal error probability $\hat{\epsilon}$ for $P_{S|Z}$, the backward channel for $P_S P_{Z|S}$. The codebook construction, which follows Feinstein's exhaustive procedure [48] with the modification that the decoding sets are defined using the distortion measure rather than the information density, is described as follows.

For the purposes of this proof, we denote for brevity

$$B_z = \{s \in \mathcal{M} : \mathbf{d}(s, z) \leq d\} \quad (2.123)$$

The first codeword $c_1 \in \widehat{\mathcal{M}}$ is chosen to satisfy

$$P_{S|Z=c_1}(B_{c_1}) > 1 - \hat{\epsilon} \quad (2.124)$$

After $1, \dots, m-1$ codewords have been selected, $c_m \in \widehat{\mathcal{M}}$ is chosen so that

$$P_{S|Z=c_m} \left(B_{c_m} \setminus \bigcup_{i=1}^{m-1} B_{c_i} \right) > 1 - \hat{\epsilon} \quad (2.125)$$

The procedure of adding codewords stops once a codeword choice satisfying (2.125) becomes impossible.

Channel encoder: The channel encoder is defined by

$$\hat{\mathbf{f}}(m) = c_m, \quad m = 1, \dots, M \quad (2.126)$$

Channel decoder: The channel decoder is defined by

$$\hat{\mathbf{g}}(s) = \begin{cases} m & s \in B_{c_m} \setminus \bigcup_{i=1}^{m-1} B_{c_i} \\ \text{arbitrary} & \text{otherwise} \end{cases} \quad (2.127)$$

Channel code analysis: It follows from (2.125) and (2.127) that we constructed an $(M, \hat{\epsilon})$ code (maximal error probability) for $P_{S|Z}$.

Moreover, denoting

$$D = \bigcup_{i=1}^M B_{c_i} \quad (2.128)$$

where M is the total number of codewords, we conclude by the codebook construction that

$$P_{S|Z=z}(B_z \cap D^c) \leq 1 - \hat{\epsilon}, \quad \forall z \in \widehat{\mathcal{M}} \quad (2.129)$$

because the left side of (2.129) is 0 for $z \in \{c_1, \dots, c_M\}$ and $\leq 1 - \hat{\epsilon}$ for $z \in \widehat{\mathcal{M}} \setminus \{c_1, \dots, c_M\}$ since otherwise we could have added z to the codebook.

Using (2.129) and the union bound, we conclude that for $\forall z \in \widehat{\mathcal{M}}$

$$P_{S|Z=z}(D) \geq \hat{\epsilon} - P_{S|Z=z}(B_z^c) \quad (2.130)$$

Taking the expectation of (2.130) with respect to P_Z , where $P_S \rightarrow P_{Z|S} \rightarrow P_Z$, we conclude that

$$P_S(D) \geq \hat{\epsilon} - \mathbb{P}[\mathbf{d}(S, Z) > d] \quad (2.131)$$

Source code: Define the source code (\mathbf{f}, \mathbf{g}) by

$$(\mathbf{f}, \mathbf{g}) = (\hat{\mathbf{g}}, \hat{\mathbf{f}}) \quad (2.132)$$

Source code analysis: The probability of excess distortion is given by

$$\mathbb{P}[\mathbf{d}(S, \mathbf{g}(\mathbf{f}(S))) > d] = \mathbb{P}[\mathbf{d}(S, \hat{\mathbf{f}}(\hat{\mathbf{g}}(S))) > d] \quad (2.133)$$

$$= \mathbb{P}[\mathbf{d}(S, \hat{\mathbf{f}}(\hat{\mathbf{g}}(S))) > d, S \in D^c] \quad (2.134)$$

$$\leq P_S(D^c) \quad (2.135)$$

$$\leq 1 - \hat{\epsilon} + \mathbb{P}[\mathbf{d}(S, Z) > d] \quad (2.136)$$

$$\leq \sup_{z \in \widehat{\mathcal{M}}} \mathbb{P}[\mathbf{d}_{S;Z}(S; z) \geq \log M - \gamma] + \exp(-\gamma) + \mathbb{P}[\mathbf{d}(S, Z) > d] \quad (2.137)$$

where (2.134) follows from (2.127), (2.136) uses (2.131), and (2.137) uses the Wolfowitz converse for channel coding [49] [3, Theorem 9] to lower bound the maximal error probability $\hat{\epsilon}$.

□

2.6 Gaussian approximation

2.6.1 Rate-dispersion function

In the spirit of [3], we introduce the following definition.

Definition 2.7. Fix $d \geq d_{\min}$. The rate-dispersion function (squared information units per source output) is defined as

$$\mathcal{V}(d) \triangleq \lim_{\epsilon \rightarrow 0} \limsup_{k \rightarrow \infty} k \left(\frac{R(k, d, \epsilon) - R(d)}{Q^{-1}(\epsilon)} \right)^2 \quad (2.138)$$

$$= \lim_{\epsilon \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{k (R(k, d, \epsilon) - R(d))^2}{2 \log_e \frac{1}{\epsilon}} \quad (2.139)$$

Fix d , $0 < \epsilon < 1$, $\eta > 0$, and suppose the target is to sustain the probability of exceeding distortion d bounded by ϵ at rate $R = (1 + \eta)R(d)$. As (1.9) implies, the required blocklength scales linearly with rate dispersion:

$$k(d, \eta, \epsilon) \approx \frac{\mathcal{V}(d)}{R^2(d)} \left(\frac{Q^{-1}(\epsilon)}{\eta} \right)^2 \quad (2.140)$$

where note that only the first factor depends on the source, while the second depends only on the design specifications.

2.6.2 Main result

In addition to the basic conditions (a)-(b) of Section 2.2, in the remainder of this section we impose the following restrictions on the source and on the distortion measure.

- (i) The source $\{S_i\}$ is stationary and memoryless, $P_{S^k} = P_S \times \dots \times P_S$.
- (ii) The distortion measure is separable, $d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k d(s_i, z_i)$.
- (iii) The distortion level satisfies $d_{\min} < d < d_{\max}$, where d_{\min} is defined in (2.5), and $d_{\max} = \inf_{z \in \mathcal{M}} \mathbb{E}[d(S, z)]$, where the expectation is with respect to the unconditional distribution of S . The excess-distortion probability satisfies $0 < \epsilon < 1$.
- (iv) $\mathbb{E}[d^9(S, Z^*)] < \infty$ where the expectation is with respect to $P_S \times P_{Z^*}$.¹⁰

The main result in this section is the following¹¹.

¹⁰Since several reviewers surmised the 9 in the exponent was a typo, it seems fitting to stress that the finiteness of the ninth moment of the random variable $d(S, Z^*)$ is indeed required in the proof of the achievability part of Theorem 2.22.

¹¹Using an approach based on typical sequences, Ingber and Kochman [50] independently found the dispersion of finite alphabet sources. The Gaussian i.i.d. source with mean-square error distortion was treated separately in [50]. The result of Theorem 2.22 is more general as it applies to sources with abstract alphabets.

Theorem 2.22 (Gaussian approximation). *Under restrictions (i)–(iv),*

$$R(k, d, \epsilon) = R(d) + \sqrt{\frac{\mathcal{V}(d)}{k}} Q^{-1}(\epsilon) + \theta \left(\frac{\log k}{k} \right) \quad (2.141)$$

$$R(d) = \mathbb{E} [J_{\mathcal{S}}(\mathbf{S}, d)] \quad (2.142)$$

$$\mathcal{V}(d) = \text{Var} [J_{\mathcal{S}}(\mathbf{S}, d)] \quad (2.143)$$

and the remainder term in (2.141) satisfies

$$-\frac{1}{2} \frac{\log k}{k} + O \left(\frac{1}{k} \right) \leq \theta \left(\frac{\log k}{k} \right) \quad (2.144)$$

$$\leq C_0 \frac{\log k}{k} + \frac{\log \log k}{k} + O \left(\frac{1}{k} \right) \quad (2.145)$$

where

$$C_0 = \frac{1}{2} + \frac{\text{Var} [J'_{Z^*}(\mathbf{S}, \lambda^*)]}{\mathbb{E} [|J''_{Z^*}(\mathbf{S}, \lambda^*)|] \log e} \quad (2.146)$$

In (2.146), $(\cdot)'$ denotes differentiation with respect to λ , $J_{Z^*}(\mathbf{s}, \lambda)$ is defined in (2.24), and $\lambda^* = -R'(d)$.

Remark 2.6. As highlighted in (2.142) (see also (2.11) in Section 2.3), the rate-distortion function can be expressed as the expectation of the random variable whose variance we take in (2.143), thereby drawing a pleasing parallel with the channel coding results in [3].

Remark 2.7. For almost lossless data compression, Theorem 2.22 still holds as long as the random variable $\imath_{\mathcal{S}}(\mathbf{S})$ has finite third moment. Moreover, using (2.94) the upper bound in (2.145) can be strengthened (Appendix B.3) to obtain for $\text{Var} [\imath_{\mathcal{S}}(\mathbf{S})] > 0$

$$R(k, 0, \epsilon) = H(\mathbf{S}) + \sqrt{\frac{\text{Var} [\imath_{\mathcal{S}}(\mathbf{S})]}{k}} Q^{-1}(\epsilon) - \frac{1}{2} \frac{\log k}{k} + O \left(\frac{1}{k} \right) \quad (2.147)$$

which is consistent with the second-order refinement for almost lossless data compression developed in [6, 9]. If $\text{Var} [\imath_{\mathcal{S}}(\mathbf{S})] = 0$ as in the case of a non-redundant source, then

$$R(k, 0, \epsilon) = H(\mathbf{S}) - \frac{1}{k} \log \frac{1}{1 - \epsilon} + o_k \quad (2.148)$$

where

$$0 \leq o_k \leq \frac{\exp(-kH(\mathbf{S}))}{(1 - \epsilon)k} \quad (2.149)$$

As we will see in Section 2.7, in contrast to the lossless case in (2.147), the remainder term in the

lossy case in (2.141) can be strictly larger than $-\frac{1}{2} \frac{\log k}{k}$ appearing in (2.147) even when $\mathcal{V}(d) > 0$.

Remark 2.8. As will become apparent in the proof of Theorem 2.22, if $\mathcal{V}(d) = 0$, the lower bound in (2.141) can be strengthened non-asymptotically:

$$R(k, d, \epsilon) \geq R(d) - \frac{1}{k} \log \frac{1}{1 - \epsilon} \quad (2.150)$$

which aligns nicely with (2.148).

Remark 2.9. Let us consider what happens if we drop restriction (b) of Section 2.2 that $R(d)$ is achieved by the unique conditional distribution $P_{Z^*|S}$. If several $P_{Z|S}$ achieve $R(d)$, writing $j_{S;Z}(s, d)$ for the d -tilted information corresponding to Z , Theorem 2.22 still holds with

$$\mathcal{V}(d) = \begin{cases} \max \text{Var} [j_{S;Z}(S, d)] & 0 < \epsilon \leq \frac{1}{2} \\ \min \text{Var} [j_{S;Z}(S, d)] & \frac{1}{2} < \epsilon < 1 \end{cases} \quad (2.151)$$

where the optimization is performed over all $P_{Z|S}$ that achieve the rate-distortion function. Moreover, as explained in Appendix B.4, Theorem 2.12 and the converse part of Theorem 2.22 do not even require existence of a minimizing $P_{Z^*|S}$.

Remark 2.10. For finite alphabet sources satisfying the regularity conditions of Theorem 2.2, the rate-dispersion function admits the following alternative representation [50] (cf. (2.17)):

$$\mathcal{V}(d) = \text{Var} \left[\dot{\mathbb{R}}_S(S, d) \right] \quad (2.152)$$

where the partial derivatives $\dot{\mathbb{R}}_S(\cdot, d)$ are defined in (2.13).

Let us consider three special cases where $\mathcal{V}(d)$ is constant as a function of d .

a) *Zero dispersion.* For a particular value of d , $\mathcal{V}(d) = 0$ if and only if $j_S(S, d)$ is deterministic with probability 1. In particular, it follows from (2.152) that for finite alphabet sources $\mathcal{V}(d) = 0$ if the source distribution P_S maximizes $\mathbb{R}_S(d)$ over all source distributions defined on the same alphabet [50]. Moreover, Dembo and Kontoyiannis [51] showed that under mild conditions, the rate-dispersion function can only vanish for at most finitely many distortion levels d unless the source is equiprobable and the distortion matrix is symmetric with rows that are permutations of one another, in which case $\mathcal{V}(d) = 0$ for all $d \in (d_{\min}, d_{\max})$.

b) *Binary source with bit error rate distortion.* Plugging $k = 1$ into (2.19), we observe that the rate-dispersion function reduces to the varentropy [2] of the source,

$$\mathcal{V}(d) = \mathcal{V}(0) = \text{Var} [\iota_S(S)] \quad (2.153)$$

c) *Gaussian source with mean-square error distortion.* Plugging $k = 1$ into (2.20), we see that

$$\mathcal{V}(d) = \frac{1}{2} \log^2 e \quad (2.154)$$

for all $0 < d < \sigma^2$. Similar to the BMS case, the rate dispersion is equal to the variance of $\log f_S(S)$, where $f_S(S)$ is the Gaussian probability density function.

2.6.3 Proof of main result

Before we proceed to proving Theorem 2.22, we state two auxiliary results. The first is an important tool in the Gaussian approximation analysis of $R(k, d, \epsilon)$.

Theorem 2.23 (Berry-Esséen Central Limit Theorem (CLT), e.g. [52, Ch. XVI.5 Theorem 2]). *Fix a positive integer k . Let W_i , $i = 1, \dots, k$ be independent. Then, for any real t*

$$\left| \mathbb{P} \left[\sum_{i=1}^k W_i > k \left(\mu_k + t \sqrt{\frac{V_k}{k}} \right) \right] - Q(t) \right| \leq \frac{B_k}{\sqrt{k}}, \quad (2.155)$$

where

$$\mu_k = \frac{1}{k} \sum_{i=1}^k \mathbb{E} [W_i] \quad (2.156)$$

$$V_k = \frac{1}{k} \sum_{i=1}^k \text{Var} [W_i] \quad (2.157)$$

$$T_k = \frac{1}{k} \sum_{i=1}^k \mathbb{E} [|W_i - \mu_i|^3] \quad (2.158)$$

$$B_k = \frac{c_0 T_k}{V_k^{3/2}} \quad (2.159)$$

and $0.4097 \leq c_0 \leq 0.5600$ ($0.4097 \leq c_0 < 0.4784$ for identically distributed W_i).

The second auxiliary result, proven in Appendix B.5, is a nonasymptotic refinement of the lossy AEP (Theorem 2.11) tailored to our purposes.

Lemma 2.24. *Under restrictions (i)–(iv), there exist constants $k_0, c, K > 0$ such that for all $k \geq k_0$,*

$$\mathbb{P} \left[\log \frac{1}{P_{Z^{k*}}(B_d(S^k))} \leq \sum_{i=1}^k \mathcal{J}_S(S_i, d) + C_0 \log k + c \right] \geq 1 - \frac{K}{\sqrt{k}} \quad (2.160)$$

where C_0 is given by (2.146).

We start with the converse part. Note that for the converse, restriction (iv) can be replaced by the following weaker one:

(iv') The random variable $\mathcal{J}_S(\mathbf{S}, d)$ has finite absolute third moment.

To verify that (iv) implies (iv'), observe that by the concavity of the logarithm,

$$0 \leq \mathcal{J}_S(\mathbf{s}, d) + \lambda^* d \leq \lambda^* \mathbb{E} [d(\mathbf{s}, Z^*)] \quad (2.161)$$

so

$$\mathbb{E} \left[|\mathcal{J}_S(\mathbf{S}, d) + \lambda^* d|^3 \right] \leq \lambda^{*3} \mathbb{E} [d^3(\mathbf{S}, Z^*)] \quad (2.162)$$

Proof of the converse part of Theorem 2.22. First, observe that due to (i) and (ii), $P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, and the d -tilted information single-letterizes, that is, for a.e. s^k ,

$$\mathcal{J}_{S^k}(s^k, d) = \sum_{i=1}^k \mathcal{J}_S(s_i, d) \quad (2.163)$$

Consider the case $\mathcal{V}(d) > 0$, so that B_k in (2.159) with $W_i = \mathcal{J}_S(S_i, d)$ is finite by restriction (iv').

Let $\gamma = \frac{1}{2} \log k$ in (2.81), and choose

$$\log M = kR(d) + \sqrt{nV(d)}Q^{-1}(\epsilon_k) - \gamma \quad (2.164)$$

$$\epsilon_k = \epsilon + \exp(-\gamma) + \frac{B_k}{\sqrt{k}} \quad (2.165)$$

so that $R = \frac{\log M}{k}$ can be written as the right side of (2.141) with (2.144) satisfied. Substituting (2.163) and (2.164) in (2.81), we conclude that for any (M, d, ϵ') code it must hold that

$$\epsilon' \geq \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_S(S_i, d) \geq kR(d) + \sqrt{nV(d)}Q^{-1}(\epsilon_k) \right] - \exp(-\gamma) \quad (2.166)$$

The proof for $\mathcal{V}(d) > 0$ is complete upon noting that the right side of (2.166) is lower bounded by ϵ by the Berry-Essén inequality (2.155) in view of (2.165).

If $\mathcal{V}(d) = 0$, it follows that $j_S(S, d) = R(d)$ almost surely. Choosing $\gamma = \log \frac{1}{1-\epsilon}$ and $\log M = kR(d) - \gamma$ in (2.81) it is obvious that $\epsilon' \geq \epsilon$. \square

Proof of the achievability part of Theorem 2.22. The proof consists of the asymptotic analysis of the bound in Corollary 2.17 using Lemma 2.24.¹² Denote

$$G_k = \log M - \sum_{i=1}^k j_S(s_i, d) - C_0 \log k - c \quad (2.167)$$

where constants c and C were defined in Lemma 2.24. Letting $S = S^k$ in (2.105) and weakening the right side of (2.105) by choosing $P_{\bar{Z}} = P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, we conclude that there exists a (k, M, d, ϵ') code with

$$\epsilon' \leq \mathbb{E} \left[e^{-MP_{Z^*}^*(B_d(S^k))} \right] \quad (2.168)$$

$$\leq \mathbb{E} \left[e^{-\exp(G_k)} \right] + \frac{K}{\sqrt{k}} \quad (2.169)$$

$$= \mathbb{E} \left[e^{-\exp(G_k)} 1 \left\{ G_k < \log \frac{\log_e k}{2} \right\} \right] + \mathbb{E} \left[e^{-\exp(G_k)} 1 \left\{ G_k \geq \log \frac{\log_e k}{2} \right\} \right] + \frac{K}{\sqrt{k}} \quad (2.170)$$

$$\leq \mathbb{P} \left[G_k < \log \frac{\log_e k}{2} \right] + \frac{1}{\sqrt{k}} \mathbb{P} \left[G_k \geq \log \frac{\log_e k}{2} \right] + \frac{K}{\sqrt{k}} \quad (2.171)$$

where (2.169) holds for $k \geq k_0$ by Lemma 2.24, and (2.171) follows by upper bounding $e^{-\exp(G_k)}$ by 1 and $\frac{1}{\sqrt{k}}$ respectively. We need to show that (2.171) is upper bounded by ϵ for some $R = \frac{\log M}{k}$ that can be written as (2.141) with the remainder satisfying (2.145). Considering first the case $\mathcal{V}(d) > 0$, let

$$\log M = kR(d) + \sqrt{k\mathcal{V}(d)}Q^{-1}(\epsilon_k) + C_0 \log k + \log \frac{\log_e k}{2} + c \quad (2.172)$$

$$\epsilon_k = \epsilon - \frac{B_k + K + 1}{\sqrt{k}} \quad (2.173)$$

where B_k is given by (2.159) and is finite by restriction (iv'). Substituting (2.172) into (2.171) and applying the Berry-Esséen inequality (2.155) to the first term in (2.171), we conclude that $\epsilon' \leq \epsilon$ for all k such that $\epsilon_k > 0$.

It remains to tackle the case $\mathcal{V}(d) = 0$, which implies $j_S(S, d) = R(d)$ almost surely. Let

$$\log M = kR(d) + C_0 \log k + c + \log \log_e \frac{1}{\epsilon - \frac{K}{\sqrt{k}}} \quad (2.174)$$

¹²Note that Theorem 2.19 also leads to the same asymptotic expansion.

Substituting M into (2.169) we obtain immediately that $\epsilon' \leq \epsilon$, as desired. \square

2.6.4 Distortion-dispersion function

One can also consider the related problem of finding the minimum excess distortion $D(k, R, \epsilon)$ achievable at blocklength k , rate R and excess-distortion probability ϵ . We define the *distortion-dispersion function* at rate R by

$$\mathcal{W}(R) \triangleq \lim_{\epsilon \rightarrow 0} \limsup_{k \rightarrow \infty} \frac{k (D(k, R, \epsilon) - D(R))^2}{2 \log_e \frac{1}{\epsilon}} \quad (2.175)$$

For a fixed k and ϵ , the functions $R(k, \cdot, \epsilon)$ and $D(k, \cdot, \epsilon)$ are functional inverses of each other. Consequently, the rate-dispersion and the distortion-dispersion functions also define each other. Under mild conditions, it is easy to find one from the other:

Theorem 2.25. (*Distortion dispersion*) *If $R(d)$ is twice differentiable, $R'(d) \neq 0$ and $V(d)$ is differentiable in some interval $(\underline{d}, \bar{d}] \subseteq (d_{\min}, d_{\max}]$ then for any rate R such that $R = R(d)$ for some $d \in (\underline{d}, \bar{d})$ the distortion-dispersion function is given by*

$$\mathcal{W}(R) = (D'(R))^2 \mathcal{V}(D(R)) \quad (2.176)$$

and

$$D(k, R, \epsilon) = D(R) + \sqrt{\frac{\mathcal{W}(R)}{k}} Q^{-1}(\epsilon) - D'(R) \theta \left(\frac{\log k}{k} \right) \quad (2.177)$$

where $\theta(\cdot)$ satisfies (2.144), (2.145).

Proof. Appendix B.6. \square

Remark 2.11. Substituting (2.152) into (2.176), it follows that for finite alphabet sources (satisfying the regularity conditions of Theorem 2.2 as well as those of Theorem 2.25) the distortion-dispersion function can be represented as

$$\mathcal{W}(R) = \text{Var} \left[\dot{\mathbb{D}}_{\mathbf{S}}(\mathbf{S}, R) \right] \quad (2.178)$$

where

$$\dot{\mathbb{D}}_{\mathbf{S}}(\mathbf{s}, R) \triangleq \frac{\partial}{\partial P_{\mathbf{S}}(s)} \mathbb{D}_{\mathbf{S}}(R) \big|_{P_{\mathbf{S}}=P_{\mathbf{S}}} \quad (2.179)$$

In parallel to (2.140), suppose that the goal is to compress at rate R while exceeding distortion $d = (1 + \eta)D(R)$ with probability not higher than ϵ . As (2.177) implies, the required blocklength

scales linearly with the distortion-dispersion function:

$$k(R, \eta, \epsilon) \approx \frac{\mathcal{W}(R)}{D^2(R)} \left(\frac{Q^{-1}(\epsilon)}{\eta} \right)^2 \quad (2.180)$$

The distortion-dispersion function assumes a particularly simple form for the Gaussian memoryless source with mean-square error distortion, in which case for any $0 < d < \sigma^2$

$$D(R) = \sigma^2 \exp(-2R) \quad (2.181)$$

$$\frac{\mathcal{W}(R)}{D^2(R)} = 2 \quad (2.182)$$

$$k(R, \eta, \epsilon) \approx 2 \left(\frac{Q^{-1}(\epsilon)}{\eta} \right)^2 \quad (2.183)$$

so in the Gaussian case, the required blocklength is essentially independent of the target distortion.

2.7 Binary memoryless source

This section particularizes the nonasymptotic bounds in Section 2.5 and the asymptotic analysis in Section 2.6 to the stationary binary memoryless source with bit error rate distortion measure, i.e. $d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k 1 \{s_i \neq z_i\}$. For convenience, we denote

$$\left\langle \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\rangle = \sum_{j=0}^{\ell} \binom{k}{j} \quad (2.184)$$

with the convention $\left\langle \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\rangle = 0$ if $\ell < 0$ and $\left\langle \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} k \\ k \end{smallmatrix} \right\rangle$ if $\ell > k$.

2.7.1 Equiprobable BMS (EBMS)

The following results pertain to the i.i.d. binary equiprobable source and hold for $0 \leq d < \frac{1}{2}$, $0 < \epsilon < 1$.

Particularizing (2.19) to the equiprobable case, one observes that for all binary k -strings s^k

$$J_{S^k}(s^k, d) = k \log 2 - kh(d) = kR(d) \quad (2.185)$$

Then, Theorem 2.12 reduces to (2.150). Theorem 2.13 leads to the following stronger converse result.

Theorem 2.26 (Converse, EBMS). *Any (k, M, d, ϵ) code must satisfy:*

$$\epsilon \geq 1 - M2^{-k} \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle \quad (2.186)$$

Proof. Invoking Theorem 2.13 with the k -dimensional source distribution P_{S^k} playing the role of P_S therein, we have

$$M \geq \sup_Q \inf_{z^k \in \{0,1\}^k} \frac{\beta_{1-\epsilon}(P_{S^k}, Q)}{\mathbb{Q}[\mathbf{d}(S^k, z^k) \leq d]} \quad (2.187)$$

$$\geq \inf_{z^k \in \{0,1\}^k} \frac{\beta_{1-\epsilon}(P_{S^k}, P_{S^k})}{\mathbb{P}[\mathbf{d}(S^k, z^k) \leq d]} \quad (2.188)$$

$$= \frac{1 - \epsilon}{\mathbb{P}[\mathbf{d}(S^k, \mathbf{0}) \leq d]} \quad (2.189)$$

$$= \frac{1 - \epsilon}{2^{-k} \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle} \quad (2.190)$$

where (2.188) is obtained by choosing $Q = P_{S^k}$. \square

Theorem 2.27 (Exact performance of random coding, EBMS). *The minimum averaged probability that bit error rate exceeds d achieved by random coding with i.i.d. M codewords is given by*

$$\min_{P_Z} \mathbb{E}[\epsilon_d(Z_1, \dots, Z_M)] = \left(1 - 2^{-k} \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle\right)^M \quad (2.191)$$

attained by P_Z equiprobable on $\{0,1\}^k$. In the left side of (2.191), Z_1, \dots, Z_M are i.i.d. with common distribution P_Z .

Proof. For all $M \geq 1$, $(1 - x)^M$ is a convex function of x on $0 \leq x < 1$, so the right side of (2.99) is lower bounded by Jensen's inequality, for an arbitrary P_{Z^k}

$$\mathbb{E} \left[(1 - P_{Z^k}(B_d(S^k)))^M \right] \geq (1 - \mathbb{E}[P_{Z^k}(B_d(S^k))])^M \quad (2.192)$$

Equality in (2.192) is attained by Z^k equiprobable on $\{0,1\}^k$, because then

$$P_{Z^k}(B_d(S^k)) = 2^{-k} \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle \text{ a.s.} \quad (2.193)$$

\square

Theorem 2.27 leads to an achievability bound since there must exist an $(M, d, \mathbb{E}[\epsilon_d(Z_1, \dots, Z_M)])$ code.

Corollary 2.28 (Achievability, EBMS). *There exists a (k, M, d, ϵ) code such that*

$$\epsilon \leq \left(1 - 2^{-k} \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle\right)^M \quad (2.194)$$

As mentioned in Section 2.6 after Theorem 2.22, the EBMS with bit error rate distortion has zero rate-dispersion function for all d . The asymptotic analysis of the bounds in (2.194) and (2.186) allows for the following more accurate characterization of $R(k, d, \epsilon)$.

Theorem 2.29 (Gaussian approximation, EBMS). *The minimum achievable rate at blocklength k satisfies*

$$R(k, d, \epsilon) = \log 2 - h(d) + \frac{1}{2} \frac{\log k}{k} + O\left(\frac{1}{k}\right) \quad (2.195)$$

if $0 < d < \frac{1}{2}$, and

$$R(k, 0, \epsilon) = \log 2 - \frac{1}{k} \log \frac{1}{1 - \epsilon} + o_k \quad (2.196)$$

where $0 \leq o_k \leq \frac{2^{-k}}{(1-\epsilon)^k}$.

Proof. Appendix B.7. □

A numerical comparison of the achievability bound (2.51) evaluated with stationary memoryless $P_{Z^k|S^k}$, the new bounds in (2.194) and (2.186) as well as the approximation in (2.195) neglecting the $O(\frac{1}{k})$ term is presented in Fig. 2.1. Note that Marton's converse (Theorem 2.10) is not applicable to the EBMS because the region in (2.69) is empty. The achievability bound in (2.51), while asymptotically optimal, is quite loose in the displayed region of blocklengths. The converse bound in (2.186) and the achievability bound in (2.194) tightly sandwich the finite blocklength fundamental limit. Furthermore, the approximation in (2.195) is quite accurate, although somewhat optimistic, for all but very small blocklengths.

2.7.2 Non-equiprobable BMS

The results in this subsection focus on the i.i.d. binary memoryless source with $\mathbb{P}[S = 1] = p < \frac{1}{2}$ and apply for $0 \leq d < p$, $0 < \epsilon < 1$. The following converse result is a simple calculation of the bound in Theorem 2.12 using (2.19).

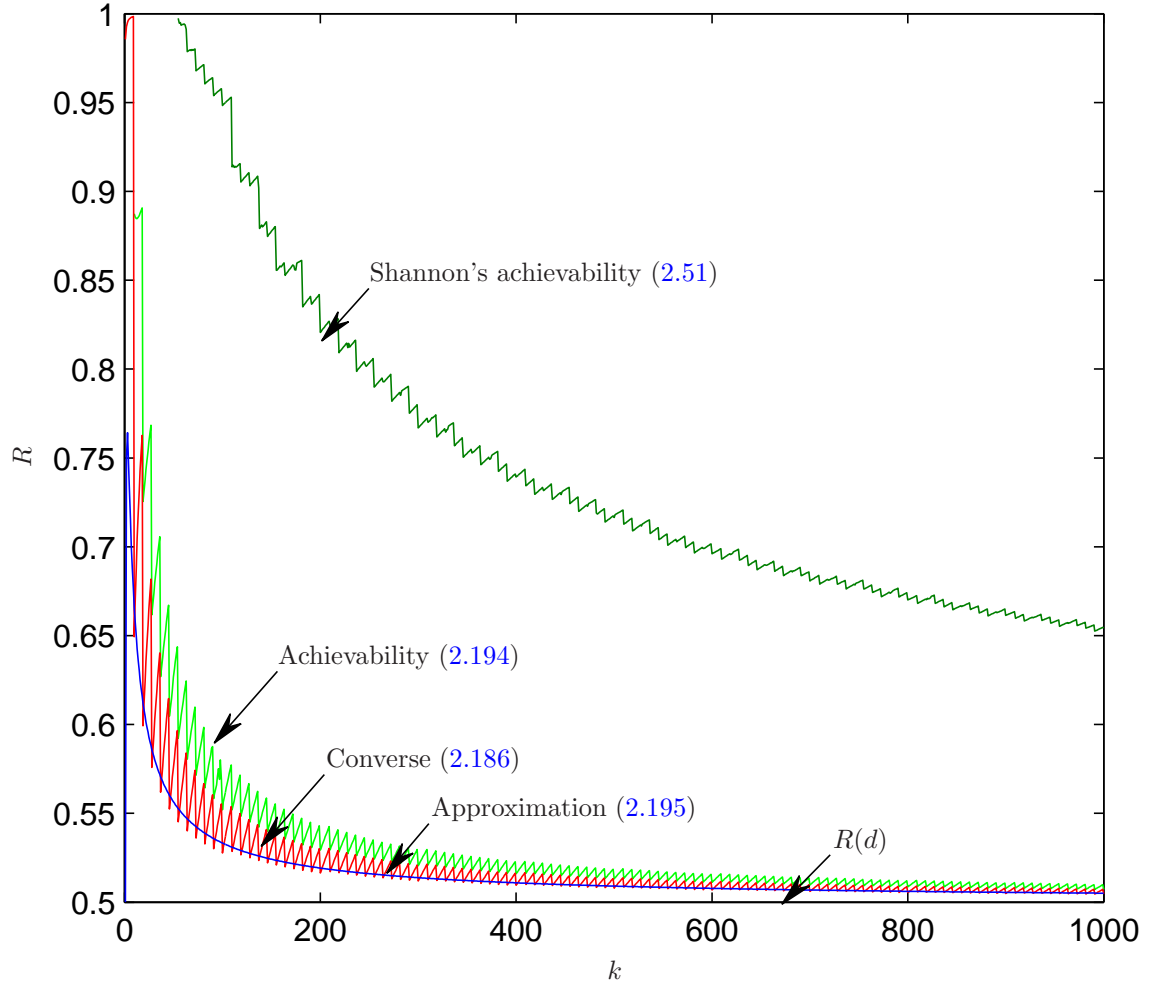


Figure 2.1: Rate-blocklength tradeoff for EBMS, $d = 0.11$, $\epsilon = 10^{-2}$.

Theorem 2.30 (Converse, BMS). *For any (k, M, d, ϵ) code, it holds that*

$$\epsilon \geq \sup_{\gamma \geq 0} \{ \mathbb{P} [g_k(W) \geq \log M + \gamma] - \exp(-\gamma) \} \quad (2.197)$$

$$g_k(W) = W \log \frac{1}{p} + (k - W) \log \frac{1}{1-p} - kh(d) \quad (2.198)$$

where W is binomial with success probability p and k degrees of freedom.

An application of Theorem 2.13 to the specific case of non-equiprobable BMS yields the following converse bound:

Theorem 2.31 (Converse, BMS). *Any (k, M, d, ϵ) code must satisfy*

$$M \geq \frac{\left\langle \binom{k}{r^*} \right\rangle + \alpha \binom{k}{r^*+1}}{\left\langle \binom{k}{\lfloor kd \rfloor} \right\rangle} \quad (2.199)$$

where we have denoted the integer

$$r^* = \max \left\{ r : \sum_{j=0}^r \binom{k}{j} p^j (1-p)^{k-j} \leq 1 - \epsilon \right\} \quad (2.200)$$

and $\alpha \in [0, 1)$ is the solution to

$$\sum_{j=0}^{r^*} \binom{k}{j} p^j (1-p)^{k-j} + \alpha p^{r^*+1} (1-p)^{k-r^*-1} \binom{k}{r^*+1} = 1 - \epsilon \quad (2.201)$$

Proof. In Theorem 2.13, the k -dimensional source distribution P_{S^k} plays the role of P_S , and we make the possibly suboptimal choice $Q = U$, the equiprobable distribution on $\mathcal{M} = \{0, 1\}^k$. The optimal randomized test to decide between P_{S^k} and U is given by

$$P_{W|S^k}(1|s^k) = \begin{cases} 0, & |s^k| > r^* + 1 \\ 1, & |s^k| \leq r^* \\ \alpha, & |s^k| = r^* + 1 \end{cases} \quad (2.202)$$

where $|s^k|$ denotes the Hamming weight of s^k , and α is such that $\sum_{s^k \in \mathcal{M}} P(s^k) P_{W|S}(1|s^k) = 1 - \epsilon$,

so

$$\begin{aligned}\beta_{1-\epsilon}(P_{S^k}, U) &= \min_{\substack{P_{W|S^k}: \\ \sum_{s^k \in \mathcal{M}} P_{S^k}(s^k) P_{W|S^k}(1|s^k) \geq 1-\epsilon}} 2^{-k} \sum_{s^k \in \mathcal{M}} P_{W|S^k}(1|s^k) \\ &= 2^{-k} \left[\left\langle \frac{k}{r^*} \right\rangle + \alpha \binom{k}{r^* + 1} \right]\end{aligned}\tag{2.203}$$

The result is now immediate from (2.90). \square

An application of Theorem 2.16 to the non-equiprobable BMS yields the following achievability bound:

Theorem 2.32 (Achievability, BMS). *There exists a (k, M, d, ϵ) code with*

$$\epsilon \leq \sum_{j=0}^k \binom{k}{j} p^j (1-p)^{k-j} \left[1 - \sum_{t=0}^k L_k(j, t) q^t (1-q)^{k-t} \right]^M \tag{2.204}$$

where

$$q = \frac{p-d}{1-2d} \tag{2.205}$$

and

$$L_k(j, t) = \binom{j}{t_0} \binom{k-j}{t-t_0} \tag{2.206}$$

with $t_0 = \left\lceil \frac{t+j-kd}{2} \right\rceil^+$ if $t-kd \leq j \leq t+kd$, and $L_k(j, t) = 0$ otherwise.

Proof. We compute an upper bound to (2.104) for the specific case of the BMS. Let $P_{\bar{Z}^k} = P_Z \times \dots \times P_Z$, where $P_Z(1) = q$. Note that P_Z is the marginal of the joint distribution that achieves the rate-distortion function (e.g. [53]). The number of binary strings of Hamming weight t that lie within Hamming distance kd from a given string of Hamming weight j is

$$\sum_{i=t_0}^j \binom{j}{i} \binom{k-j}{t-i} \geq \binom{j}{t_0} \binom{k-j}{t-t_0} \tag{2.207}$$

as long as $t-kd \leq j \leq t+kd$ and is 0 otherwise. It follows that if s^k has Hamming weight j ,

$$P_{\bar{Z}^k}(B_d(s^k)) \geq \sum_{t=0}^k L_k(j, t) q^t (1-q)^{k-t} \tag{2.208}$$

Relaxing (2.104) using (2.208), (2.204) follows. \square

The following bound shows that good constant composition codes exist.

Theorem 2.33 (Achievability, BMS). *There exists a (k, M, d, ϵ) constant composition code with*

$$\epsilon \leq \sum_{j=0}^k \binom{k}{j} p^j (1-p)^{k-j} \left[1 - \binom{k}{\lceil kq \rceil}^{-1} L_k(j, \lceil kq \rceil) \right]^M \quad (2.209)$$

where q and $L_k(\cdot, \cdot)$ are defined in (2.205) and (2.206) respectively.

Proof. The proof is along the lines of the proof of Theorem 2.32, except that now we let $P_{\mathbb{Z}^k}$ be equiprobable on the set of binary strings of Hamming weight $\lceil kq \rceil$. \square

The following asymptotic analysis of $R(k, d, \epsilon)$ strengthens Theorem 2.22.

Theorem 2.34 (Gaussian approximation, BMS). *The minimum achievable rate at blocklength k satisfies (2.141) where*

$$R(d) = h(p) - h(d) \quad (2.210)$$

$$\mathcal{V}(d) = \text{Var}[\iota_S(S)] = p(1-p) \log^2 \frac{1-p}{p} \quad (2.211)$$

and the remainder term in (2.141) satisfies

$$O\left(\frac{1}{k}\right) \leq \theta\left(\frac{\log k}{k}\right) \quad (2.212)$$

$$\leq \frac{1}{2} \frac{\log k}{k} + \frac{\log \log k}{k} + O\left(\frac{1}{k}\right) \quad (2.213)$$

if $0 < d < p$, and

$$\theta\left(\frac{\log k}{k}\right) = -\frac{1}{2} \frac{\log k}{k} + O\left(\frac{1}{k}\right) \quad (2.214)$$

if $d = 0$.

Proof. The case $d = 0$ follows immediately from (2.147). For $0 < d < p$, the dispersion (2.211) is easily obtained plugging $k = 1$ into (2.19). The tightened upper bound for the remainder (2.213) follows via the asymptotic analysis of Theorem 2.33 shown in Appendix B.8. We proceed to show the converse part, which yields a better $\frac{\log k}{k}$ term than Theorem 2.22.

According to the definition of r^* in (2.200),

$$\mathbb{P}\left[\sum_{i=1}^k S_i > r\right] \geq \epsilon \quad (2.215)$$

for any $r \leq r^*$, where $\{S_i\}$ are binary i.i.d. with $P_{S_i}(1) = p$. In particular, due to (2.155), (2.215)

holds for

$$r = np + \sqrt{kp(1-p)}Q^{-1}\left(\epsilon + \frac{B_k}{\sqrt{k}}\right) \quad (2.216)$$

$$= np + \sqrt{kp(1-p)}Q^{-1}(\epsilon) + O(1) \quad (2.217)$$

where (2.217) follows because in the present case $B_k = 6\frac{1-2p+2p^2}{\sqrt{p(1-p)}}$, which does not depend on k .

Using (2.199), we have

$$M \geq \frac{\left\langle \begin{smallmatrix} k \\ \lfloor r \rfloor \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} k \\ \lfloor kd \rfloor \end{smallmatrix} \right\rangle} \quad (2.218)$$

Taking logarithms of both sides of (2.218), we have

$$\log M \geq \log \left\langle \begin{smallmatrix} k \\ \lfloor r \rfloor \end{smallmatrix} \right\rangle - \log \left\langle \begin{smallmatrix} k \\ \lfloor kd \rfloor \end{smallmatrix} \right\rangle \quad (2.219)$$

$$= kh\left(p + \frac{1}{\sqrt{k}}\sqrt{p(1-p)}Q^{-1}(\epsilon)\right) - kh(d) + O(1) \quad (2.220)$$

$$= kh(p) - kh(d) + \sqrt{k}\sqrt{p(1-p)}h'(p)Q^{-1}(\epsilon) + O(1)$$

where (2.220) is due to (B.147) in Appendix B.7. The desired bound (2.213) follows since $h'(p) = \log \frac{1-p}{p}$. \square

Figures 2.2 and 2.3 present a numerical comparison of Shannon's achievability bound (2.51), the new bounds in (2.204), (2.199) and (2.197) as well as the Gaussian approximation in (2.141) in which we have neglected $\theta\left(\frac{\log k}{k}\right)$. The achievability bound (2.51) is very loose and so is Marton's converse which is essentially indistinguishable from $R(d)$. The new finite blocklength bounds (2.204) and (2.199) are fairly tight unless the blocklength is very small. In Fig. 2.3 obtained with a more stringent ϵ , the approximation of Theorem 2.34 is essentially halfway between the converse and achievability bounds.

2.8 Discrete memoryless source

This section particularizes the bounds in Section 2.5 to stationary memoryless sources with countable alphabet \mathcal{S} where $|\mathcal{S}| = m$ (possibly ∞) and symbol error rate distortion measure, i.e. $d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k 1\{s_i \neq z_i\}$. For convenience, we denote the number of strings within Hamming distance ℓ

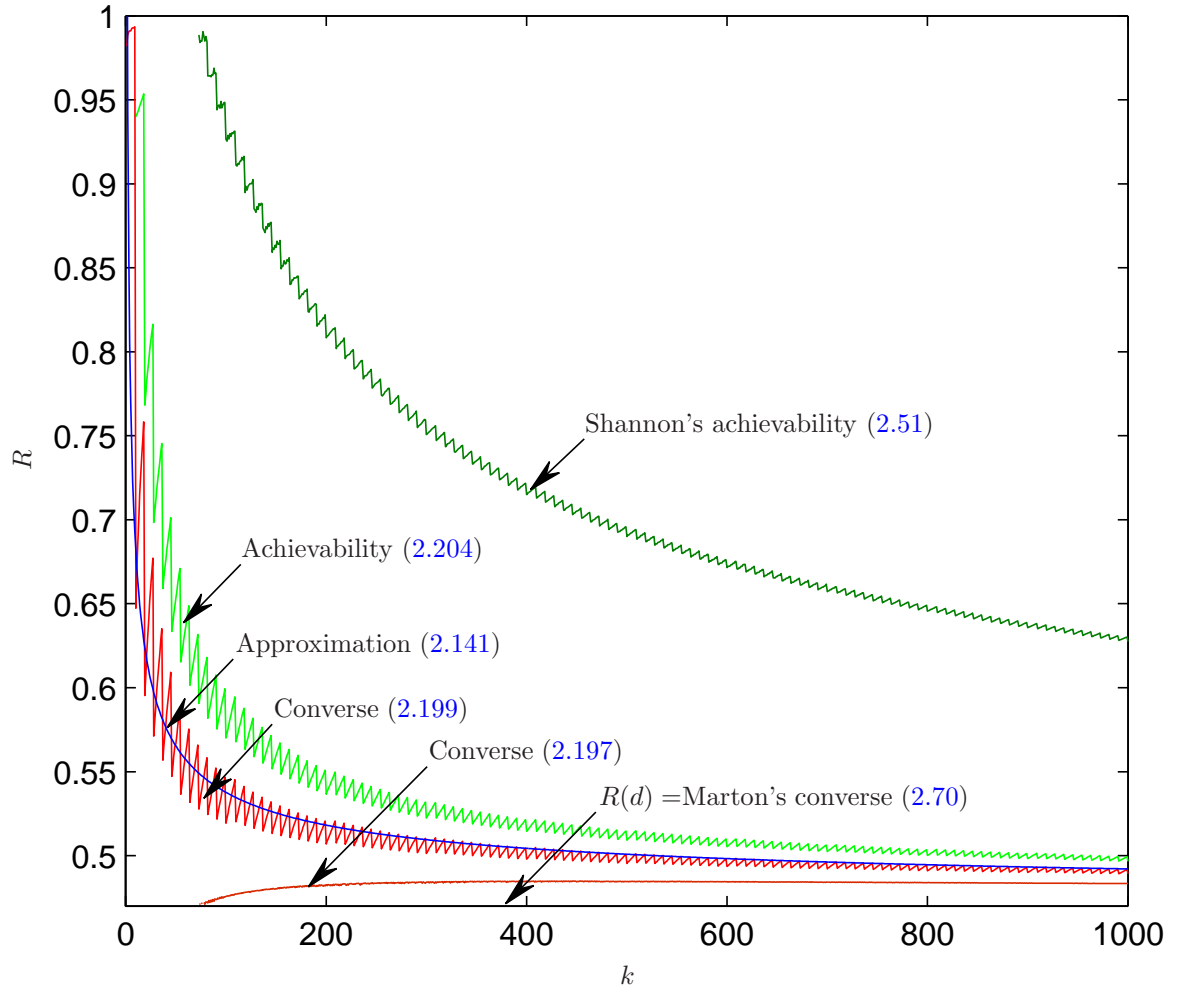


Figure 2.2: Rate-blocklength tradeoff for BMS with $p = 2/5$, $d = 0.11$, $\epsilon = 10^{-2}$.

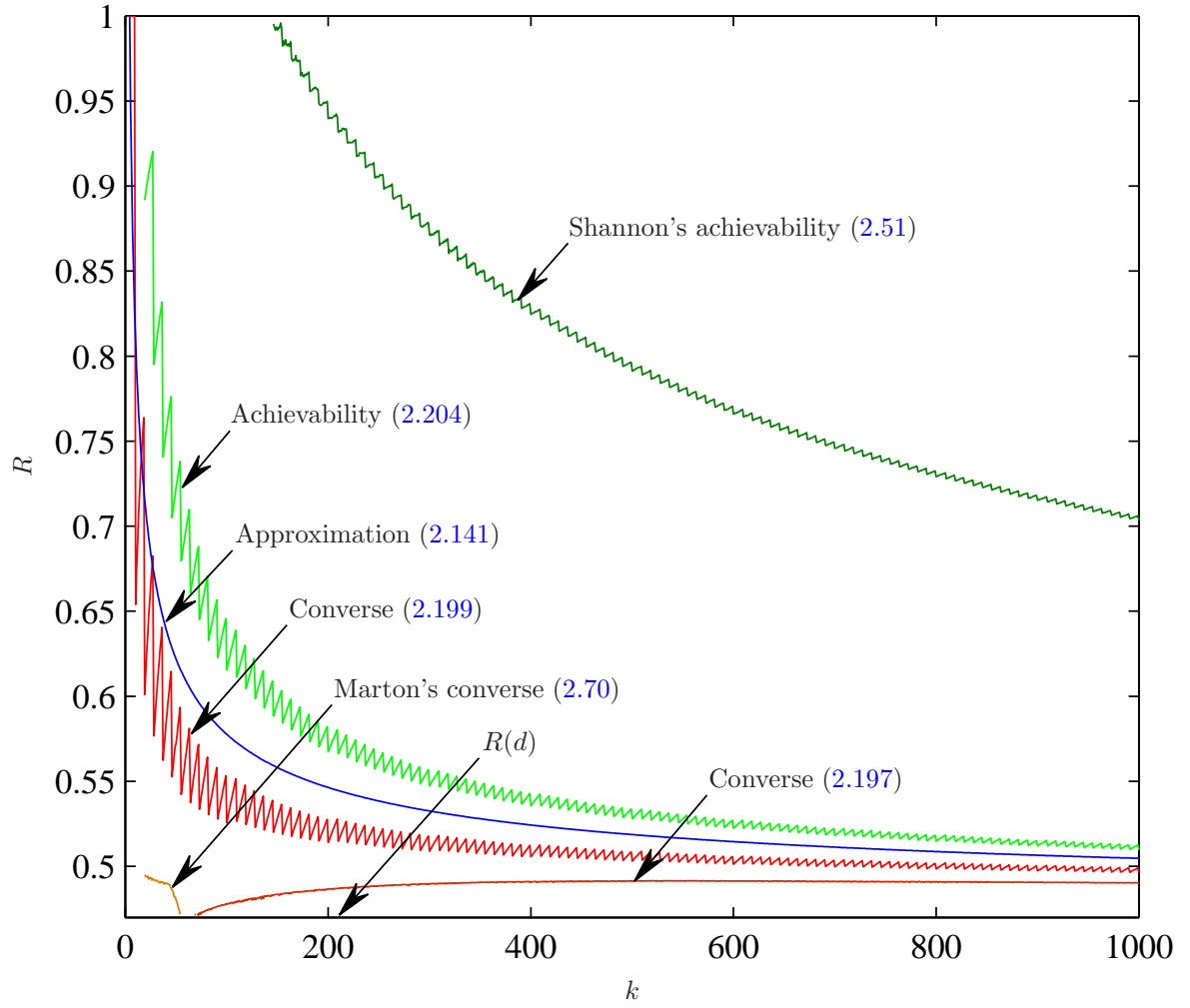


Figure 2.3: Rate-blocklength tradeoff for BMS with $p = 2/5$, $d = 0.11$, $\epsilon = 10^{-4}$.

from a given string by

$$S_\ell = \sum_{j=0}^{\ell} \binom{k}{j} (m-1)^j \quad (2.221)$$

2.8.1 Equiprobable DMS (EDMS)

In this subsection we fix $0 \leq d < 1 - \frac{1}{m}$, $0 < \epsilon < 1$ and assume that all source letters are equiprobable, in which case the rate-distortion function is given by [54]

$$R(d) = \log m - h(d) - d \log(m-1) \quad (2.222)$$

As in the equiprobable binary case, Theorem 2.12 reduces to (2.150). A stronger converse bound is obtained using Theorem 2.13 in a manner analogous to that of Theorem 2.26.

Theorem 2.35 (Converse, EDMS). *Any (k, M, d, ϵ) code must satisfy:*

$$\epsilon \geq 1 - Mm^{-k}S_{\lfloor kd \rfloor} \quad (2.223)$$

The following result is a straightforward generalization of Theorem 2.27 to the non-binary case.

Theorem 2.36 (Exact performance of random coding, EDMS). *The minimal averaged probability that symbol error rate exceeds d achieved by random coding with M codewords is*

$$\min_{P_Z} \mathbb{E} [\epsilon_d(Z_1, \dots, Z_M)] = (1 - m^{-k}S_{\lfloor kd \rfloor})^M \quad (2.224)$$

attained by P_Z equiprobable on \mathcal{S}^k . In the left side of (2.224), Z_1, \dots, Z_M are independent distributed according to P_Z .

Theorem 2.36 leads to the following achievability bound.

Theorem 2.37 (Achievability, EDMS). *There exists a (k, M, d, ϵ) code such that*

$$\epsilon \leq (1 - S_{\lfloor kd \rfloor} m^{-k})^M \quad (2.225)$$

The asymptotic analysis of the bounds in (2.225) and (2.223) yields the following tight approximation.

Theorem 2.38 (Gaussian approximation, EDMS). *The minimum achievable rate at blocklength k*

satisfies

$$R(k, d, \epsilon) = R(d) + \frac{1}{2} \frac{\log k}{k} + O\left(\frac{1}{k}\right) \quad (2.226)$$

if $0 < d < 1 - \frac{1}{m}$, and

$$R(k, 0, \epsilon) = \log m - \frac{1}{k} \log \frac{1}{1 - \epsilon} + o_k \quad (2.227)$$

where $0 \leq o_k \leq \frac{m^{-k}}{(1-\epsilon)^k}$.

Proof. Appendix [B.9](#). □

2.8.2 Nonequiprobable DMS

Assume without loss of generality that the source letters are labeled by $\mathcal{S} = \{1, 2, \dots\}$ so that

$$P_{\mathcal{S}}(1) \geq P_{\mathcal{S}}(2) \geq \dots \quad (2.228)$$

Assume further that $0 \leq d < 1 - P_{\mathcal{S}}(1)$, $0 < \epsilon < 1$.

Recall that the rate-distortion function is achieved by [\[54\]](#)

$$P_{\mathbf{Z}^*}(b) = \begin{cases} \frac{P_{\mathcal{S}}(b) - \eta}{1 - d - \eta} & b \leq M_{\mathcal{S}}(\eta) \\ 0 & \text{otherwise} \end{cases} \quad (2.229)$$

$$P_{\mathcal{S}|\mathbf{Z}^*}(a|b) = \begin{cases} 1 - d & a = b, a \leq M_{\mathcal{S}}(\eta) \\ \eta & a \neq b, a \leq M_{\mathcal{S}}(\eta) \\ P_{\mathcal{S}}(a) & a > M_{\mathcal{S}}(\eta) \end{cases} \quad (2.230)$$

where $M_{\mathcal{S}}(\eta)$ is the number of masses with probability strictly larger than η :

$$M_{\mathcal{S}}(\eta) = \max \left\{ s \in \mathcal{M} : \iota_{\mathcal{S}}(\mathbf{s}) < \log \frac{1}{\eta} \right\} \quad (2.231)$$

and $0 \leq \eta \leq 1$ is the solution to

$$d = \mathbb{P} \left[\iota_{\mathcal{S}}(\mathbf{S}) \geq \log \frac{1}{\eta} \right] + (M_{\mathcal{S}}(\eta) - 1)\eta \quad (2.232)$$

The rate-distortion function can be expressed as [54]

$$R(d) = \mathbb{E} \left[\iota_S(S) 1 \left\{ \iota_S(S) < \log \frac{1}{\eta} \right\} \right] - (M_S(\eta) - 1) \eta \log \frac{1}{\eta} - (1 - d) \log \frac{1}{1 - d} \quad (2.233)$$

Note that if the source alphabet is finite and $0 \leq d < (m - 1)P_S(m)$, then $M_S(\eta) = m$, $\eta = \frac{d}{m-1}$, and (2.229), (2.230) and (2.233) can be simplified. In particular, the rate-distortion function on that region is given by

$$R(d) = H(S) - h(d) - d \log(m - 1) \quad (2.234)$$

which generalizes (2.222).

The first result of this section is a particularization of the bound in Theorem 2.12 to the DMS case.

Theorem 2.39 (Converse, DMS). *For any (k, M, d, ϵ) code, it holds that*

$$\epsilon \geq \sup_{\gamma \geq 0} \left\{ \mathbb{P} \left[\sum_{i=1}^k J_S(S_i, d) \geq \log M + \gamma \right] - \exp \{-\gamma\} \right\} \quad (2.235)$$

where

$$J_S(a, d) = (1 - d) \log(1 - d) + d \log \eta + \min \left\{ \iota_S(a), \log \frac{1}{\eta} \right\} \quad (2.236)$$

and η is defined in (2.232).

Proof. Case $d = 0$ is obvious. For $0 < d < 1 - P_S(1)$, differentiating (2.233) with respect to d yields

$$\lambda^* = \log \frac{1 - d}{\eta} \quad (2.237)$$

Plugging (2.230) and λ^* into (2.8), one obtains (2.236). \square

We now assume that the source alphabet is finite, $m < \infty$ and adopt the notation of [55]:

- type of the string: $\mathbf{j} = (j_1, \dots, j_m)$, $j_1 + \dots + j_m = k$
- probability of a given string of type \mathbf{j} : $p^{\mathbf{j}} = P_S(1)^{j_1} \dots P_S(m)^{j_m}$
- type ordering: $\mathbf{i} \preceq \mathbf{j}$ if and only if $p^{\mathbf{i}} \geq p^{\mathbf{j}}$
- type $\mathbf{1}$ denotes $[k, 0, \dots, 0]$
- previous and next types: $\mathbf{j} - 1$ and $\mathbf{j} + 1$, respectively

- multinomial coefficient: $\binom{k}{\mathbf{j}} = \frac{k!}{j_1! \dots j_m!}$

The next converse result is a particularization of Theorem 2.13.

Theorem 2.40 (Converse, DMS). *Any (k, M, d, ϵ) code must satisfy*

$$M \geq \frac{\sum_{\mathbf{i}=1}^{\mathbf{j}^*} \binom{k}{\mathbf{i}} + \alpha \binom{k}{\mathbf{j}^* + 1}}{S_{[kd]}} \quad (2.238)$$

where

$$\mathbf{j}^* = \max \left\{ \mathbf{j} : \sum_{\mathbf{i}=1}^{\mathbf{j}} \binom{k}{\mathbf{i}} p^{\mathbf{i}} \leq 1 - \epsilon \right\} \quad (2.239)$$

and $\alpha \in [0, 1)$ is the solution to

$$\sum_{\mathbf{i}=1}^{\mathbf{j}^*} \binom{k}{\mathbf{i}} p^{\mathbf{i}} + \alpha \binom{k}{\mathbf{j}^* + 1} p^{\mathbf{j}^* + 1} = 1 - \epsilon \quad (2.240)$$

Proof. Consider a binary hypothesis test between the k -dimensional source distribution P_{S^k} and U , the equiprobable distribution on \mathcal{S}^k . From Theorem 2.13,

$$M \geq m^k \frac{\beta_{1-\epsilon}(P_{S^k}, U)}{S_{[kd]}} \quad (2.241)$$

The calculation of $\beta_{1-\epsilon}(P_{S^k}, U)$ is analogous to the BMS case. \square

The following result guarantees existence of a good code with all codewords of type $\mathbf{t}^* = ([kP_{Z^*}(1)], \dots, [kP_{Z^*}(M_S(\eta))], 0, \dots, 0)$ where $[\cdot]$ denotes rounding off to a neighboring integer so that $\sum_{b=1}^{M_S(\eta)} [nP_{Z^*}(b)] = k$ holds.

Theorem 2.41 (Achievability, DMS). *There exists a (k, M, d, ϵ) fixed composition code with codewords of type \mathbf{t}^* and*

$$\epsilon \leq \sum_{\mathbf{j}} \binom{k}{\mathbf{j}} p^{\mathbf{j}} \left(1 - \binom{k}{\mathbf{t}^*}^{-1} L_k(\mathbf{j}, \mathbf{t}^*) \right)^M \quad (2.242)$$

$$L_k(\mathbf{j}, \mathbf{t}^*) = \prod_{a=1}^m \binom{j_a}{t_a^*} \quad (2.243)$$

where $\mathbf{j} = [j_1, \dots, j_m]$ ranges over all k -types, and j_a -types $\mathbf{t}_a = (t_{a,1}, \dots, t_{a,M_S(\eta)})$ are given by

$$t_{a,b} = \lceil P_{S|Z^*}(a|b)t_b^* + \delta(a,b)k \rceil \quad (2.244)$$

where

$$\delta(a,b) = \frac{\Delta_a}{M_S(\eta)} + \begin{cases} \frac{1}{M_S(\eta)^2} \sum_{i=M_S(\eta)+1}^m \Delta_i & a = b, a \leq M_S(\eta) \\ \frac{-1}{M_S(\eta)^2(M_S(\eta)-1)} \sum_{i=M_S(\eta)+1}^m \Delta_i & a \neq b, a \leq M_S(\eta) \\ 0 & a > M_S(\eta) \end{cases} \quad (2.245)$$

$$k\Delta_a = j_a - kP_S(a), \quad a = 1, \dots, m \quad (2.246)$$

In (2.244), $a = 1, \dots, m$, $b = 1, \dots, M_S(\eta)$ and $\lceil \cdot \rceil$ denotes rounding off to a neighboring nonnegative integer so that

$$\sum_{b=1}^{M_S(\eta)} t_{b,b} \geq k(1-d) \quad (2.247)$$

$$\sum_{b=1}^{M_S(\eta)} t_{a,b} = j_a \quad (2.248)$$

$$\sum_{a=1}^m t_{a,b} = t_b^* \quad (2.249)$$

and among all possible choices the one that results in the largest value for (2.243) is adopted. If no such choice exists, $L_k(\mathbf{j}, \mathbf{t}^*) = 0$.

Proof. We compute an upper bound to (2.104) for the specific case of the DMS. Let $P_{\bar{Z}^k}$ be equiprobable on the set of m -ary strings of type \mathbf{t}^* . To compute the number of strings of type \mathbf{t}^* that are within distortion d from a given string s^k of type \mathbf{j} , observe that by fixing s^k we have divided an k -string into m bins, the a -th bin corresponding to the letter a and having size j_a . If $t_{a,b}$ is the number of the letters b in a sequence z^k of type \mathbf{t}^* that fall into a -th bin, the strings s^k and z^k are within Hamming distance kd from each other as long as (2.247) is satisfied. Therefore, the number of strings of type \mathbf{t}^* that are within Hamming distance kd from a given string of type \mathbf{j} is bounded by

$$\sum \prod_{a=1}^m \binom{j_a}{t_a} \geq L_k(\mathbf{j}, \mathbf{t}^*) \quad (2.250)$$

where the summation in the left side is over all collections of j_a -types $\mathbf{t}_a = (t_{a,1}, \dots, t_{a,M_S(\eta)})$, $a = 1, \dots, m$ that satisfy (2.247)-(2.249), and inequality (2.250) is obtained by lower bounding the

sum by the term with $t_{a,b}$ given by (2.244). It follows that if s^k has type \mathbf{j} ,

$$P_{Z^k}(B_d(s^k)) \geq \binom{k}{\mathbf{t}^*}^{-1} L_k(\mathbf{j}, \mathbf{t}^*) \quad (2.251)$$

Relaxing (2.104) using (2.251), (2.242) follows. \square

Remark 2.12. As k increases, the bound in (2.250) becomes increasingly tight. This is best understood by checking that all strings with $k_{a,b}$ given by (2.244) lie at a Hamming distance of approximately kd from some fixed string of type \mathbf{j} , and recalling [41] that most of the volume of an k -dimensional ball is concentrated near its surface (a similar phenomenon occurs in Euclidean spaces as well), so that the largest contribution to the sum on the left side of (2.250) comes from the strings satisfying (2.244).

The following second-order analysis makes use of Theorem 2.22 and, to strengthen the bounds for the remainder term, of Theorems 2.40 and 2.41.

Theorem 2.42 (Gaussian approximation, DMS). *The minimum achievable rate at blocklength k , $R(k, d, \epsilon)$, satisfies (2.141) where $R(d)$ is given by (2.233), and $\mathcal{V}(d)$ can be characterized parametrically:*

$$\mathcal{V}(d) = \text{Var} \left[\min \left\{ \imath_{\mathcal{S}}(\mathbf{S}), \log \frac{1}{\eta} \right\} \right] \quad (2.252)$$

where η depends on d through (2.232), (2.231). Moreover, (2.145) can be replaced by:

$$\theta \left(\frac{\log k}{k} \right) \leq \frac{(m-1)(M_{\mathcal{S}}(\eta) - 1)}{2} \frac{\log k}{k} + \frac{\log \log k}{k} + O \left(\frac{1}{k} \right) \quad (2.253)$$

If $0 \leq d < (m-1)P_{\mathcal{S}}(m)$, (2.252) reduces to

$$\mathcal{V}(d) = \text{Var} [\imath_{\mathcal{S}}(\mathbf{S})] \quad (2.254)$$

and if $d > 0$, (2.144) can be strengthened to

$$O \left(\frac{1}{k} \right) \leq \theta \left(\frac{\log k}{k} \right) \quad (2.255)$$

while if $d = 0$,

$$\theta \left(\frac{\log k}{k} \right) = -\frac{1}{2} \frac{\log k}{k} + O \left(\frac{1}{k} \right) \quad (2.256)$$

Proof. Using the expression for d -tilted information (2.236), we observe that

$$\text{Var}[j_S(S, d)] = \text{Var}\left[\min\left\{i_S(S), \log \frac{1}{\eta}\right\}\right] \quad (2.257)$$

and (2.252) follows. The case $d = 0$ is verified using (2.147). Theorem 2.41 leads to (2.253), as we show in Appendix B.10.

When $0 < d < (m-1)P_S(m)$, not only (2.233) and (2.252) reduce to (2.234) and (2.254) respectively, but a tighter converse for the $\frac{\log k}{k}$ term (2.255) can be shown. Recall the asymptotics of $S_{\lfloor kd \rfloor}$ in (B.176) (Appendix B.9). Furthermore, it can be shown [55] that

$$\sum_{\mathbf{i}=1}^{\mathbf{j}} \binom{k}{\mathbf{i}} = \frac{C}{\sqrt{k}} \exp\left\{kH\left(\frac{\mathbf{j}}{k}\right)\right\} \quad (2.258)$$

for some constant C . Armed with (2.258) and (B.176), we are ready to proceed to the second-order analysis of (2.238). From the definition of \mathbf{j}^* in (2.239),

$$\mathbb{P}\left[\frac{1}{k} \sum_{i=1}^k i_S(S_i) > H(S) + \sum_{a=1}^m \Delta_a i_S(a)\right] \geq \epsilon \quad (2.259)$$

for any Δ with $\sum_{a=1}^m \Delta_a = 0$ satisfying $k(\mathbf{p} + \Delta) \preceq \mathbf{j}^*$, where $\mathbf{p} = [P_S(1), \dots, P_S(m)]$ (we slightly abused notation here as $k(\mathbf{p} + \Delta)$ is not always precisely an k -type; naturally, the definition of the type ordering \preceq extends to such cases). Noting that $\mathbb{E}[i_S(S_i)] = H(S)$ and $\text{Var}[i_S(S_i)] = \text{Var}[i_S(S)]$, we conclude from the Berry-Esséen CLT (2.155) that (2.259) holds for

$$\sum_{a=1}^m \Delta_a i_S(a) = \sqrt{\frac{\text{Var}[i_S(S)]}{k}} Q^{-1}\left(\epsilon - \frac{B_k}{\sqrt{k}}\right) \quad (2.260)$$

where B_k is given by (2.159). Taking logarithms of both sides of (2.238), we have

$$\log M \geq \log \left[\sum_{\mathbf{i}=1}^{\mathbf{j}^*} \binom{k}{\mathbf{i}} + \alpha \binom{k}{\mathbf{j}^*} \right] - \log S_{\lfloor kd \rfloor} \quad (2.261)$$

$$\geq \log \sum_{\mathbf{i}=1}^{\mathbf{j}^*} \binom{k}{\mathbf{i}} - \log S_{\lfloor kd \rfloor} \quad (2.262)$$

$$\geq kH(\mathbf{p} + \Delta) - kh(d) - kd \log(m-1) + O(1) \quad (2.263)$$

$$= kH(\mathbf{p}) + k \sum_{a=1}^m \Delta_a i_S(a) - kh(d) - kd \log(m-1) + O(1) \quad (2.264)$$

where we used (B.176) and (2.258) to obtain (2.263), and (2.264) is obtained by applying a Taylor

series expansion to $H(\mathbf{p} + \mathbf{\Delta})$. The desired result in (2.255) follows by substituting (2.260) in (2.264), applying a Taylor series expansion to $Q^{-1}\left(\epsilon - \frac{B_k}{\sqrt{k}}\right)$ in the vicinity of ϵ and noting that B_k is a finite constant. \square

The rate-dispersion function and the blocklength (2.140) required to sustain $R = 1.1R(d)$ are plotted in Fig. 2.4 for a quaternary source with distribution $[\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}]$. Note that according to (2.140), the blocklength required to approach $1.1R(d)$ with a given probability of excess distortion grows rapidly as $d \rightarrow d_{\max}$.

2.9 Gaussian memoryless source

This section applies Theorems 2.12, 2.13 and 2.16 to the i.i.d. Gaussian source with mean-square error distortion, $d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k (s_i - z_i)^2$, and refines the second-order analysis in Theorem 2.22. Throughout this section, it is assumed that $S_i \sim \mathcal{N}(0, \sigma^2)$, $0 < d < \sigma^2$ and $0 < \epsilon < 1$.

The particularization of Theorem 2.12 to the GMS using (2.20) yields the following result.

Theorem 2.43 (Converse, GMS). *Any (k, M, d, ϵ) code must satisfy*

$$\epsilon \geq \sup_{\gamma \geq 0} \{\mathbb{P}[g_k(W) \geq \log M + \gamma] - \exp(-\gamma)\} \quad (2.265)$$

$$g_k(W) = \frac{k}{2} \log \frac{\sigma^2}{d} + \frac{W - k}{2} \log e \quad (2.266)$$

where $W \sim \chi_2^k$ (i.e. chi-squared distributed with k degrees of freedom).

The following result can be obtained by an application of Theorem 2.13 to the GMS.

Theorem 2.44 (Converse, GMS). *Any (k, M, d, ϵ) code must satisfy*

$$M \geq \left(\frac{\sigma}{\sqrt{d}} r_k(\epsilon) \right)^k \quad (2.267)$$

where $r_k(\epsilon)$ is the solution to

$$\mathbb{P}[W < k r_k^2(\epsilon)] = 1 - \epsilon, \quad (2.268)$$

and $W \sim \chi_k^2$.

Proof. Inequality (2.267) simply states that the minimum number of k -dimensional balls of radius \sqrt{kd} required to cover an k -dimensional ball of radius $\sqrt{k}\sigma r_k(\epsilon)$ cannot be smaller than the ratio of

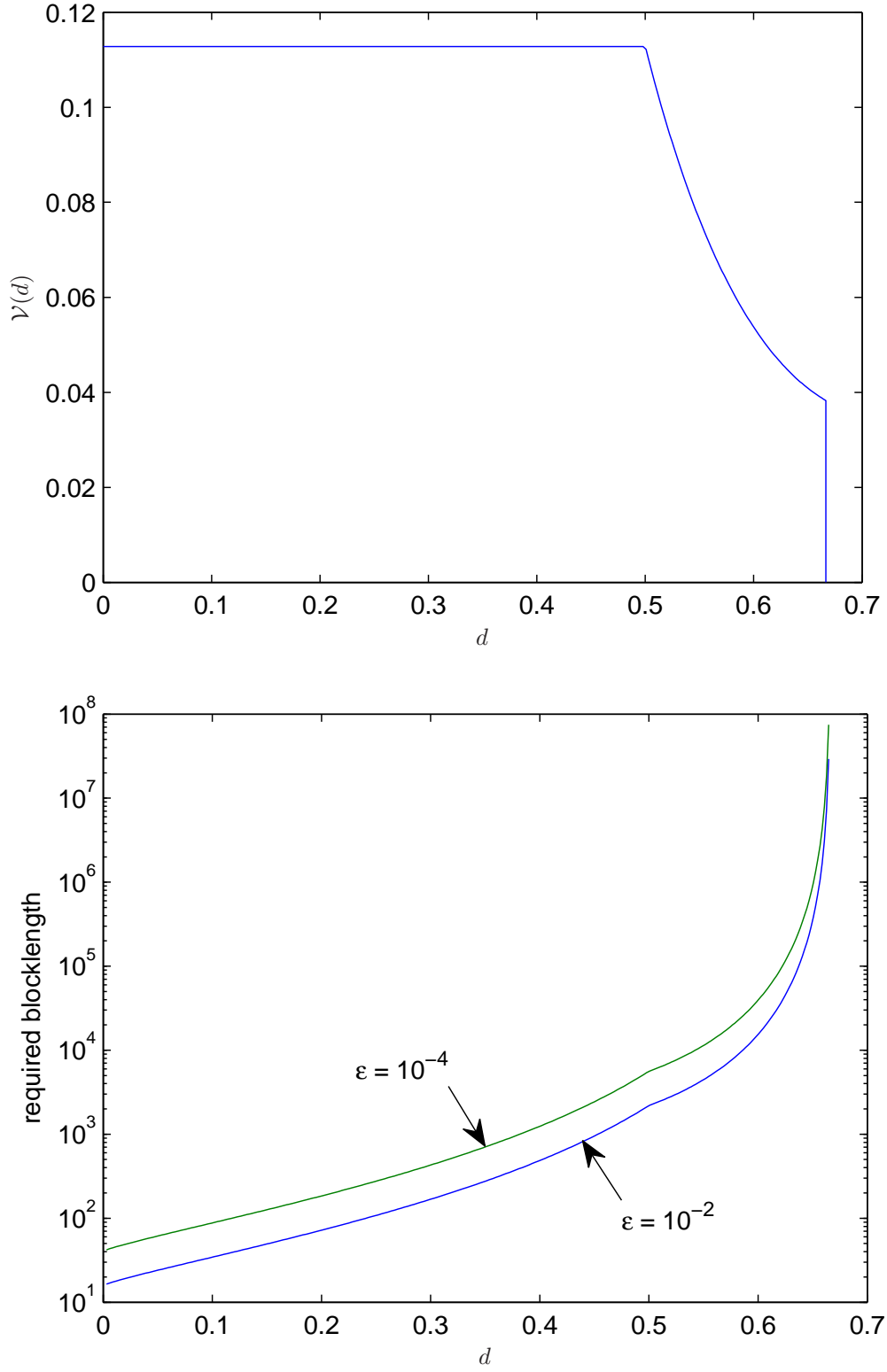


Figure 2.4: Rate-dispersion function (bits) and the blocklength (2.140) required to sustain $R = 1.1R(d)$ provided that excess-distortion probability is bounded by ϵ for DMS with $P_S = [\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}]$.

their volumes. Since

$$W = \frac{1}{\sigma^2} \sum_{i=1}^k S_i^2 \quad (2.269)$$

is χ_k^2 -distributed, the left side of (2.268) is the probability that the source produces a sequence that falls inside \mathbb{B} , the k -dimensional ball of radius $\sqrt{k}\sigma r_k(\epsilon)$ with center at $\mathbf{0}$. But as follows from the spherical symmetry of the Gaussian distribution, \mathbb{B} has the smallest volume among all sets in \mathbb{R}^k having probability $1 - \epsilon$. Since any (k, M, d, ϵ) -code is a covering of a set that has total probability of at least $1 - \epsilon$, the result follows. \square

Note that the proof of Theorem 2.44 can be formulated in the hypothesis testing language of Theorem 2.13 by choosing Q to be the Lebesgue measure on \mathbb{R}^k .

The following achievability result can be regarded as the rate-distortion counterpart to Shannon's geometric analysis of optimal coding for the Gaussian channel [10].

Theorem 2.45 (Achievability, GMS). *There exists a (k, M, d, ϵ) code with*

$$\epsilon \leq k \int_0^\infty [1 - \rho(k, x)]^M f_{\chi_k^2}(kx) dx \quad (2.270)$$

where $f_{\chi_k^2}(\cdot)$ is the χ_k^2 probability density function, and

$$\rho(k, x) = \frac{\Gamma(\frac{k}{2} + 1)}{\sqrt{\pi}k\Gamma(\frac{k-1}{2} + 1)} \left(1 - \frac{(1 + x - 2\frac{d}{\sigma^2})^2}{4(1 - \frac{d}{\sigma^2})x} \right)^{\frac{k-1}{2}} \quad (2.271)$$

if $a^2 \leq x \leq b^2$, where

$$a = \sqrt{1 - \frac{d}{\sigma^2}} - \sqrt{\frac{d}{\sigma^2}} \quad (2.272)$$

$$b = \sqrt{1 - \frac{d}{\sigma^2}} + \sqrt{\frac{d}{\sigma^2}} \quad (2.273)$$

and $\rho(k, x) = 0$ otherwise.

Proof. We compute an upper bound to (2.104) for the specific case of the GMS. Let P_{Z^k} be the uniform distribution on the surface of the k -dimensional sphere with center at $\mathbf{0}$ and radius

$$r_0 = \sqrt{k}\sigma \sqrt{1 - \frac{d}{\sigma^2}} \quad (2.274)$$

This choice corresponds to an asymptotically-optimal positioning of representation points in the limit of large k , see Fig. 2.5(a), [29, 42]. Indeed, for large k , most source sequences will be concentrated

within a thin shell near the surface of the sphere of radius $\sqrt{k}\sigma$. The center of the sphere of radius \sqrt{kd} must be at distance r_0 from the origin in order to cover the largest area of the surface of the sphere of radius $\sqrt{k}\sigma$.

We proceed to lower-bound $P_{Z^k}(B_d(s^k))$, $s^k \in \mathbb{R}^k$. Observe that $P_{Z^k}(B_d(s^k)) = 0$ if s^k is either too close or too far from the origin, that is, if $|s^k| < \sqrt{k}\sigma a$ or $|s^k| > \sqrt{k}\sigma b$, where $|\cdot|$ denotes the Euclidean norm. To treat the more interesting case $\sqrt{k}\sigma a \leq |s^k| \leq \sqrt{k}\sigma b$, it is convenient to introduce the following notation.

- $S_k(r) = \frac{k\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}r^{k-1}$: surface area of an k -dimensional sphere of radius r ;
- $S_k(r, \theta)$: surface area of an k -dimensional polar cap of radius r and polar angle θ .

Similar to [29, 42], from Fig. 2.5(b),

$$S_k(r, \theta) \geq \frac{\pi^{\frac{k-1}{2}}}{\Gamma(\frac{k-1}{2}+1)}(r \sin \theta)^{k-1} \quad (2.275)$$

where the right side of (2.275) is the area of a $(k-1)$ -dimensional disc of radius $r \sin \theta$. So if $\sqrt{k}\sigma a \leq |s^k| = r \leq \sqrt{k}\sigma b$,

$$P_{Z^k}(B_d(s^k)) = \frac{S_k(|s^k|, \theta)}{S_k(|s^k|)} \quad (2.276)$$

$$\geq \frac{\Gamma(\frac{k}{2}+1)}{\sqrt{\pi k} \Gamma(\frac{k-1}{2}+1)} (\sin \theta)^{k-1} \quad (2.277)$$

where θ is the angle in Fig. 2.5(b); by the law of cosines

$$\cos \theta = \frac{r^2 + r_0^2 - kd}{2rr_0} \quad (2.278)$$

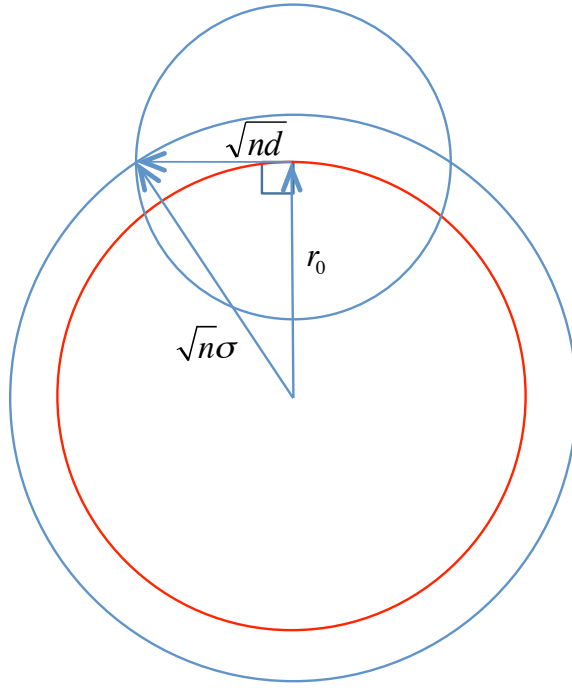
Finally, by Theorem 2.16, there exists a (k, M, d, ϵ) code with

$$\epsilon \leq \mathbb{E} \left[(1 - P_{Z^k}(B_d(S^k)))^M \right] \quad (2.279)$$

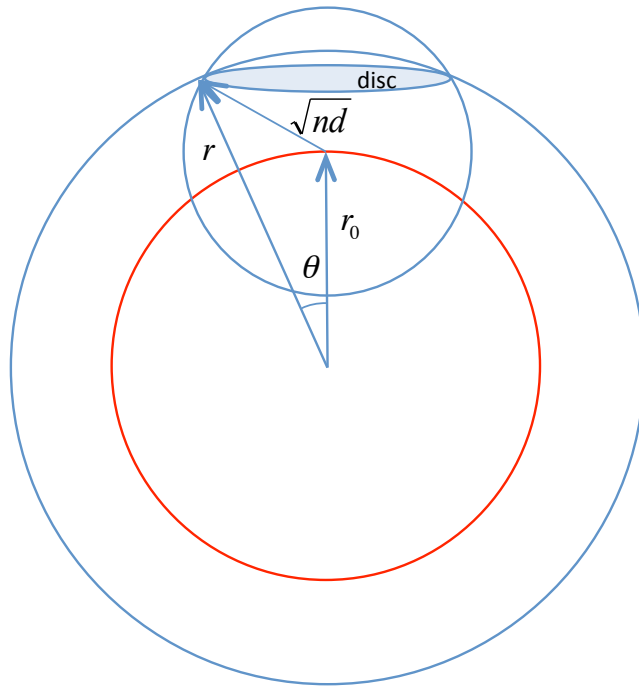
$$\begin{aligned} &= \mathbb{E} \left[(1 - P_{Z^k}(B_d(S^k)))^M \mid \sqrt{k}\sigma a \leq |S^k| \leq \sqrt{k}\sigma b \right] \\ &\quad + \mathbb{P} \left[|S^k| < \sqrt{k}\sigma a \right] + \mathbb{P} \left[|S^k| > \sqrt{k}\sigma b \right] \end{aligned} \quad (2.280)$$

Since $\frac{|S^k|^2}{\sigma^2}$ is χ_k^2 -distributed, one obtains (2.270) by plugging $\sin^2 \theta = 1 - \cos^2 \theta$ into (2.277) and substituting the latter in (2.280). \square

Essentially Theorem 2.45 evaluates the performance of Shannon's random code with all codewords



(a)



(b)

Figure 2.5: Optimum positioning of the representation sphere (a) and the geometry of the excess-distortion probability calculation (b).

lying on the surface of a sphere contained inside the sphere of radius $\sqrt{k}\sigma$. The following result allows us to bound the performance of a code whose codewords lie inside a ball of radius slightly larger than $\sqrt{k}\sigma$.

Theorem 2.46 (Rogers [56] - Verger-Gaugry [57]). *If $r > 1$ and $k \geq 2$, an k -dimensional sphere of radius r can be covered by $\lfloor \mathbb{M}(r) \rfloor$ spheres of radius 1, where*

$$\mathbb{M}(r) = \begin{cases} e(k \log_e k + k \log_e \log_e k + 5k) r^k & r \geq k \\ k(k \log_e k + k \log_e \log_e k + 5k) r^k & \frac{k}{\log_e k} \leq r < k \\ \frac{7^4 \log_e 7 / 7}{4} \sqrt{2\pi} \frac{k \sqrt{k} \left[(k-1) \log_e r k + (k-1) \log_e \log_e k + \frac{1}{2} \log_e k + \log_e \frac{\pi \sqrt{2k}}{\sqrt{\pi k - 2}} \right]}{r \left(1 - \frac{2}{\log_e k} \right) \left(1 - \frac{2}{\sqrt{\pi k}} \right) \log_e^2 k} r^k & 2 < r < \frac{k}{\log_e k} \\ \sqrt{2\pi} \frac{\sqrt{k} \left[(k-1) \log_e r k + (k-1) \log_e \log_e k + \frac{1}{2} \log_e k + \log_e \frac{\pi \sqrt{2k}}{\sqrt{\pi k - 2}} \right]}{r \left(1 - \frac{2}{\log_e k} \right) \left(1 - \frac{2}{\sqrt{\pi k}} \right)} r^k & 1 < r \leq 2 \end{cases} \quad (2.281)$$

The first two cases in (2.281) are encompassed by the classical result of Rogers [56] that appears not to have been improved since 1963, while the last two are due to the recent improvement by Verger-Gaugry [57]. An immediate corollary to Theorem 2.46 is the following:

Theorem 2.47 (Achievability, GMS). *For $k \geq 2$, there exists a (k, M, d, ϵ) code such that*

$$M \leq \mathbb{M} \left(\frac{\sigma}{\sqrt{d}} r_k(\epsilon) \right) \quad (2.282)$$

where $r_k(\epsilon)$ is the solution to (2.268).

Proof. Theorem 2.46 implies that there exists a code with no more than $\mathbb{M} \left(\frac{\sigma}{\sqrt{d}} r_k(\epsilon) \right)$ codewords such that all source sequences that fall inside \mathbb{B} , the k -dimensional ball of radius $\sqrt{k}\sigma r_k(\epsilon)$ with center at $\mathbf{0}$, are reproduced within distortion d . The excess-distortion probability is therefore given by the probability that the source produces a sequence that falls outside \mathbb{B} . \square

Note that Theorem 2.47 studies the number of balls of radius \sqrt{kd} to cover \mathbb{B} that is provably achievable, while the converse in Theorem 2.44 lower bounds the minimum number of balls of radius \sqrt{kd} required to cover \mathbb{B} by the ratio of their volumes.

Theorem 2.48 (Gaussian approximation, GMS). *The minimum achievable rate at blocklength k satisfies*

$$R(k, d, \epsilon) = \frac{1}{2} \log \frac{\sigma^2}{d} + \sqrt{\frac{1}{2k}} Q^{-1}(\epsilon) \log e + \theta \left(\frac{\log k}{k} \right) \quad (2.283)$$

where the remainder term satisfies

$$O\left(\frac{1}{k}\right) \leq \theta\left(\frac{\log k}{k}\right) \quad (2.284)$$

$$\leq \frac{1}{2} \frac{\log k}{k} + \frac{\log \log k}{k} + O\left(\frac{1}{k}\right) \quad (2.285)$$

Proof. We start with the converse part, i.e. (2.284).

Since in Theorem 2.44 $W = \frac{1}{\sigma^2} \sum_{i=1}^k S_i^2$, $S_i \sim \mathcal{N}(0, \sigma^2)$, we apply the Berry-Esséen CLT (Theorem 2.23) to $\frac{1}{\sigma^2} S_i^2$. Each $\frac{1}{\sigma^2} S_i^2$ has mean, second and third central moments equal to 1, 2 and 8, respectively. Let

$$r^2 = 1 + \sqrt{\frac{2}{k}} Q^{-1}\left(\epsilon + \frac{2c_0\sqrt{2}}{\sqrt{k}}\right) \quad (2.286)$$

$$= 1 + \sqrt{\frac{2}{k}} Q^{-1}(\epsilon) + O\left(\frac{1}{k}\right) \quad (2.287)$$

where c_0 is that in (2.159). Then by the Berry-Esséen inequality (2.155)

$$\mathbb{P}[W > k\bar{r}^2] \geq \epsilon \quad (2.288)$$

and therefore $r_k(\epsilon)$ that achieves the equality in (2.268) must satisfy $r_k(\epsilon) \geq r$. Weakening (2.267) by plugging r instead of $r_k(\epsilon)$ and taking logarithms of both sides therein, one obtains:

$$\log M \geq \frac{k}{2} \log \frac{\sigma^2 r^2}{d} \quad (2.289)$$

$$= \frac{k}{2} \log \frac{\sigma^2}{d} + \sqrt{\frac{k}{2}} Q^{-1}(\epsilon) \log e + O(1) \quad (2.290)$$

where (2.290) is a Taylor approximation of the right side of (2.289).

The achievability part (2.285) is proven in Appendix B.11 using Theorem 2.45. Theorem 2.47 leads to the correct rate-dispersion term but a weaker remainder term. \square

Figures 2.6 and 2.7 present a numerical comparison of Shannon's achievability bound (2.51) and the new bounds in (2.270), (2.282), (2.267) and (2.265) as well as the Gaussian approximation in (2.283) in which we took $\theta\left(\frac{\log k}{k}\right) = \frac{1}{2} \frac{\log k}{k}$. The achievability bound in (2.282) is tighter than the one in (2.270) at shorter blocklengths. Unsurprisingly, the converse bound in (2.267) is quite a bit tighter than the one in (2.265).

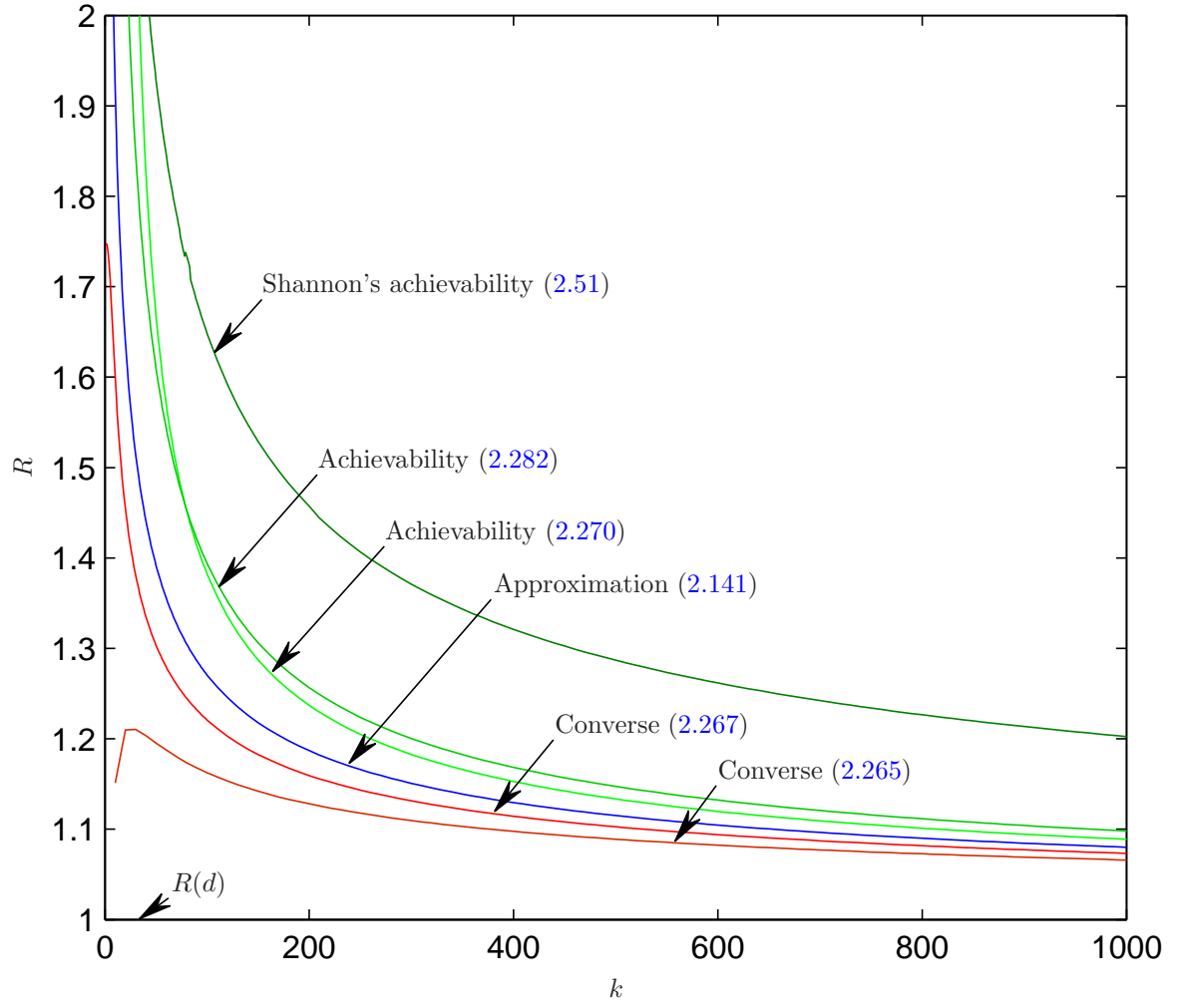


Figure 2.6: Rate-blocklength tradeoff for GMS with $\sigma = 1$, $d = \frac{1}{4}$, $\epsilon = 10^{-2}$.

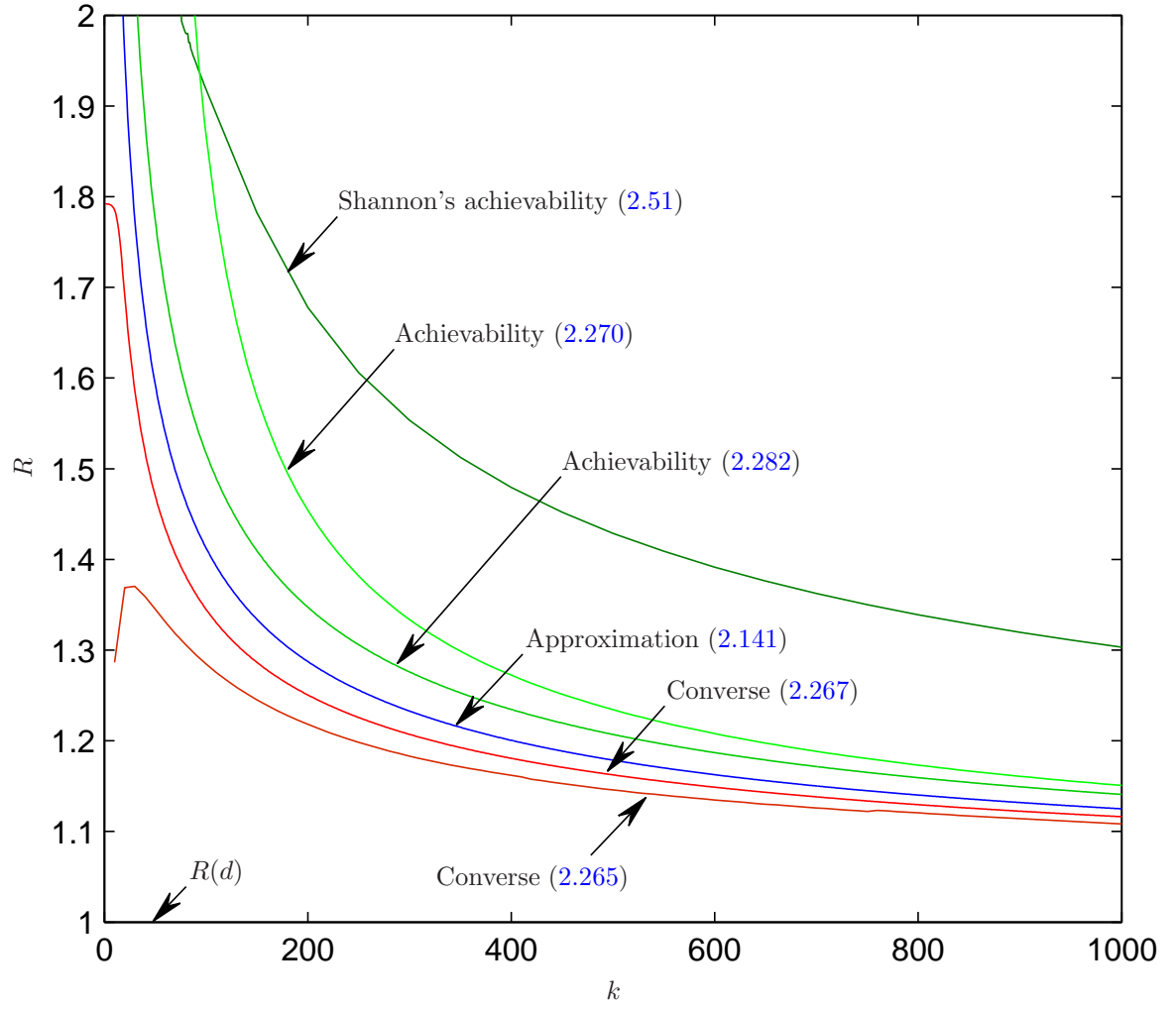


Figure 2.7: Rate-blocklength tradeoff for GMS with $\sigma = 1$, $d = \frac{1}{4}$, $\epsilon = 10^{-4}$.

2.10 Conclusion

To estimate the minimum rate required to sustain a given fidelity at a given blocklength, we have shown new achievability and converse bounds, which apply in full generality and which are tighter than existing bounds. The tightness of these bounds for stationary memoryless sources allowed us to obtain a compact closed-form expression that approximates the excess rate over the rate-distortion function incurred in the nonasymptotic regime (Theorem 2.22). For those sources and unless the blocklength is small, the rate-dispersion function (along with the rate-distortion function) serves to give tight approximations to the fundamental fidelity-rate tradeoff in the non-asymptotic regime.

The major results and insights are highlighted below.

- 1) A general new converse bound (Theorem 2.12) leverages the concept of \mathbf{d} -tilted information (Definition 2.1), a random variable which corresponds (in a sense that can be formalized, see Section 2.6 and [32]) to the number of bits required to represent a given source outcome within distortion d and whose role in lossy compression is on a par with that of information (in (2.2)) in lossless compression.
- 2) A tight achievability result (Theorem 2.121) in terms of generalized \mathbf{d} -tilted information underlines the importance of \mathbf{d} -tilted information in the nonasymptotic regime.
- 3) Since $\mathbb{E}[\mathbf{d}(S, Z)] = \int_0^\infty \mathbb{P}[\mathbf{d}(S, Z) > \xi] d\xi$, bounds for average distortion can be obtained by integrating our bounds on excess distortion. Note, however, that the code that minimizes $\mathbb{P}[\mathbf{d}(S, Z) > \xi]$ depends on ξ . Since the distortion cdf of any single code does not majorize the cdfs of all possible codes, the converse bound on the average distortion obtained through this approach, although asymptotically tight, may be loose at short blocklengths. Likewise, regarding achievability bounds (e.g. (2.104)), the optimization over channel and source random codes, P_X and P_Z , must be performed after the integration, so that the choice of code does not depend on the distortion threshold ξ .
- 4) For stationary memoryless sources, we have shown a concise closed-form approximation to the nonasymptotic fidelity-rate tradeoff in terms of the mean and the variance of the \mathbf{d} -tilted information (Theorem 2.22). As evidenced by our numerical results, that expression approximates well the excess rate over the rate-distortion function incurred in the nonasymptotic regime.

Chapter 3

Lossy joint source-channel coding

3.1 Introduction

In this chapter we study the nonasymptotic fundamental limits of lossy JSCC. After summarizing basic definitions and notation in Section 3.2, we proceed to show the new converse and achievability bounds to the maximum achievable coding rate in Sections 3.3 and 3.4, respectively. A Gaussian approximation analysis of the new bounds is presented in Section 3.5. The evaluation of the bounds and the approximation is performed for two important special cases:

- the transmission of a binary memoryless source (BMS) over a binary symmetric channel (BSC) with bit error rate distortion (Section 3.6);
- the transmission of a Gaussian memoryless source (GMS) with mean-square error distortion over an AWGN channel with a total power constraint (Section 3.7).

Section 3.8 identifies the dispersion on symbol-by-symbol transmission and performs a numerical comparison of symbol-by-symbol transmission to our block coding bounds. The material in this chapter was presented in part in [58–60].

Prior research relating to finite blocklength analysis of JSCC includes the work of Csiszár [61, 62] who demonstrated that the error exponent of joint source-channel coding outperforms that of separate source-channel coding. For discrete source-channel pairs with average distortion criterion, Pilc’s achievability bound [40, 63] applies. For the transmission of a Gaussian source over a discrete channel under the average mean square error constraint, Wyner’s achievability bound [42, 64] applies. Non-asymptotic achievability and converse bounds for a graph-theoretic model of JSCC have been

obtained by Csiszár [65]. Most recently, Tauste Campo et al. [66] showed a number of finite-blocklength random-coding bounds applicable to the almost-lossless JSCC setup, while Wang et al. [67] found the dispersion of JSCC for sources and channels with finite alphabets, independently and simultaneously with our work.

3.2 Definitions

A lossy source-channel code is a pair of (possibly randomized) mappings $f: \mathcal{M} \mapsto \mathcal{X}$ and $g: \mathcal{Y} \mapsto \widehat{\mathcal{M}}$. A distortion measure $d: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]$ is used to quantify the performance of the lossy code. A cost function $c: \mathcal{X} \mapsto [0, +\infty]$ may be imposed on the channel inputs. The channel is used without feedback.

Definition 3.1. *The pair (f, g) is a (d, ϵ, β) lossy source-channel code for $\{\mathcal{M}, \mathcal{X}, \mathcal{Y}, \widehat{\mathcal{M}}, P_S, d, P_{Y|X}, b\}$ if $\mathbb{P}[d(S, g(Y)) > d] \leq \epsilon$ and either $\mathbb{E}[b(X)] \leq \beta$ (average cost constraint) or $b(X) \leq \beta$ a.s. (maximal cost constraint), where $f(S) = X$. In the absence of an input cost constraint we simplify the terminology and refer to the code as (d, ϵ) lossy source-channel code.*

The special case $d = 0$ and $d(s, z) = 1\{s \neq z\}$ corresponds to almost-lossless compression. If, in addition, P_S is equiprobable on an alphabet of cardinality $|\mathcal{M}| = |\widehat{\mathcal{M}}| = M$, a $(0, \epsilon, \beta)$ code in Definition 3.1 corresponds to an (M, ϵ, β) channel code (i.e. a code with M codewords and average error probability ϵ and cost β). On the other hand, if $P_{Y|X}$ is an identity mapping on an alphabet of cardinality M without cost constraints, a (d, ϵ) code in Definition 3.1 corresponds to an (M, d, ϵ) lossy compression code (in Definition 2.2).

As our bounds in Sections 3.3 and 3.4 do not foist a Cartesian structure on the underlying alphabets, we state them in the one-shot paradigm of Definition 3.1. When we apply those bounds to the block coding setting, transmitted objects indeed become vectors, and the following definition comes into play.

Definition 3.2. *In the conventional fixed-to-fixed (or block) setting in which \mathcal{X} and \mathcal{Y} are the n -fold Cartesian products of alphabets \mathcal{A} and \mathcal{B} , while \mathcal{M} and $\widehat{\mathcal{M}}$ are the k -fold Cartesian products of alphabets \mathcal{S} and $\hat{\mathcal{S}}$, and $d_k: \mathcal{S}^k \times \hat{\mathcal{S}}^k \mapsto [0, +\infty]$, $c_n: \mathcal{A}^n \mapsto [0, +\infty]$, a (d, ϵ, β) code for $\{\mathcal{S}^k, \mathcal{A}^n, \mathcal{B}^n, \hat{\mathcal{S}}^k, P_{S^k}, d_k, P_{Y^n|X^n}, c_n\}$ is called a $(k, n, d, \epsilon, \beta)$ code (or a (k, n, d, ϵ) code if there is no cost constraint).*

Definition 3.3. *Fix ϵ, d, β and the channel blocklength n . The maximum achievable source block-*

length and coding rate (source symbols per channel use) are defined by, respectively

$$k^*(n, d, \epsilon, \beta) \triangleq \sup \{k: \exists(k, n, d, \epsilon, \beta) \text{ code}\} \quad (3.1)$$

$$R(n, d, \epsilon, \beta) \triangleq \frac{1}{n} k^*(n, d, \epsilon, \beta) \quad (3.2)$$

Alternatively, fix ϵ , β , source blocklength k and channel blocklength n . The minimum achievable excess distortion is defined by

$$D(k, n, \epsilon, \beta) \triangleq \inf \{d: \exists(k, n, d, \epsilon, \beta) \text{ code}\} \quad (3.3)$$

Denote, for a given $P_{Y|X}$ and a cost function $\mathbf{b}: \mathcal{X} \mapsto [0, +\infty]$,

$$\mathbb{C}(\beta) \triangleq \sup_{\substack{P_X: \\ \mathbb{E}[\mathbf{b}(X)] \leq \beta}} I(X; Y) \quad (3.4)$$

In addition to the basic restrictions (a)–(b) in Section 2.2 on the source and the distortion measure, we assume that

(c) The supremum in (3.4) is achieved by a unique P_{X^*} such that $\mathbb{E}[\mathbf{b}(X^*)] = \beta$.

The function (recall notation (2.22))

$$\imath_{Y|X\|\bar{Y}}(x; y) = \log \frac{dP_{Y|X=x}}{dP_{\bar{Y}}}(y) \quad (3.5)$$

defined with an arbitrary $P_{\bar{Y}}$, which need not be generated by any input distribution, will play a major role in our development. If $P_{X^*} \rightarrow P_{Y|X} \rightarrow P_{Y^*}$ we abbreviate the notation as

$$\imath_{X;Y}^*(x; y) \triangleq \imath_{Y|X\|Y^*}(x; y) \quad (3.6)$$

The dispersion, which serves to quantify the penalty on the rate of the best JSCC code induced by the finite blocklength, is defined as follows.

Definition 3.4. Fix β and $d \geq d_{\min}$. The rate-dispersion function of joint source-channel coding (source samples squared per channel use) is defined as

$$\mathcal{V}(d, \beta) \triangleq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{n \left(\frac{\mathbb{C}(\beta)}{R(d)} - R(n, d, \epsilon, \beta) \right)^2}{2 \log_e \frac{1}{\epsilon}} \quad (3.7)$$

where $C(\beta)$ and $R(d)$ are the channel capacity-cost and source rate-distortion functions, respectively.¹

The distortion-dispersion function of joint source-channel coding is defined as

$$\mathcal{W}(R, \beta) \triangleq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{n \left(D \left(\frac{C(\beta)}{R} \right) - D(nR, n, \epsilon, \beta) \right)^2}{2 \log_e \frac{1}{\epsilon}} \quad (3.8)$$

where $D(\cdot)$ is the distortion-rate function of the source.

If there is no cost constraint, we will simplify notation by dropping β from (3.1), (3.2), (3.3), (3.4), (3.7) and (3.8).

So as not to clutter notation, in Sections 3.3 and 3.4 we assume that there are no cost constraints. However, all results in those sections generalize to the case of a maximal cost constraint by considering X whose distribution is supported on the subset of allowable channel inputs:

$$\mathcal{F}(\beta) \triangleq \{x \in \mathcal{X} : \mathbf{b}(x) \leq \beta\} \quad (3.9)$$

rather than the entire channel input alphabet \mathcal{X} .

3.3 Converses

3.3.1 Converses via d-tilted information

Our first result is a general converse bound.

Theorem 3.1 (Converse). *The existence of a (d, ϵ) code for P_S , \mathbf{d} and $P_{Y|X}$ requires that*

$$\epsilon \geq \inf_{P_{X|S}} \sup_{\gamma > 0} \left\{ \sup_{P_{\bar{Y}}} \mathbb{P} [J_S(S, d) - \imath_{Y|X||\bar{Y}}(X; Y) \geq \gamma] - \exp(-\gamma) \right\} \quad (3.10)$$

$$\geq \sup_{\gamma > 0} \left\{ \sup_{P_{\bar{Y}}} \mathbb{E} \left[\inf_{x \in \mathcal{X}} \mathbb{P} [J_S(S, d) - \imath_{Y|X||\bar{Y}}(x; Y) \geq \gamma \mid S] \right] - \exp(-\gamma) \right\} \quad (3.11)$$

where in (3.10), $S - X - Y$, and the conditional probability in (3.11) is with respect to Y distributed according to $P_{Y|X=x}$ (independent of S).

Proof. Fix γ and the (d, ϵ) code $(P_{X|S}, P_{Z|Y})$. Fix an arbitrary probability measure $P_{\bar{Y}}$ on \mathcal{Y} . Let $P_{\bar{Y}} \rightarrow P_{Z|Y} \rightarrow P_{\bar{Z}}$. Recalling notation (2.33), we can write the probability in the right side of (3.10)

¹While for memoryless sources and channels, $R(d) = \mathbb{R}_S(d)$ and $C(\beta) = \mathbb{C}(\beta)$ given by (2.3) and (3.4) evaluated with single-letter distributions, it is important to distinguish between the operational definitions and the extremal mutual information quantities, since the core results in this thesis allow for memory.

as

$$\begin{aligned} & \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(X; Y) \geq \gamma] \\ &= \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(X; Y) \geq \gamma, \mathbf{d}(S; Z) > d] + \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(X; Y) \geq \gamma, \mathbf{d}(S; Z) \leq d] \end{aligned} \quad (3.12)$$

$$\begin{aligned} & \leq \epsilon + \sum_{s \in \mathcal{M}} P_S(s) \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \sum_{y \in \mathcal{Y}} \sum_{z \in B_d(s)} P_{Z|Y}(z|y) P_{Y|X}(y|x) \\ & \quad \cdot 1 \{P_{Y|X}(y|x) \leq P_{\bar{Y}}(y) \exp(j_S(s, d) - \gamma)\} \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \leq \epsilon + \exp(-\gamma) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d)) \sum_{y \in \mathcal{Y}} P_{\bar{Y}}(y) \sum_{z \in B_d(s)} P_{Z|Y}(z|y) \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \\ &= \epsilon + \exp(-\gamma) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d)) \sum_{y \in \mathcal{Y}} P_{\bar{Y}}(y) \sum_{z \in B_d(s)} P_{Z|Y}(z|y) \end{aligned} \quad (3.14)$$

$$= \epsilon + \exp(-\gamma) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d)) P_{\bar{Z}}(B_d(s)) \quad (3.15)$$

$$\leq \epsilon + \exp(-\gamma) \sum_{z \in \widehat{\mathcal{M}}} P_{\bar{Z}}(z) \sum_{s \in \mathcal{M}} P_S(s) \exp(j_S(s, d) + \lambda^* d - \lambda^* \mathbf{d}(s, z)) \quad (3.16)$$

$$\leq \epsilon + \exp(-\gamma) \quad (3.17)$$

where (3.17) is due to (2.12). Optimizing over $\gamma > 0$ and $P_{\bar{Y}}$, we get the best possible bound for a given encoder $P_{X|S}$. To obtain a code-independent converse, we simply choose $P_{X|S}$ that gives the weakest bound, and (3.10) follows. To show (3.11), we weaken (3.10) as

$$\epsilon \geq \sup_{\gamma > 0} \left\{ \sup_{P_{\bar{Y}}} \inf_{P_{X|S}} \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(X; Y) \geq \gamma] - \exp(-\gamma) \right\} \quad (3.18)$$

and observe that for any $P_{\bar{Y}}$,

$$\begin{aligned} & \inf_{P_{X|S}} \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(X; Y) \geq \gamma] \\ &= \sum_{s \in \mathcal{M}} P_S(s) \inf_{P_{X|S=s}} \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) 1 \{j_S(s, d) - \imath_{Y|X\|\bar{Y}}(x; y) \geq \gamma\} \end{aligned} \quad (3.19)$$

$$= \sum_{s \in \mathcal{M}} P_S(s) \inf_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) 1 \{j_S(s, d) - \imath_{Y|X\|\bar{Y}}(x; y) \geq \gamma\} \quad (3.20)$$

$$= \mathbb{E} \left[\inf_{x \in \mathcal{X}} \mathbb{P} [j_S(S, d) - \imath_{Y|X\|\bar{Y}}(x; Y) \geq \gamma \mid S] \right] \quad (3.21)$$

□

An immediate corollary to Theorem 3.1 is the following result.

Theorem 3.2 (Converse). *Assume that there exists a distribution $P_{\bar{Y}}$ such that the distribution of $\iota_{Y|X\|\bar{Y}}(x; Y)$ (according to $P_{Y|X=x}$) does not depend on the choice of $x \in \mathcal{X}$. If a (d, ϵ) code for P_S , \mathbf{d} and $P_{Y|X}$ exists, then*

$$\epsilon \geq \sup_{\gamma > 0} \left\{ \mathbb{P} [j_S(S, d) - \iota_{Y|X\|\bar{Y}}(x; Y) \geq \gamma] - \exp(-\gamma) \right\} \quad (3.22)$$

for an arbitrary $x \in \mathcal{X}$. The probability measure \mathbb{P} in (3.22) is generated by $P_S P_{Y|X=x}$.

Proof. Under the assumption, the conditional probability in the right side of (3.11) is the same regardless of the choice of $x \in \mathcal{X}$. \square

Remark 3.1. Our converse for lossy source coding in Theorem 2.12 can be viewed as a particular case of the result in Theorem 3.2. Indeed, if $\mathcal{X} = \mathcal{Y} = \{1, \dots, M\}$ and $P_{Y|X}(m|m) = 1$, $P_Y(1) = \dots = P_Y(M) = \frac{1}{M}$, then (3.22) becomes

$$\epsilon \geq \sup_{\gamma > 0} \{ \mathbb{P} [j_S(S, d) \geq \log M + \gamma] - \exp(-\gamma) \} \quad (3.23)$$

which is precisely (2.81).

Remark 3.2. Theorems 3.1 and 3.2 still hold in the case $d = 0$ and $d(x, y) = 1 \{x \neq y\}$, which corresponds to almost-lossless data compression. Indeed, recalling (2.18), it is easy to see that the proof of Theorem 3.1 applies, skipping the now unnecessary step (3.16), and, therefore, (3.10) reduces to

$$\epsilon \geq \inf_{P_{X|S}} \sup_{\gamma > 0} \left\{ \sup_{P_{\bar{Y}}} \mathbb{P} [\iota_S(S) - \iota_{Y|X\|\bar{Y}}(X; Y) \geq \gamma] - \exp(-\gamma) \right\} \quad (3.24)$$

The next result follows from Theorem 3.1. When we apply Corollary 3.3 in Section 3.5 to find the dispersion of JSCC, we will let T be the number of channel input types, and we will let Y_t be generated by the type of the channel input block. If $T = 1$, the bound in (3.25) reduces to (3.11).

Corollary 3.3 (Converse). *The existence of a (d, ϵ) code for P_S , \mathbf{d} and $P_{Y|X}$ requires that*

$$\epsilon \geq \max_{\gamma > 0, T} \left\{ \sup_{\bar{Y}_1, \dots, \bar{Y}_T} \mathbb{E} \left[\sup_t \inf_{x \in \mathcal{X}} \mathbb{P} [j_S(S, d) - \iota_{Y|X\|\bar{Y}_t}(x; Y) \geq \gamma \mid S] \right] - T \exp(-\gamma) \right\} \quad (3.25)$$

where T is a positive integer, and $\bar{Y}_1, \dots, \bar{Y}_T$ are defined on \mathcal{Y} .

Proof. We weaken (3.11) by letting P_Y to be the following convex combination of distributions:

$$P_Y(y) = \frac{1}{T} \sum_{t=1}^T P_{Y_t}(y) \quad (3.26)$$

For an arbitrary t , applying

$$P_Y(y) \geq \frac{1}{T} P_{Y_t}(y) \quad (3.27)$$

to lower bound the probability in (3.11), we obtain (3.25). \square

Remark 3.3. As we will see in Chapter 5, a more careful choice of the weights in the convex combination (3.26) leads to a tighter bound.

3.3.2 Converses via hypothesis testing and list decoding

To show a joint source-channel converse in [62], Csiszár used a list decoder, which outputs a list of L elements drawn from \mathcal{M} . While traditionally list decoding has only been considered in the context of finite alphabet sources, we generalize the setting to sources with abstract alphabets. In our setup, the encoder is the random transformation $P_{X|S}$, and the decoder is defined as follows.

Definition 3.5 (List decoder). *Let L be a positive real number, and let Q_S be a measure on \mathcal{M} . An (L, Q_S) list decoder is a random transformation $P_{\tilde{S}|Y}$, where \tilde{S} takes values on Q_S -measurable sets with Q_S -measure not exceeding L :*

$$Q_S(\tilde{S}) \leq L \quad (3.28)$$

Even though we keep the standard “list” terminology, the decoder output need not be a finite or countably infinite set. The error probability with this type of list decoding is the probability that the source outcome S does not belong to the decoder output list for Y :

$$1 - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \sum_{\tilde{s} \in \mathcal{M}^{(L)}} \sum_{s \in \tilde{s}} P_{\tilde{S}|Y}(\tilde{s}|y) P_{Y|X}(y|x) P_{X|S}(x|s) P_S(s) \quad (3.29)$$

where $\mathcal{M}^{(L)}$ is the set of all Q_S -measurable subsets of \mathcal{M} with Q_S -measure not exceeding L .

Definition 3.6 (List code). *An (ϵ, L, Q_S) list code is a pair of random transformations $(P_{X|S}, P_{\tilde{S}|Y})$ such that (3.28) holds and the list error probability (3.29) does not exceed ϵ .*

Of course, letting $Q_S = U_S$, where U_S is the counting measure on \mathcal{M} , we recover the conventional list decoder definition where the smallest scalar that satisfies (3.28) is an integer. The almost-lossless

JSCC setting ($d = 0$) in Definition 3.1 corresponds to $L = 1$, $Q_S = U_S$. If the source is analog (has a continuous distribution), it is reasonable to let Q_S be the Lebesgue measure.

Any converse for list decoding implies a converse for conventional decoding. To see why, observe that any (d, ϵ) lossy code can be converted to a list code with list error probability not exceeding ϵ by feeding the lossy decoder output to a function that outputs the set of all source outcomes within distortion d from the output $z \in \widehat{\mathcal{M}}$ of the original lossy decoder. In this sense, the set of all (d, ϵ) lossy codes is included in the set of all list codes with list error probability $\leq \epsilon$ and list size

$$L = \max_{z \in \widehat{\mathcal{M}}} Q_S(\{s: d(s, z) \leq d\}) \quad (3.30)$$

Recalling notation (2.89), we generalize the hypothesis testing converse for channel coding [3, Theorem 27] to joint source-channel coding with list decoding as follows.

Theorem 3.4 (Converse). *Fix P_S and $P_{Y|X}$, and let Q_S be a σ -finite measure. The existence of an (ϵ, L, Q_S) list code requires that*

$$\inf_{P_{X|S}} \sup_{P_{\tilde{Y}}} \beta_{1-\epsilon}(P_S P_{X|S} P_{Y|X}, Q_S P_{X|S} P_{\tilde{Y}}) \leq L \quad (3.31)$$

where the supremum is over all probability measures $P_{\tilde{Y}}$ defined on the channel output alphabet \mathcal{Y} .

Proof. Fix Q_S , the encoder $P_{X|S}$, and an auxiliary σ -finite conditional measure $Q_{Y|XS}$. Consider the (not necessarily optimal) test for deciding between $P_{SXY} = P_S P_{X|S} P_{Y|X}$ and $Q_{SXY} = Q_S P_{X|S} Q_{Y|XS}$ which chooses P_{SXY} if S belongs to the decoder output list. Note that this is a hypothetical test, which has access to both the source outcome and the decoder output.

According to \mathbb{P} , the probability measure generated by P_{SXY} , the probability that the test chooses P_{SXY} is given by

$$\mathbb{P}[S \in \tilde{S}] \geq 1 - \epsilon \quad (3.32)$$

Since $\mathbb{Q}[S \in \tilde{S}]$ is the measure of the event that the test chooses P_{SXY} when Q_{SXY} is true, and the optimal test cannot perform worse than the possibly suboptimal one that we selected, it follows that

$$\beta_{1-\epsilon}(P_S P_{X|S} P_{Y|X}, Q_S P_{X|S} Q_{Y|XS}) \leq \mathbb{Q}[S \in \tilde{S}] \quad (3.33)$$

Now, fix an arbitrary probability measure $P_{\tilde{Y}}$ on \mathcal{Y} . Choosing $Q_{Y|XS} = P_{\tilde{Y}}$, the inequality in (3.33)

can be weakened as follows.

$$\mathbb{Q}[S \in \tilde{S}] = \sum_{y \in \mathcal{Y}} P_{\tilde{Y}}(y) \sum_{\tilde{s} \in \mathcal{M}^{(L)}} P_{\tilde{S}|Y}(\tilde{s}|y) \sum_{s \in \tilde{s}} Q_S(s) \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \quad (3.34)$$

$$= \sum_{y \in \mathcal{Y}} P_{\tilde{Y}}(y) \sum_{\tilde{s} \in \mathcal{M}^{(L)}} P_{\tilde{S}|Y}(\tilde{s}|y) \sum_{s \in \tilde{s}} Q_S(s) \quad (3.35)$$

$$\leq \sum_{y \in \mathcal{Y}} P_{\tilde{Y}}(y) \sum_{\tilde{s} \in \mathcal{M}^{(L)}} P_{\tilde{S}|Y}(\tilde{s}|y) L \quad (3.36)$$

$$= L \quad (3.37)$$

Optimizing the bound over $P_{\tilde{Y}}$ and choosing $P_{X|S}$ that yields the weakest bound in order to obtain a code-independent converse, (3.31) follows. \square

Remark 3.4. Similar to how Wolfowitz's converse for channel coding can be obtained from the meta-converse for channel coding [3], the converse for almost-lossless joint source-channel coding in (3.24) can be obtained by appropriately weakening (3.31) with $L = 1$. Indeed, invoking [3]

$$\beta_\alpha(P, Q) \geq \frac{1}{\gamma} \left(\alpha - \mathbb{P} \left[\frac{dP}{dQ} > \gamma \right] \right) \quad (3.38)$$

and letting $Q_S = U_S$ in (3.31), where U_S is the counting measure on \mathcal{M} , we have

$$1 \geq \inf_{P_{X|S}} \sup_{P_{\tilde{Y}}} \beta_{1-\epsilon}(P_S P_{X|S} P_{Y|X}, U_S P_{X|S} P_{\tilde{Y}}) \quad (3.39)$$

$$\geq \inf_{P_{X|S}} \sup_{P_{\tilde{Y}}} \sup_{\gamma > 0} \frac{1}{\gamma} (1 - \epsilon - \mathbb{P}[\iota_{Y|X\|\tilde{Y}}(X; Y) - \iota_S(S) > \log \gamma]) \quad (3.40)$$

which upon rearranging yields (3.24).

In general, computing the infimum in (3.31) is challenging. However, if the channel is symmetric (in a sense formalized in the next result), $\beta_{1-\epsilon}(P_S P_{X|S} P_{Y|X}, U_S P_{X|S} P_{\tilde{Y}})$ is independent of $P_{X|S}$.

Theorem 3.5 (Converse). *Fix a probability measure $P_{\tilde{Y}}$. Assume that the distribution of $\iota_{Y|X\|\tilde{Y}}(x; Y)$ does not depend on $x \in \mathcal{X}$ under either $P_{Y|X=x}$ or $P_{\tilde{Y}}$. Then, the existence of an (ϵ, L, Q_S) list code requires that*

$$\beta_{1-\epsilon}(P_S P_{Y|X=x}, Q_S P_{\tilde{Y}}) \leq L \quad (3.41)$$

where $x \in \mathcal{X}$ is arbitrary.

Proof. The Neyman-Pearson lemma (e.g. [68]) implies that the outcome of the optimum binary hypothesis test between P and Q only depends on the observation through $\frac{dP}{dQ}$. In particular, the

optimum binary hypothesis test W^* for deciding between $P_S P_{X|S} P_{Y|X}$ and $Q_S P_{X|S} P_{\bar{Y}}$ satisfies

$$W^* - (S, \iota_{Y|X\|\bar{Y}}(X; Y)) - (S, X, Y) \quad (3.42)$$

For all $s \in \mathcal{M}$, $x \in \mathcal{X}$, we have

$$\mathbb{P}[W^* = 1 | S = s, X = x] = \mathbb{E}[\mathbb{P}[W^* = 1 | X = x, S = s, Y]] \quad (3.43)$$

$$= \mathbb{E}[\mathbb{P}[W^* = 1 | S = s, \iota_{Y|X\|\bar{Y}}(X; Y) = \iota_{Y|X\|\bar{Y}}(x; Y)]] \quad (3.44)$$

$$= \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) P_{W^*|S, \iota_{Y|X\|\bar{Y}}(X; Y)}(1 | s, \iota_{Y|X\|\bar{Y}}(x; y)) \quad (3.45)$$

$$= \mathbb{P}[W^* = 1 | S = s] \quad (3.46)$$

and

$$\mathbb{Q}[W^* = 1 | S = s, X = x] = \mathbb{Q}[W^* = 1 | S = s] \quad (3.47)$$

where

- (3.44) is due to (3.42),
- (3.45) uses the Markov property $S - X - Y$,
- (3.46) follows from the symmetry assumption on the distribution of $\iota_{Y|X\|\bar{Y}}(x, Y)$,
- (3.47) is obtained similarly to (3.45).

Since (3.46), (3.47) imply that the optimal test achieves the same performance (that is, the same $\mathbb{P}[W^* = 1]$ and $\mathbb{Q}[W^* = 1]$) regardless of $P_{X|S}$, we choose $P_{X|S} = 1_X(x)$ for some $x \in \mathcal{X}$ in the left side of (3.31) to obtain (3.41). \square

Remark 3.5. In the case of finite channel input and output alphabets, the channel symmetry assumption of Theorem 3.5 holds, in particular, if the rows of the channel transition probability matrix are permutations of each other, and $P_{\bar{Y}^n}$ is the equiprobable distribution on the (n -dimensional) channel output alphabet, which, coincidentally, is also the capacity-achieving output distribution. For Gaussian channels with equal power constraint, which corresponds to requiring the channel inputs to lie on the power sphere, any spherically-symmetric $P_{\bar{Y}^n}$ satisfies the assumption of Theorem 3.5.

3.4 Achievability

Given a source code $(f_s^{(M)}, g_s^{(M)})$ of size M , and a channel code $(f_c^{(M)}, g_c^{(M)})$ of size M , we may concatenate them to obtain the following sub-class of the source-channel codes introduced in Definition 3.1:

Definition 3.7. An (M, d, ϵ) source-channel code is a (d, ϵ) source-channel code such that the encoder and decoder mappings satisfy

$$f = f_c^{(M)} \circ f_s^{(M)} \quad (3.48)$$

$$g = g_c^{(M)} \circ g_s^{(M)} \quad (3.49)$$

where

$$f_s^{(M)}: \mathcal{M} \mapsto \{1, \dots, M\} \quad (3.50)$$

$$f_c^{(M)}: \{1, \dots, M\} \mapsto \mathcal{X} \quad (3.51)$$

$$g_c^{(M)}: \mathcal{Y} \mapsto \{1, \dots, M\} \quad (3.52)$$

$$g_s^{(M)}: \{1, \dots, M\} \mapsto \widehat{\mathcal{M}} \quad (3.53)$$

(see Fig. 3.1).

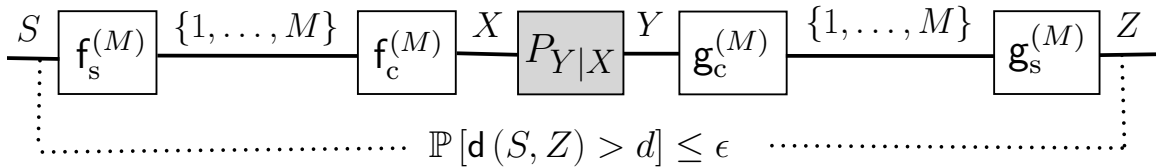


Figure 3.1: An (M, d, ϵ) joint source-channel code.

Note that an (M, d, ϵ) code is an $(M + 1, d, \epsilon)$ code.

The conventional separate source-channel coding paradigm corresponds to the special case of Definition 3.7 in which the source code $(f_s^{(M)}, g_s^{(M)})$ is chosen without knowledge of $P_{Y|X}$ and the channel code $(f_c^{(M)}, g_c^{(M)})$ is chosen without knowledge of P_S and the distortion measure d . A pair of source and channel codes is separation-optimal if the source code is chosen so as to minimize the distortion (average or excess) when there is no channel, whereas the channel code is chosen so as to

minimize the worst-case (over source distributions) average error probability:

$$\max_{P_U} \mathbb{P} \left[U \neq \mathbf{g}_c^{(M)}(Y) \right] \quad (3.54)$$

where $X = \mathbf{f}_c^{(M)}(U)$ and U takes values on $\{1, \dots, M\}$. If both the source and the channel code are chosen separation-optimally for their given sizes, the separation principle guarantees that under certain quite general conditions (which encompass the memoryless setting, see [69]) the asymptotic fundamental limit of joint source-channel coding is achievable. In the finite blocklength regime, however, such SSCC construction is, in general, only suboptimal. Within the SSCC paradigm, we can obtain an achievability result by further optimizing with respect to the choice of M :

Theorem 3.6 (Achievability, SSCC). *Fix $P_{Y|X}$, \mathbf{d} and P_S . Denote by $\epsilon^*(M)$ the minimum achievable worst-case average error probability among all transmission codes of size M , and the minimum achievable probability of exceeding distortion d with a source code of size M by $\epsilon^*(M, d)$.*

Then, there exists a (d, ϵ) source-channel code with

$$\epsilon \leq \min_M \{ \epsilon^*(M) + \epsilon^*(M, d) \} \quad (3.55)$$

Bounds on $\epsilon^*(M)$ have been obtained recently in [3],² while those on $\epsilon^*(M, d)$ are covered in Chapter 2. Definition 3.7 does not rule out choosing the source code based on the knowledge of $P_{Y|X}$ or the channel code based on the knowledge of P_S , \mathbf{d} and d . One of the interesting conclusions in the present chapter is that the optimal dispersion of JSCC is achievable within the class of (M, d, ϵ) source-channel codes introduced in Definition 3.7. However, the dispersion achieved by the conventional SSCC approach is in fact suboptimal.

To shed light on the reason behind the suboptimality of SSCC at finite blocklength despite its asymptotic optimality, we recall the reason SSCC achieves the asymptotic fundamental limit. The output of the optimum source encoder is, for large k , approximately equiprobable over a set of roughly $\exp(kR(d))$ distinct messages, which allows the encoder to represent most of the source outcomes within distortion d . From the channel coding theorem we know that there exists a channel code that is capable of distinguishing, with high probability, $M = \exp(kR(d)) < \exp(nC)$ messages when equipped with the maximum likelihood decoder. Therefore, a simple concatenation of the

²As the maximal (over source outputs) error probability cannot be lower than the worst-case error probability, the maximal error probability achievability bounds of [3] apply to upper-bound $\epsilon^*(M)$. Moreover, most channel random coding bounds on average error probability, in particular, the random coding union (RCU) bound of [3], although stated assuming equiprobable source, are oblivious to the distribution of the source and thus upper-bound the worst-case average error probability $\epsilon^*(M)$ as well.

source code and the channel code achieves vanishing probability of distortion exceeding d , for any $d > D\left(\frac{nC}{k}\right)$. However, at finite n , the output of the optimum source encoder need not be nearly equiprobable, so there is no reason to expect that a separated scheme employing a maximum-likelihood channel decoder, which does not exploit unequal message probabilities, would achieve near-optimal non-asymptotic performance. Indeed, in the non-asymptotic regime the gain afforded by taking into account the residual encoded source redundancy at the channel decoder is appreciable. The following achievability result, obtained using independent random source codes and random channel codes within the paradigm of Definition 3.7, capitalizes on this intuition.

Theorem 3.7 (Achievability). *There exists a (d, ϵ) source-channel code with*

$$\epsilon \leq \inf_{P_X, P_Z, P_{W|S}} \left\{ \mathbb{E} \left[\exp \left(-|I_{X;Y}(X;Y) - \log W|^+ \right) \right] + \mathbb{E} \left[(1 - P_Z(B_d(S)))^W \right] \right\} \quad (3.56)$$

where the expectations are with respect to $P_S P_X P_{Y|X} P_Z P_{W|S}$ defined on $\mathcal{M} \times \mathcal{X} \times \mathcal{Y} \times \widehat{\mathcal{M}} \times \mathbb{N}$, where \mathbb{N} is the set of natural numbers.

Proof. Fix a positive integer M . Fix a positive integer-valued random variable W that depends on other random variables only through S and that satisfies $W \leq M$. We will construct a code with separate encoders for source and channel and separate decoders for source and channel as in Definition 3.7. We will perform a random coding analysis by choosing random independent source and channel codes which will lead to the conclusion that there exists an (M, d, ϵ) code with error probability ϵ guaranteed in (3.56) with $W \leq M$. Observing that increasing M can only tighten the bound in (3.56) in which W is restricted to not exceed M , we will let $M \rightarrow \infty$ and conclude, by invoking the bounded convergence theorem, that the support of W in (3.56) need not be bounded.

Source Encoder. Given an ordered list of representation points $z^M = (z_1, \dots, z_M) \in \widehat{\mathcal{M}}^M$, and having observed the source outcome s , the (probabilistic) source encoder generates W from $P_{W|S=s}$ and selects the lowest index $m \in \{1, \dots, W\}$ such that s is within distance d of z_m . If no such index can be found, the source encoder outputs a pre-selected arbitrary index, e.g. M . Therefore,

$$f_s^{(M)}(s) = \begin{cases} \min\{m, W\} & d(s, z_m) \leq d < \min_{i=1, \dots, m-1} d(s, z_i) \\ M & d < \min_{i=1, \dots, W} d(s, z_i) \end{cases} \quad (3.57)$$

In a good (M, d, ϵ) JSCC code, M would be chosen so large that with overwhelming probability, a source outcome would be encoded successfully within distortion d . It might seem counterproductive to let the source encoder in (3.57) give up before reaching the end of the list of representation points,

but in fact, such behavior helps the channel decoder by skewing the distribution of $\mathbf{f}_s^{(M)}(S)$.

Channel Encoder. Given a codebook $(x_1, \dots, x_M) \in \mathcal{X}^M$, the channel encoder outputs x_m if m is the output of the source encoder:

$$\mathbf{f}_c^{(M)}(m) = x_m \quad (3.58)$$

Channel Decoder. Define the random variable $U \in \{1, \dots, M+1\}$ which is a function of S , W and z^M only:

$$U = \begin{cases} \mathbf{f}_s^{(M)}(S) & d(S, \mathbf{g}_s(\mathbf{f}_s(S))) \leq d \\ M+1 & \text{otherwise} \end{cases} \quad (3.59)$$

Having observed $y \in \mathcal{Y}$, the channel decoder chooses arbitrarily among the members of the set³:

$$\mathbf{g}_c^{(M)}(y) = m \in \arg \max_{j \in \{1, \dots, M\}} P_{U|Z^M}(j|z^M) P_{Y|X}(y|x_j) \quad (3.60)$$

A MAP decoder would multiply $P_{Y|X}(y|x_j)$ by $P_X(x_j)$. While that decoder would be too hard to analyze, the product in (3.60) is a good approximation because $P_{U|Z^M}(j|z^M)$ and $P_X(x_j)$ are related by

$$P_X(x_j) = \sum_{m: x_m = x_j} P_{U|Z^M}(m|z^M) + P_{U|Z^M}(M+1|z^M) \mathbf{1}\{j = M\} \quad (3.61)$$

so the decoder in (3.60) differs from a MAP decoder only when either several x_m are identical, or there is no representation point among the first W points within distortion d of the source, both unusual events.

Source Decoder. The source decoder outputs z_m if m is the output of the channel decoder:

$$\mathbf{g}_s^{(M)}(m) = z_m \quad (3.62)$$

Error Probability Analysis. We now proceed to analyze the performance of the code described above. If there were no source encoding error, a channel decoding error can occur if and only if

$$\exists j \neq m: P_{U|Z^M}(j|z^M) P_{Y|X}(Y|x_j) \geq P_{U|Z^M}(m|z^M) P_{Y|X}(Y|x_m) \quad (3.63)$$

Let the channel codebook (X_1, \dots, X_M) be drawn i.i.d. from P_X , and independent of the source

³The elegant decoder in (3.60), which leads to the simplification of our achievability bound in [59] with the tighter version in Theorem 3.7, was suggested by Dr. Oliver Kosut.

codebook (Z_1, \dots, Z_M) , which is drawn i.i.d. from P_Z . Denote by $\epsilon(x^M, z^M)$ the excess-distortion probability attained with the source codebook z^M and the channel codebook x^M . Conditioned on the event $\{d(S, \mathbf{g}_s(\mathbf{f}_s(S)) \leq d\} = \{U \leq W\} = \{U \neq M+1\}$ (no failure at the source encoder), the probability of excess distortion is upper bounded by the probability that the channel decoder does not choose $\mathbf{f}_s^{(M)}(S)$, so

$$\begin{aligned} \epsilon(x^M, z^M) &\leq \sum_{m=1}^M P_{U|Z^M}(m|z^M) \mathbb{P} \left[\bigcup_{j \neq m} \left\{ \frac{P_{U|Z^M}(j|z^M) P_{Y|X}(Y|x_j)}{P_{U|Z^M}(m|z^M) P_{Y|X}(Y|x_m)} \geq 1 \right\} \mid X = x_m \right] \\ &\quad + P_{U|Z^M}(U > W|z^M) \end{aligned} \quad (3.64)$$

We now average (3.64) over the source and channel codebooks. Averaging the m -th term of the sum in (3.64) with respect to the channel codebook yields

$$P_{U|Z^M}(m|z^M) \mathbb{P} \left[\bigcup_{j \neq m} \left\{ \frac{P_{U|Z^M}(j|z^M) P_{Y|X}(Y|X_j)}{P_{U|Z^M}(m|z^M) P_{Y|X}(Y|X_m)} \geq 1 \right\} \right] \quad (3.65)$$

where Y, X_1, \dots, X_M are distributed according to

$$P_{Y X_1 \dots X_M}(y, x_1, \dots, x_M) = P_{Y|X_m}(y|x_m) \prod_{j \neq m} P_X(x_j) \quad (3.66)$$

Letting \bar{X} be an independent copy of X and applying the union bound to the probability in (3.65), we have that for any given (m, z^M) ,

$$\begin{aligned} &\mathbb{P} \left[\bigcup_{j \neq m} \left\{ \frac{P_{U|Z^M}(j|z^M) P_{Y|X}(Y|X_j)}{P_{U|Z^M}(m|z^M) P_{Y|X}(Y|X_m)} \geq 1 \right\} \right] \\ &\leq \mathbb{E} \left[\min \left\{ 1, \sum_{j=1}^M \mathbb{P} \left[\frac{P_{U|Z^M}(j|z^M) P_{Y|X}(Y|\bar{X})}{P_{U|Z^M}(m|z^M) P_{Y|X}(Y|X)} \geq 1 \mid X, Y \right] \right\} \right] \end{aligned} \quad (3.67)$$

$$\leq \mathbb{E} \left[\min \left\{ 1, \sum_{j=1}^M \frac{P_{U|Z^M}(j|z^M)}{P_{U|Z^M}(m|z^M)} \frac{\mathbb{E}[P_{Y|X}(Y|\bar{X})|Y]}{P_{Y|X}(Y|X)} \right\} \right] \quad (3.68)$$

$$= \mathbb{E} \left[\min \left\{ 1, \sum_{j=1}^M \frac{P_{U|Z^M}(j|z^M)}{P_{U|Z^M}(m|z^M)} \frac{P_Y(Y)}{P_{Y|X}(Y|X)} \right\} \right] \quad (3.69)$$

$$= \mathbb{E} \left[\min \left\{ 1, \frac{\mathbb{P}[U \leq W \mid Z^M = z^M]}{P_{U|Z^M}(m|z^M)} \frac{P_Y(Y)}{P_{Y|X}(Y|X)} \right\} \right] \quad (3.70)$$

$$= \mathbb{E} \left[\min \left\{ 1, \frac{1}{P_{U|Z^M, 1\{U \leq W\}}(m|z^M, 1)} \frac{P_Y(Y)}{P_{Y|X}(Y|X)} \right\} \right] \quad (3.71)$$

where (3.68) is due to $1\{a \geq 1\} \leq a$.

Applying (3.71) to (3.64) and averaging with respect to the source codebook, we may write

$$\mathbb{E} [\epsilon(X^M, Z^M)] \leq \mathbb{E} [\min \{1, G\}] + \mathbb{P} [U > W] \quad (3.72)$$

where for brevity we denoted the random variable

$$G = \frac{1}{P_{U|Z^M, 1\{U \leq W\}}(U|Z^M, 1)} \frac{P_Y(Y)}{P_{Y|X}(Y|X)} \quad (3.73)$$

The expectation in the right side of (3.72) is with respect to $P_{Z^M} P_{U|Z^M} P_{W|U, Z^M} P_X P_{Y|X}$. It is equal to

$$\begin{aligned} \mathbb{E} [\min \{1, G\}] &= \mathbb{E} [\mathbb{E} [\min \{1, G\} \mid X, Y, Z^M, 1\{U \leq W\}]] \\ &\leq \mathbb{E} [\min \{1, \mathbb{E} [G \mid X, Y, Z^M, 1\{U \leq W\}]]] \end{aligned} \quad (3.74)$$

$$= \mathbb{E} \left[\min \left\{ 1, W \frac{P_Y(Y)}{P_{Y|X}(Y|X)} \right\} \right] \quad (3.75)$$

$$= \mathbb{E} \left[\exp \left(-|z_{X;Y}(X; Y) - \log W|^+ \right) \right] \quad (3.76)$$

where

- (3.74) applies Jensen's inequality to the concave function $\min\{1, a\}$;
- (3.75) uses $P_{U|X, Y, Z^M, 1\{U \leq W\}} = P_{U|Z^M, 1\{U \leq W\}}$;
- (3.76) is due to $\min\{1, a\} = \exp \left(-|\log \frac{1}{a}|^+ \right)$, where a is nonnegative.

To evaluate the probability in the right side of (3.72), note that conditioned on $S = s$, $W = w$, U is distributed as:

$$P_{U|S, W}(m|s, w) = \begin{cases} \rho(s)(1 - \rho(s))^{m-1} & m = 1, 2, \dots, w \\ (1 - \rho(s))^w & m = M + 1 \end{cases} \quad (3.77)$$

where we denoted for brevity

$$\rho(s) = P_Z(B_d(s)) \quad (3.78)$$

Therefore,

$$\mathbb{P} [U > W] = \mathbb{E} [\mathbb{P} [U > W | S, W]] \quad (3.79)$$

$$= \mathbb{E} \left[(1 - \rho(S))^W \right] \quad (3.80)$$

Applying (3.76) and (3.80) to (3.72) and invoking Shannon's random coding argument, (3.56) follows. \square

Remark 3.6. As we saw in the proof of Theorem 3.7, if we restrict W to take values on $\{1, \dots, M\}$, then the bound on the error probability ϵ in (3.56) is achieved in the class of (M, d, ϵ) codes. The code size M that leads to tight achievability bounds following from Theorem 3.7 is in general much larger than the size that achieves the minimum in (3.55). In that case, M is chosen so that $\log M$ lies between $kR(d)$ and nC so as to minimize the sum of source and channel decoding error probabilities without the benefit of a channel decoder that exploits residual source redundancy. In contrast, Theorem 3.8 is obtained with an approximate MAP decoder that allows a larger choice for $\log M$, even beyond nC . Still we can achieve a good (d, ϵ) tradeoff because the channel code employs unequal error protection: those codewords with higher probabilities are more reliably decoded.

Remark 3.7. Had we used the ML channel decoder in lieu of (3.60) in the proof of Theorem 3.7, we would conclude that a (d, ϵ) code exists with

$$\epsilon \leq \inf_{P_X, P_Z, M} \left\{ \mathbb{E} \left[\exp \left(-|\iota_{X;Y}(X;Y) - \log(M-1)|^+ \right) \right] + \mathbb{E} \left[(1 - P_Z(B_d(S)))^M \right] \right\} \quad (3.81)$$

which corresponds to the SSCC bound in (3.55) with the worst-case average channel error probability $\epsilon^*(M)$ upper bounded using a relaxation of the random coding union (RCU) bound of [3] using Markov's inequality and the source error probability $\epsilon^*(M, d)$ upper bounded using the random coding achievability bound in Theorem 2.16.

Remark 3.8. Weakening (3.56) by letting $W = M$, we obtain a slightly looser version of (3.81) in which $M-1$ in the exponent is replaced by M . To get a generally tighter bound than that afforded by SSCC, a more intelligent choice of W is needed, as detailed next in Theorem 3.8.

Theorem 3.8 (Achievability). *There exists a (d, ϵ) source-channel code with*

$$\epsilon \leq \inf_{P_X, P_Z, \gamma > 0} \left\{ \mathbb{E} \left[\exp \left(-\left| \iota_{X;Y}(X;Y) - \log \frac{\gamma}{P_Z(B_d(S))} \right|^+ \right) \right] + e^{1-\gamma} \right\} \quad (3.82)$$

where the expectation is with respect to $P_S P_X P_{Y|X} P_Z$ defined on $\mathcal{M} \times \mathcal{X} \times \mathcal{Y} \times \widehat{\mathcal{M}}$.

Proof. We fix an arbitrary $\gamma > 0$ and choose

$$W = \left\lfloor \frac{\gamma}{\rho(S)} \right\rfloor \quad (3.83)$$

where $\rho(\cdot)$ is defined in (3.78). Observing that

$$(1 - \rho(s))^{\lfloor \frac{\gamma}{\rho(s)} \rfloor} \leq (1 - \rho(s))^{\frac{\gamma}{\rho(s)} - 1} \quad (3.84)$$

$$\leq e^{-\rho(s)(\frac{\gamma}{\rho(s)} - 1)} \quad (3.85)$$

$$\leq e^{1-\gamma} \quad (3.86)$$

we obtain (3.82) by weakening (3.56) using (3.83) and (3.86). \square

In the case of almost-lossless JSCC, the bound in Theorem 3.8 can be sharpened as shown recently by Tauste Campo et al. [66]:

Theorem 3.9 (Achievability, almost-lossless JSCC [66]). *There exists a $(0, \epsilon)$ code with*

$$\epsilon \leq \inf_{P_X} \mathbb{E} [\exp(-|\imath_{X;Y}(X;Y) - \imath_S(S)|^+)] \quad (3.87)$$

where the expectation is with respect to $P_S P_X P_{Y|X}$ defined on $\mathcal{M} \times \mathcal{X} \times \mathcal{Y}$.

3.5 Gaussian Approximation

In addition to the basic conditions (a)-(c) of Sections 2.2 and 3.2 and to the standard stationarity and memorylessness assumptions on the source spelled out in (i)-(iv) in Section 2.6.2, we assume the following.

- (v) The channel is stationary and memoryless, $P_{Y^n|X^n} = P_{Y|X} \times \dots \times P_{Y|X}$. If the channel has an input cost function then it satisfies $\mathbf{b}_n(x^n) = \frac{1}{n} \sum_{i=1}^n \mathbf{b}(x_i)$.

Theorem 3.10 (Gaussian approximation). *Under restrictions (i)-(v), the parameters of the optimal (k, n, d, ϵ) code satisfy*

$$nC(\beta) - kR(d) = \sqrt{nV(\beta) + k\mathcal{V}(d)}Q^{-1}(\epsilon) + \theta(n) \quad (3.88)$$

where

1. $\mathcal{V}(d)$ is the source dispersion given by

$$\mathcal{V}(d) = \text{Var}[\jmath_S(S, d)] \quad (3.89)$$

2. $V(\beta)$ is the channel dispersion given by:

a) If \mathcal{A} and \mathcal{B} are finite,

$$V(\beta) = \text{Var} [\imath_{\mathbf{X}, \mathbf{Y}}^*(\mathbf{X}^*; \mathbf{Y}^*)] \quad (3.90)$$

$$\imath_{\mathbf{X}, \mathbf{Y}}^*(x; y) \triangleq \log \frac{dP_{\mathbf{Y}|\mathbf{X}=x}}{dP_{\mathbf{Y}^*}}(y) \quad (3.91)$$

where $\mathbf{X}^*, \mathbf{Y}^*$ are the capacity-achieving input and output random variables.

b) If the channel is Gaussian with either equal or maximal power constraint,

$$V = \frac{1}{2} \left(1 - \frac{1}{(1+P)^2} \right) \log^2 e \quad (3.92)$$

where P is the signal-to-noise ratio.

3. The remainder term $\theta(n)$ satisfies:

(a) If \mathcal{A} and \mathcal{B} are finite, the channel has no cost constraints and $V > 0$,

$$-\underline{c} \log n + O(1) \leq \theta(n) \quad (3.93)$$

$$\leq C_0 \log n + \log \log n + O(1) \quad (3.94)$$

where

$$\underline{c} = |\mathcal{A}| - \frac{1}{2} \quad (3.95)$$

and C_0 is the constant specified in (2.146).

(b) If \mathcal{A} and \mathcal{B} are finite and $V = 0$, (3.94) still holds, while (3.93) is replaced with

$$\liminf_{n \rightarrow \infty} \frac{\theta(n)}{\sqrt{n}} \geq 0 \quad (3.96)$$

(c) If the channel is such that the (conditional) distribution of $\imath_{\mathbf{X}, \mathbf{Y}}^*(\mathbf{x}; \mathbf{Y})$ does not depend on $x \in \mathcal{A}$ (no cost constraint), then $\underline{c} = \frac{1}{2}$.⁴

(d) If the channel is Gaussian with equal or maximal power constraint, (3.94) still holds, and (3.93) holds with $\underline{c} = \frac{1}{2}$.

⁴ As we show in Chapter 5, the symmetricity condition is actually superfluous.

(e) In the almost-lossless case, $R(d) = H(S)$, and provided that the third absolute moment of $\mathfrak{v}_S(S)$ is finite, (3.88) and (3.93) still hold, while (3.94) strengthens to

$$\theta(n) \leq O(1) \quad (3.97)$$

Proof. In this chapter we limit the proof to the case of no cost constraints (unless the channel is Gaussian). The validity of Theorem 3.10 for the DMC with cost will be shown in Chapter 5.

- Appendices C.1.1 and C.1.2 show the converses in (3.93) and (3.96) for cases $V > 0$ and $V = 0$, respectively, using Corollary 3.3.
- Appendix C.1.3 shows the converse for the symmetric channel (3c) using Theorem 3.2.
- Appendix C.1.4 shows the converse for the Gaussian channel (3d) using Theorem 3.2.
- Appendix C.2.1 shows the achievability result for almost lossless coding (3e) using Theorem 3.9.
- Appendix C.2.2 shows the achievability result in (3.94) for the DMC using Theorem 3.8.
- Appendix C.2.3 shows the achievability result for the Gaussian channel (3d) using Theorem 3.8.

□

Remark 3.9. If the channel and the data compression codes are designed separately, we can invoke the channel coding [3] and lossy compression results in (1.2) and (1.9) to show that (cf. (1.11))

$$nC(\beta) - kR(d) \leq \min_{\eta + \zeta \leq \epsilon} \left\{ \sqrt{nV(\beta)}Q^{-1}(\eta) + \sqrt{kV(d)}Q^{-1}(\zeta) \right\} + O(\log n) \quad (3.98)$$

Comparing (3.98) to (3.88), observe that if either the channel or the source (or both) have zero dispersion, the joint source-channel coding dispersion can be achieved by separate coding. In that special case, either the \mathfrak{d} -tilted information or the channel information density are so close to being deterministic that there is no need to account for the true distributions of these random variables, as a good joint source-channel code would do.

The Gaussian approximations of JSCC and SSCC in (3.88) and (3.98), respectively, admit the following heuristic interpretation when n is large (and thus, so is k): since the source is stationary and memoryless, the normalized \mathbf{d} -tilted information $J = \frac{1}{n} \mathcal{I}_{S^k} (S^k, d)$ becomes approximately Gaussian with mean $\frac{k}{n} R(d)$ and variance $\frac{k}{n} \frac{\mathcal{V}(d)}{n}$. Likewise, the conditional normalized channel information density $I = \frac{1}{n} \mathcal{I}_{X^n; Y^n}^*(x^n; Y^n)$ is, for large k, n , approximately Gaussian with mean $C(\beta)$ and variance $\frac{V(\beta)}{n}$ for all $x^n \in \mathcal{A}^n$ typical according to the capacity-achieving distribution. Since a good encoder chooses such inputs for (almost) all source realizations, and the source and the channel are independent, the random variable $I - J$ is approximately Gaussian with mean $C(\beta) - \frac{k}{n} R(d)$ and variance $\frac{1}{n} \left(\frac{k}{n} \mathcal{V}(d) + V(\beta) \right)$, and (3.88) reflects the intuition that under JSCC, the source is reconstructed successfully within distortion d if and only if the channel information density exceeds the source \mathbf{d} -tilted information, that is, $\{I > J\}$. In contrast, in SSCC, the source is reconstructed successfully with high probability if (I, J) falls in the intersection of half-planes $\{I > r\} \cap \{J < r\}$ for some $r = \frac{\log M}{n}$, which is the capacity of the noiseless link between the source and the channel code block that can be chosen so as to maximize the probability of that intersection, as reflected in (3.98). Since in JSCC the successful transmission event is strictly larger than in SSCC, i.e. $\{I > r\} \cap \{J < r\} \subset \{I > J\}$, separate source/channel code design incurs a performance loss. It is worth pointing out that $\{I > J\}$ leads to successful reconstruction even within the paradigm of the codes in Definition 3.7 because, as explained in Remark 3.6, unlike the SSCC case, it is not necessary that $\frac{\log M}{n}$ lie between I and J for successful reconstruction.

Remark 3.10. Using Theorem 3.10, it can be shown that

$$R(n, d, \epsilon, \beta) = \frac{C(\beta)}{R(d)} - \sqrt{\frac{\mathcal{V}(d, \beta)}{n}} Q^{-1}(\epsilon) - \frac{1}{R(d)} \frac{\theta(n)}{n} \quad (3.99)$$

where the rate-dispersion function of JSCC is found as (recall Definition 3.4)

$$\mathcal{V}(d, \beta) = \frac{1}{R^2(d)} \left(V(\beta) + \frac{C(\beta)}{R(d)} \mathcal{V}(d) \right) \quad (3.100)$$

Remark 3.11. Under regularity conditions similar to those in Theorem 2.25, it can be shown that

$$D(nR, n, \epsilon, \beta) = D\left(\frac{C(\beta)}{R}\right) + \sqrt{\frac{\mathcal{W}(R, \beta)}{n}} Q^{-1}(\epsilon) - \frac{\partial}{\partial R} D\left(\frac{C(\beta)}{R}\right) \frac{\theta(n)}{n} \quad (3.101)$$

where the distortion-dispersion function of JSCC is given by

$$\mathcal{W}(R, \beta) = \left(\frac{\partial}{\partial R} D \left(\frac{C(\beta)}{R} \right) \right)^2 \left(V(\beta) + R \mathcal{V} \left(\frac{C(\beta)}{R} \right) \right) \quad (3.102)$$

Remark 3.12. Fix d , ϵ , and suppose the goal is to sustain the probability of exceeding distortion d bounded by ϵ at a given fraction $1 - \eta$ of the asymptotic limit, i.e. at rate $R = (1 - \eta) \frac{C(\beta)}{R(d)}$. If $\eta \ll 1$, (3.99) implies that the required channel blocklength scales as:

$$n(d, \eta, \epsilon, \beta) \approx \frac{1}{C^2(\beta)} \left(V(\beta) + \frac{C(\beta)}{R(d)} \mathcal{V}(d) \right) \left(\frac{Q^{-1}(\epsilon)}{\eta} \right)^2 \quad (3.103)$$

or, equivalently, the required source blocklength scales as:

$$k(d, \eta, \epsilon, \beta) \approx \frac{1}{R^2(d)} \left(\mathcal{V}(d) + \frac{R(d)}{C(\beta)} V(\beta) \right) \left(\frac{Q^{-1}(\epsilon)}{\eta} \right)^2 \quad (3.104)$$

Remark 3.13. If the basic conditions (b) and/or (c) fail so that there are several distributions $P_{Z^*|S}$ and/or several P_{X^*} that achieve the rate-distortion function and the capacity, respectively, then, for $\epsilon < \frac{1}{2}$,

$$\mathcal{V}(d) \leq \min \mathcal{V}_{Z^*, X^*}(d) \quad (3.105)$$

$$\mathcal{W}(R) \leq \min \mathcal{W}_{Z^*, X^*}(R) \quad (3.106)$$

where the minimum is taken over the rate-distortion and capacity-achieving distributions $P_{Z^*|S}$ and P_{X^*} , and $\mathcal{V}_{Z^*, X^*}(d)$ (resp. $\mathcal{W}_{Z^*, X^*}(R)$) denotes (3.100) (resp. (3.102)) computed with $P_{Z^*|S}$ and P_{X^*} . The reason for possibly lower achievable dispersion in this case is that we have the freedom to map the unlikely source realizations leading to high probability of failure to those codewords resulting in the maximum variance so as to increase the probability that the channel output escapes the decoding failure region.

Remark 3.14. The dispersion of the Gaussian channel is given by (3.92), regardless of whether an equal or a maximal power constraint is imposed. An equal power constraint corresponds to the subset of allowable channel inputs being the power sphere:

$$\mathcal{F}(P) = \left\{ x^n \in \mathbb{R}^n : \frac{|x^n|^2}{\sigma_N^2} = nP \right\} \quad (3.107)$$

where σ_N^2 is the noise power. In a maximal power constraint, (3.107) is relaxed replacing ‘=’ with

‘ \leq ’.

Specifying the nature of the power constraint in the subscript, we remark that the bounds for the maximal constraint can be obtained from the bounds for the equal power constraint via the following relation

$$k_{\text{eq}}^*(n, d, \epsilon) \leq k_{\text{max}}^*(n, d, \epsilon) \leq k_{\text{eq}}^*(n+1, d, \epsilon) \quad (3.108)$$

where the right-most inequality is due to the following idea dating back to Shannon: a (k, n, d, ϵ) code with a maximal power constraint can be converted to a $(k, n+1, d, \epsilon)$ code with an equal power constraint by appending an $(n+1)$ -th coordinate to each codeword to equalize its total power to $n\sigma_{\text{N}}^2 P$. From (3.108) it is immediate that the channel dispersions for maximal or equal power constraints must be the same.

3.6 Lossy transmission of a BMS over a BSC

In this section we particularize the bounds in Sections 3.3, 3.4 and the approximation in Section 3.5 to the transmission of a BMS with bias p over a BSC with crossover probability δ . The target bit error rate satisfies $d \leq p$.

The rate-distortion function of the source and the channel capacity are given by, respectively,

$$R(d) = h(p) - h(d) \quad (3.109)$$

$$C = 1 - h(\delta) \quad (3.110)$$

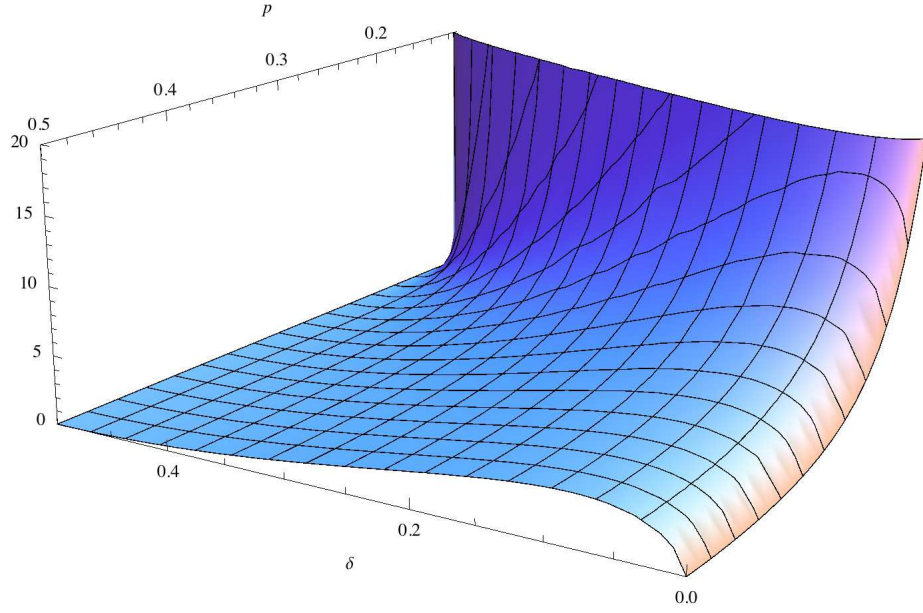
The source and the channel dispersions are given by (see [3] and (2.211)):

$$\mathcal{V}(d) = p(1-p) \log^2 \frac{1-p}{p} \quad (3.111)$$

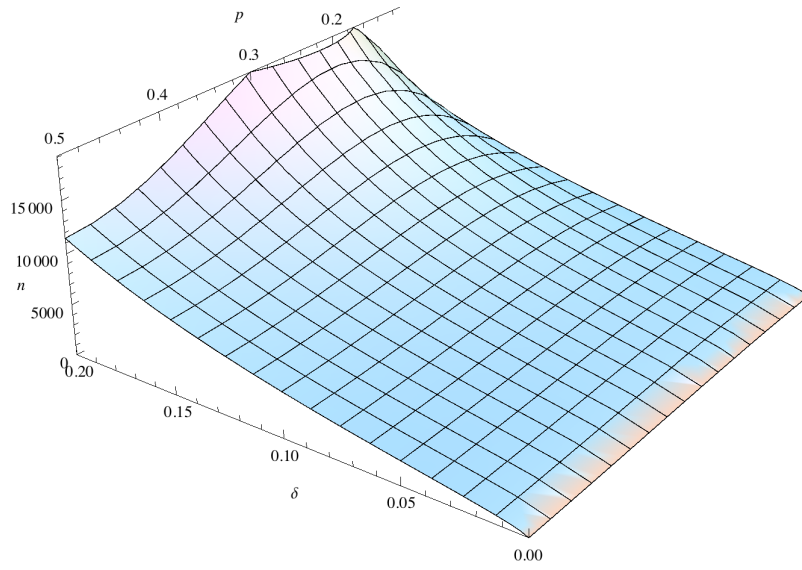
$$V = \delta(1-\delta) \log^2 \frac{1-\delta}{\delta} \quad (3.112)$$

where note that (3.111) does not depend on d . The rate-dispersion function in (3.100) together with the blocklength (3.103) required to achieve 90% of the asymptotic limit are plotted in Fig. 3.2. The rate-dispersion function vanishes as $\delta \rightarrow \frac{1}{2}$ or as $(\delta, p) \rightarrow (0, \frac{1}{2})$. The required blocklength increases fast as $\delta \rightarrow \frac{1}{2}$, a consequence of the fact that the asymptotic limit is very small in that case.

Throughout this section, $w(a^\ell)$ denotes the Hamming weight of the binary ℓ -vector a^ℓ , and T_α^ℓ denotes a binomial random variable with parameters ℓ and α , independent of all other random



(a)



(b)

Figure 3.2: The rate-dispersion function (a) in (3.100) and the channel blocklength (b) in (3.103) required to sustain $R = 0.9 \frac{C}{R(d)}$ and excess-distortion probability 10^{-4} for the transmission of a BMS over a BSC with $d = 0.11$ as functions of (δ, p) .

variables.

For convenience, we define the discrete random variable $U_{\alpha,\beta}$ by

$$U_{\alpha,\beta} = (T_\alpha^k - kp) \log \frac{1-p}{p} + (T_\beta^n - n\delta) \log \frac{1-\delta}{\delta} \quad (3.113)$$

In particular, substituting $\alpha = p$ and $\beta = \delta$ in (3.113), we observe that the terms in the right side of (3.113) are zero-mean random variables whose variances are equal to $k\mathcal{V}(d)$ and nV , respectively.

Furthermore, recall from Chapter 2 that the binomial sum is denoted by

$$\left\langle \begin{matrix} k \\ \ell \end{matrix} \right\rangle = \sum_{i=0}^{\ell} \binom{k}{i} \quad (3.114)$$

A straightforward particularization of the \mathbf{d} -tilted information converse in Theorem 3.2 leads to the following result.

Theorem 3.11 (Converse, BMS-BSC). *Any (k, n, d, ϵ) code for transmission of a BMS with bias p over a BSC with bias δ must satisfy*

$$\epsilon \geq \sup_{\gamma \geq 0} \left\{ \mathbb{P}[U_{p,\delta} \geq nC - kR(d) + \gamma] - \exp(-\gamma) \right\} \quad (3.115)$$

Proof. Let $P_{\bar{Y}^n} = P_{Y^{n*}}$, which is the equiprobable distribution on $\{0, 1\}^n$. An easy exercise reveals that

$$J_{S^k}(s^k, d) = \iota_{S^k}(s^k) - kh(d) \quad (3.116)$$

$$\iota_{S^k}(s^k) = kh(p) + (w(s^k) - kp) \log \frac{1-p}{p} \quad (3.117)$$

$$\iota_{X^n; Y^n}^*(x^n; y^n) = n(\log 2 - h(\delta)) - (w(y^n - x^n) - n\delta) \log \frac{1-\delta}{\delta} \quad (3.118)$$

Since $w(Y^n - x^n)$ is distributed as T_δ^n regardless of $x^n \in \{0, 1\}^n$, and $w(S^k)$ is distributed as T_p^k , the condition in Theorem 3.2 is satisfied, and (3.22) becomes (3.115). \square

The hypothesis-testing converse in Theorem 3.4 particularizes to the following result:

Theorem 3.12 (Converse, BMS-BSC). *Any (k, n, d, ϵ) code for transmission of a BMS with bias p*

over a BSC with bias δ must satisfy

$$\mathbb{P}\left[U_{\frac{1}{2}, \frac{1}{2}} < r\right] + \lambda \mathbb{P}\left[U_{\frac{1}{2}, \frac{1}{2}} = r\right] \leq \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle 2^{-k} \quad (3.119)$$

where $0 \leq \lambda < 1$ and scalar r are uniquely defined by

$$\mathbb{P}[U_{p, \delta} < r] + \lambda \mathbb{P}[U_{p, \delta} = r] = 1 - \epsilon \quad (3.120)$$

Proof. As in the proof of Theorem 3.11, we let $P_{\bar{Y}^n}$ be the equiprobable distribution on $\{0, 1\}^n$, $P_{\bar{Y}^n} = P_{Y^{n*}}$. Since under $P_{Y^n|X^n=x^n}$, $w(Y^n - x^n)$ is distributed as T_δ^n , and under $P_{Y^{n*}}$, $w(Y^n - x^n)$ is distributed as $T_{\frac{1}{2}}^n$, irrespective of the choice of $x^n \in A^n$, the distribution of the information density in (3.118) does not depend on the choice of x^n under either measure, so Theorem 3.5 can be applied. Further, we choose Q_{S^k} to be the equiprobable distribution on $\{0, 1\}^k$ and observe that under P_{S^k} , the random variable $w(S^k)$ in (3.117) has the same distribution as T_p^k , while under Q_{S^k} it has the same distribution as $T_{\frac{1}{2}}^k$. Therefore, the log-likelihood ratio for testing between $P_{S^k}P_{Y^n|X^n=x^n}$ and $Q_{S^k}P_{Y^{n*}}$ has the same distribution as (' \sim ' denotes equality in distribution)

$$\log \frac{P_{S^k}(S^k)P_{Y^n|X^n=x^n}(Y^n)}{Q_{S^k}(S^k)P_{Y^{n*}}(Y^n)} = i_{X^n; Y^n}^*(x^n; Y^n) - i_{S^k}(S^k) + k \log 2 \quad (3.121)$$

$$\sim n \log 2 - nh(\delta) - kh(p) - \begin{cases} U_{p, \delta} & \text{under } P_{S^k}P_{Y^n|X^n=x^n} \\ U_{\frac{1}{2}, \frac{1}{2}} & \text{under } Q_{S^k}P_{Y^{n*}} \end{cases} \quad (3.122)$$

so $\beta_{1-\epsilon}(P_{S^k}P_{Y^n|X^n=x^n}, Q_{S^k}P_{Y^{n*}})$ is equal to the left side of (3.119). Finally, matching the size of the list to the fidelity of reproduction using (3.30), we find that L is equal to the right side of (3.119). \square

If the source is equiprobable, the bound in Theorem 3.12 becomes particularly simple, as the following result details.

Theorem 3.13 (Converse, EBMS-BSC). *For $p = \frac{1}{2}$, if there exists a (k, n, d, ϵ) joint source-channel code, then*

$$\lambda \binom{n}{r^* + 1} + \left\langle \begin{matrix} n \\ r^* \end{matrix} \right\rangle \leq \left\langle \begin{matrix} k \\ \lfloor kd \rfloor \end{matrix} \right\rangle 2^{n-k} \quad (3.123)$$

where

$$r^* = \max \left\{ r : \sum_{t=0}^r \binom{n}{t} \delta^t (1-\delta)^{n-t} \leq 1 - \epsilon \right\} \quad (3.124)$$

and $\lambda \in [0, 1)$ is the solution to

$$\sum_{j=0}^{r^*} \binom{n}{j} \delta^j (1-\delta)^{n-j} + \lambda \delta^{r^*+1} (1-\delta)^{n-r^*-1} \binom{n}{r^*+1} = 1 - \epsilon \quad (3.125)$$

The achievability result in Theorem 3.8 is particularized as follows.

Theorem 3.14 (Achievability, BMS-BSC). *There exists a (k, n, d, ϵ) joint source-channel code with*

$$\epsilon \leq \inf_{\gamma > 0} \left\{ \mathbb{E} \left[\exp \left(-|U - \log \gamma|^+ \right) \right] + e^{1-\gamma} \right\} \quad (3.126)$$

where

$$U = nC - (T_\delta^n - n\delta) \log \frac{1-\delta}{\delta} - \log \frac{1}{\rho(T_p^k)} \quad (3.127)$$

and $\rho: \{0, 1, \dots, k\} \mapsto [0, 1]$ is defined as

$$\rho(T) = \sum_{t=0}^k L(T, t) q^t (1-q)^{k-t} \quad (3.128)$$

with

$$L(T, t) = \begin{cases} \binom{T}{t_0} \binom{k-T}{t-t_0} & t - kd \leq T \leq t + kd \\ 0 & \text{otherwise} \end{cases} \quad (3.129)$$

$$t_0 = \left\lceil \frac{t + T - kd}{2} \right\rceil^+ \quad (3.130)$$

$$q = \frac{p-d}{1-2d} \quad (3.131)$$

Proof. We weaken the infima over P_{X^n} and P_{Z^k} in (3.82) by choosing them to be the product distributions generated by the capacity-achieving channel input distribution and the rate-distortion function-achieving reproduction distribution, respectively, i.e. P_{X^n} is equiprobable on $\{0, 1\}^n$, and

$P_{Z^k} = P_{Z^*} \times \dots \times P_{Z^*}$, where $P_{Z^*}(1) = q$. As shown in (2.208),

$$P_{Z^k}(B_d(s^k)) \geq \rho(w(s^k)) \quad (3.132)$$

On the other hand, $|Y^n - X^n|_0$ is distributed as T_δ^n , so (3.126) follows by substituting (3.118) and (3.132) into (3.82). \square

In the special case of the BMS-BSC, Theorem 3.10 can be strengthened as follows.

Theorem 3.15 (Gaussian approximation, BMS-BSC). *The parameters of the optimal (k, n, d, ϵ) code satisfy (3.88) where $R(d)$, C , $\mathcal{V}(d)$, V are given by (3.109), (3.110), (3.111), (3.112), respectively, and the remainder term in (3.88) satisfies*

$$O(1) \leq \theta(n) \quad (3.133)$$

$$\leq \frac{1}{2} \log n + \log \log n + O(1) \quad (3.134)$$

if $0 < d < p$, and

$$-\frac{1}{2} \log n + O(1) \leq \theta(n) \quad (3.135)$$

$$\leq O(1) \quad (3.136)$$

if $d = 0$.

Proof. An asymptotic analysis of the converse bound in Theorem 3.12 akin to that found in the proof of Theorem 2.34 leads to (3.133) and (3.135). An asymptotic analysis of the achievability bound in Theorem 3.14 similar to the one found in Appendix B.8 leads to (3.134). Finally, (3.136) is the same as (3.97). \square

The bounds and the Gaussian approximation (in which we take $\theta(n) = 0$) are plotted in Fig. 3.3 ($d = 0$), Fig. 3.4 (fair binary source, $d > 0$) and Fig. 3.5 (biased binary source, $d > 0$). A source of fair coin flips has zero dispersion, and as anticipated in Remark 3.9, JSSC does not afford much gain in the finite blocklength regime (Fig. 3.4). Moreover, in that case the JSSC achievability bound in Theorem 3.8 is worse than the SSCC achievability bound. However, the more general achievability bound in Theorem 3.7 with the choice $W = M$, as detailed in Remark 3.8, nearly coincides with the SSCC curve in Fig. 3.4, providing an improvement over Theorem 3.8. The situation is different if the source is biased, with JSSC showing significant gain over SSCC (Figures 3.3 and 3.5).

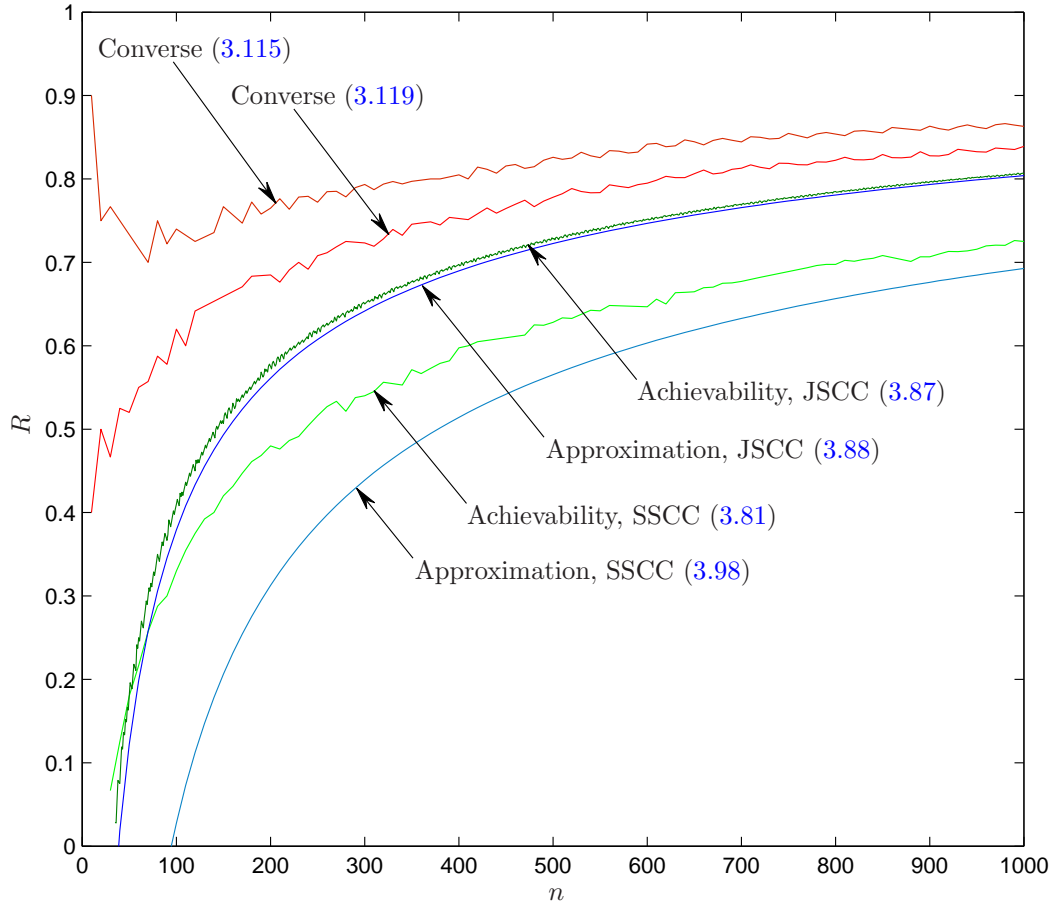


Figure 3.3: Rate-blocklength tradeoff for the transmission of a BMS with bias $p = 0.11$ over a BSC with crossover probability $\delta = p = 0.11$ and $d = 0$, $\epsilon = 10^{-2}$.

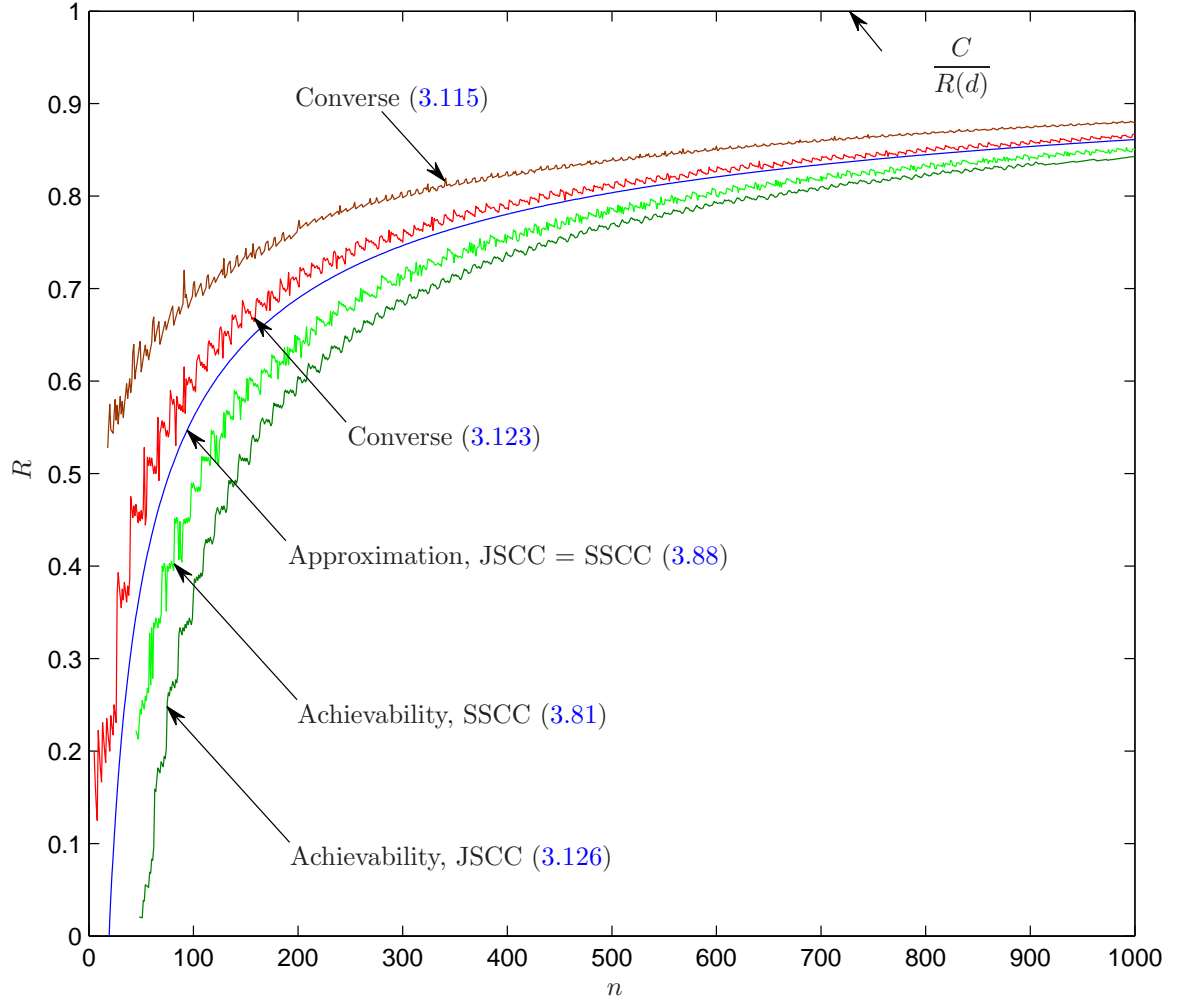


Figure 3.4: Rate-blocklength tradeoff for the transmission of a fair BMS over a BSC with crossover probability $\delta = d = 0.11$ and $\epsilon = 10^{-2}$.

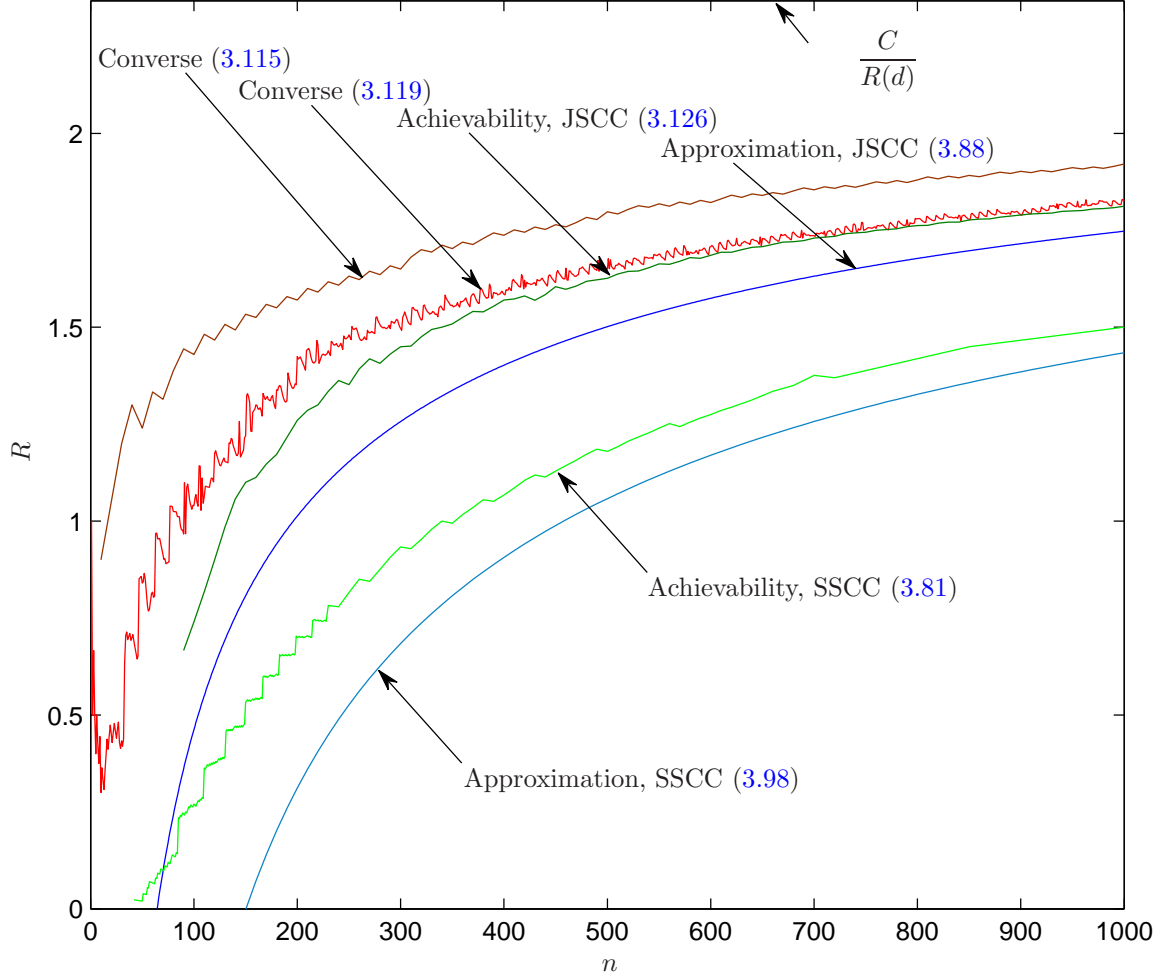


Figure 3.5: Rate-blocklength tradeoff for the transmission of a BMS with bias $p = 0.11$ over a BSC with crossover probability $\delta = p = 0.11$ and $d = 0.05$, $\epsilon = 10^{-2}$.

3.7 Transmission of a GMS over an AWGN channel

In this section we analyze the setup where the Gaussian memoryless source $S_i \sim \mathcal{N}(0, \sigma_S^2)$ is transmitted over an AWGN channel, which, upon receiving an input x^n , outputs $Y^n = x^n + N^n$, where $N^n \sim \mathcal{N}(0, \sigma_N^2 \mathbf{I})$. The encoder/decoder must satisfy two constraints, the fidelity constraint and the cost constraint:

- the MSE distortion exceeds $0 \leq d \leq \sigma_S^2$ with probability no greater than $0 < \epsilon < 1$;
- each channel codeword satisfies the equal power constraint in (3.107).⁵

The rate-distortion function and the capacity-cost function are given by

$$R(d) = \frac{1}{2} \log \left(\frac{\sigma_S^2}{d} \right) \quad (3.137)$$

$$C(P) = \frac{1}{2} \log (1 + P) \quad (3.138)$$

The source dispersion is given by (2.154):

$$\mathcal{V}(d) = \frac{1}{2} \log^2 e \quad (3.139)$$

while the channel dispersion is given by (3.92) [3].

In the rest of the section, W_λ^ℓ denotes a noncentral chi-square distributed random variable with ℓ degrees of freedom and non-centrality parameter λ , independent of all other random variables, and $f_{W_\lambda^\ell}$ denotes its probability density function.

A straightforward particularization of the d-tilted information converse in Theorem 3.2 leads to the following result.

Theorem 3.16 (Converse, GMS-AWGN). *If there exists a (k, n, d, ϵ) code, then*

$$\epsilon \geq \sup_{\gamma \geq 0} \left\{ \mathbb{P}[U \geq nC(P) - kR(d) + \gamma] - \exp(-\gamma) \right\} \quad (3.140)$$

where

$$U = \frac{\log e}{2} (W_0^k - k) + \frac{\log e}{2} \left(\frac{P}{1+P} W_{\frac{n}{P}}^n - n \right) \quad (3.141)$$

⁵See Remark 3.14 in Section 3.5 for a discussion of the close relation between an equal and a maximal power constraint.

Observe that the terms to the left of the ‘ \geq ’ sign inside the probability in (3.140) are zero-mean random variables whose variances are equal to $k\mathcal{V}(d)$ and nV , respectively.

Proof. The spherically-symmetric $P_{\bar{Y}^n} = P_{Y^{n*}} = P_{Y^*} \times \dots \times P_{Y^*}$, where $Y^* \sim \mathcal{N}(0, \sigma_{\mathbf{N}}^2(1+P))$ is the capacity-achieving output distribution, satisfies the symmetry assumption of Theorem 3.2. More precisely, it is not hard to show (see [3, (205)]) that for all $x^n \in \mathcal{F}(\alpha)$, $\iota_{X^n; Y^n}^*(x^n; Y^n)$ has the same distribution under $P_{Y^{n*}|X^n=x^n}$ as

$$\frac{n}{2} \log(1+P) - \frac{\log e}{2} \left(\frac{P}{1+P} W_{\frac{n}{P}}^n - n \right) \quad (3.142)$$

The d -tilted information in s^k is given by

$$J_{S^k}(s^k, d) = \frac{k}{2} \log \frac{\sigma_S^2}{d} + \left(\frac{|s^k|^2}{\sigma_S^2} - k \right) \frac{\log e}{2} \quad (3.143)$$

Plugging (3.142) and (3.143) into (3.22), (3.140) follows. \square

The hypothesis testing converse in Theorem 3.5 is particularized as follows.

Theorem 3.17 (Converse, GMS-AWGN).

$$k \int_0^\infty r^{k-1} \mathbb{P} \left[P W_{n(1+\frac{1}{P})}^n + k \frac{d}{\sigma^2} r^2 \leq n\tau \right] dr \leq 1 \quad (3.144)$$

where τ is the solution to

$$\mathbb{P} \left[\frac{P}{1+P} W_{\frac{n}{P}}^n + W_0^k \leq n\tau \right] = 1 - \epsilon \quad (3.145)$$

Proof. As in the proof of Theorem 3.16, we let $\bar{Y}^n \sim Y^{n*} \sim \mathcal{N}(0, \sigma_{\mathbf{N}}^2(1+P)\mathbf{I})$. Under $P_{Y^n|X^n=x^n}$, the distribution of $\iota_{X^n; Y^n}^*(x^n; Y^{n*})$ is that of (3.142), while under $P_{Y^{n*}}$, it has the same distribution as (cf. [3, (204)])

$$\frac{n}{2} \log(1+P) - \frac{\log e}{2} \left(P W_{n(1+\frac{1}{P})}^n - n \right) \quad (3.146)$$

Since the distribution of $\iota_{X^n; Y^n}^*(x^n; Y^{n*})$ does not depend on the choice of $x^n \in \mathbb{R}^n$ according to either measure, Theorem 3.5 applies. Further, choosing Q_{S^k} to be the Lebesgue measure on \mathbb{R}^k , i.e. $dQ_{S^k} = ds^k$, observe that

$$\log f_{S^k}(s^k) = \log \frac{dP_{S^k}(s^k)}{ds^k} = -\frac{k}{2} \log(2\pi\sigma_S^2) - \frac{\log e}{2\sigma_S^2} |s^k|^2 \quad (3.147)$$

Now, (3.144) and (3.145) are obtained by integrating

$$1 \left\{ \log f_{S^k}(s^k) + \imath_{X^n; Y^n}^*(x^n; y^n) > \frac{n}{2} \log(1+P) + \frac{n}{2} \log e - \frac{k}{2} \log(2\pi\sigma_5^2) - \frac{\log e}{2} n\tau \right\} \quad (3.148)$$

with respect to $ds^k dP_{Y^{n*}}(y^n)$ and $dP_{S^k}(s^k) dP_{Y^n|X^n=x^n}(y^n)$, respectively. \square

The bound in Theorem 3.8 can be computed as follows.

Theorem 3.18 (Achievability, GMS-AWGN). *There exists a (k, n, d, ϵ) code such that*

$$\epsilon \leq \inf_{\gamma > 0} \left\{ \mathbb{E} \left[\exp \left\{ -|U - \log \gamma|^+ \right\} \right] + e^{1-\gamma} \right\} \quad (3.149)$$

where

$$U = nC(P) - \frac{\log e}{2} \left(\frac{P}{1+P} W_{\frac{n}{P}}^n - n \right) - \log \frac{F}{\rho(W_0^k)} \quad (3.150)$$

$$F = \max_{n \in \mathbb{N}, t \in \mathbb{R}^+} \frac{f_{W_{nP}^n}(t)}{f_{W_0^n}\left(\frac{t}{1+P}\right)} < \infty \quad (3.151)$$

and $\rho : \mathbb{R}^+ \mapsto [0, 1]$ is defined by

$$\rho(t) = \frac{\Gamma\left(\frac{k}{2} + 1\right)}{\sqrt{\pi} k \Gamma\left(\frac{k-1}{2} + 1\right)} \left(1 - L\left(\sqrt{\frac{t}{k}}\right) \right)^{\frac{k-1}{2}} \quad (3.152)$$

where

$$L(r) = \begin{cases} 0 & r < \sqrt{\frac{d}{\sigma_5^2}} - \sqrt{1 - \frac{d}{\sigma_5^2}} \\ 1 & \left| r - \sqrt{1 - \frac{d}{\sigma_5^2}} \right| > \sqrt{\frac{d}{\sigma_5^2}} \\ \frac{\left(1 + r^2 - 2\sqrt{\frac{d}{\sigma_5^2}}\right)^2}{4\left(1 - \frac{d}{\sigma_5^2}\right)r^2} & \text{otherwise} \end{cases} \quad (3.153)$$

Proof. We compute an upper bound to (3.82) for the specific case of the transmission of a GMS over an AWGN channel. First, we weaken the infimum over P_{Z^k} in (3.82) by choosing P_{Z^k} to be the uniform distribution on the surface of the k -dimensional sphere with center at $\mathbf{0}$ and radius $r_0 = \sqrt{k}\sigma\sqrt{1 - \frac{d}{\sigma_5^2}}$. We showed in the proof of Theorem 2.45 (see also [29, 42]) that

$$P_{Z^k}(B_d(s^k)) \geq \rho(|s^k|^2) \quad (3.154)$$

which takes care of the source random variable in (3.82).

We proceed to analyze the channel random variable $\iota_{X^n;Y^n}(X^n;Y^n)$. Observe that since X^n lies on the power sphere and the noise is spherically symmetric, $|Y^n|^2 = |X^n + N^n|^2$ has the same distribution as $|x_0^n + N^n|^2$, where x_0^n is an arbitrary point on the surface of the power sphere. Letting $x_0^n = \sigma_N \sqrt{P} (1, 1, \dots, 1)$, we see that $\frac{1}{\sigma_N^2} |x_0^n + N^n|^2 = \sum_{i=1}^n \left(\frac{1}{\sigma_N^2} N_i + \sqrt{P} \right)^2$ has the non-central chi-squared distribution with n degrees of freedom and noncentrality parameter nP . To simplify calculations, we express the information density as

$$\iota_{X^n;Y^n}(x_0^n; y^n) = \iota_{X^n;Y^n}^*(x_0^n; y^n) - \log \frac{dP_{Y^n}}{dP_{Y^{n*}}}(y^n) \quad (3.155)$$

where $Y^{n*} \sim \mathcal{N}(0, \sigma_N^2(1+P)\mathbf{I})$. The distribution of $\iota_{X^n;Y^n}^*(x_0^n; Y^n)$ is the same as (3.142). Further, due to the spherical symmetry of both P_{Y^n} and $P_{Y^{n*}}$, as discussed above, we have (recall that ‘ \sim ’ denotes equality in distribution)

$$\frac{dP_{Y^n}}{dP_{Y^{n*}}}(Y^n) \sim \frac{f_{W_{nP}^n}(W_{nP}^n)}{f_{W_0^n}\left(\frac{W_{nP}^n}{1+P}\right)} \quad (3.156)$$

which is bounded uniformly in n as observed in [3, (425), (435)], thus (3.151) is finite, and (3.149) follows. \square

The following result strengthens Theorem 3.10 in the special case of the GMS-AWGN.

Theorem 3.19 (Gaussian approximation, GMS-AWGN). *The parameters of the optimal (k, n, d, ϵ) code satisfy (3.88) where $R(d)$, C , $\mathcal{V}(d)$, V are given by (3.137), (3.138), (3.139), (3.92), respectively, and the remainder term in (3.88) satisfies*

$$O(1) \leq \theta(n) \quad (3.157)$$

$$\leq \frac{1}{2} \log n + \log \log n + O(1) \quad (3.158)$$

Proof. An asymptotic analysis of the converse bound in Theorem 3.17 similar to that found in the proof of Theorem 2.48 leads to (3.157). An asymptotic analysis of the achievability bound in Theorem 3.18 similar to that given in Appendix B.11 leads to (3.158). \square

Numerical evaluation of the bounds reveals that JSCC noticeably outperforms SSCC in the displayed region of blocklengths (Fig. 3.6).

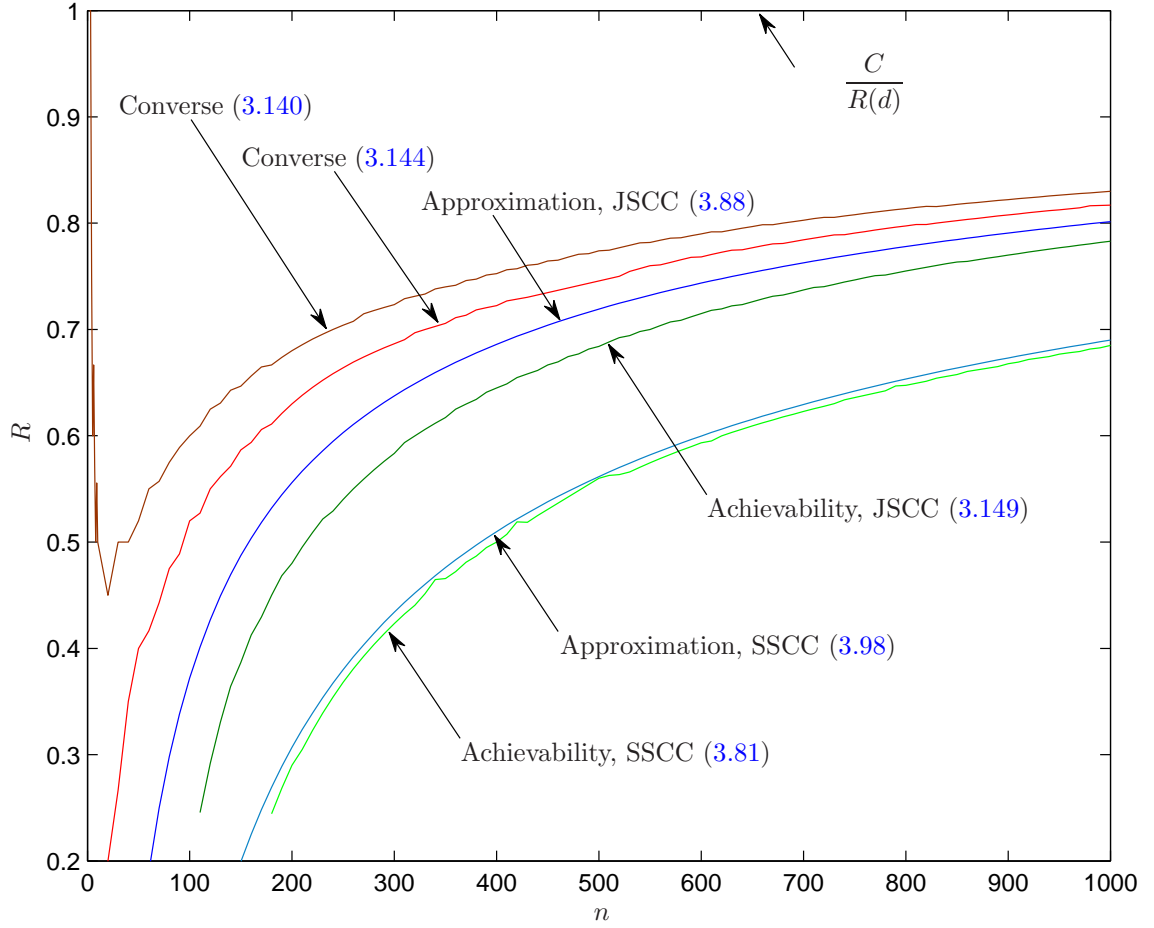


Figure 3.6: Rate-blocklength tradeoff for the transmission of a GMS with $\frac{d}{\sigma_s^2} = 0.5$ over an AWGN channel with $P = 1$, $\epsilon = 10^{-2}$.

3.8 To code or not to code

Our goal in this section is to compare the excess distortion performance of the optimal code of rate 1 at channel blocklength n with that of the optimal symbol-by-symbol code, evaluated after n channel uses, leveraging the bounds in Sections 3.3 and 3.4 and the approximation in Section 3.5. We show certain examples in which symbol-by-symbol coding is, in fact, either optimal or very close to being optimal. A general conclusion drawn from this section is that even when no coding is asymptotically suboptimal it can be a very attractive choice for short blocklengths.

3.8.1 Performance of symbol-by-symbol source-channel codes

Definition 3.8. An (n, d, ϵ, α) symbol-by-symbol code is an $(n, n, d, \epsilon, \alpha)$ code (\mathbf{f}, \mathbf{g}) (according to Definition 3.2) that satisfies

$$\mathbf{f}(s^n) = (\mathbf{f}_1(s_1), \dots, \mathbf{f}_1(s_n)) \quad (3.159)$$

$$\mathbf{g}(y^n) = (\mathbf{g}_1(y_1), \dots, \mathbf{g}_1(y_n)) \quad (3.160)$$

for some pair of functions $\mathbf{f}_1: \mathcal{S} \mapsto \mathcal{A}$ and $\mathbf{g}_1: \mathcal{B} \mapsto \hat{\mathcal{S}}$.

The minimum excess distortion achievable with symbol-by-symbol codes at channel blocklength n , excess probability ϵ and cost α is defined by

$$D_1(n, \epsilon, \alpha) = \inf \{d: \exists (n, d, \epsilon, \alpha) \text{ symbol-by-symbol code}\}. \quad (3.161)$$

Definition 3.9. The distortion-dispersion function of symbol-by-symbol joint source-channel coding is defined as

$$\mathcal{W}_1(\alpha) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{n(D(C(\alpha)) - D_1(n, \epsilon, \alpha))^2}{2 \log_e \frac{1}{\epsilon}} \quad (3.162)$$

where $D(\cdot)$ is the distortion-rate function of the source.

As before, if there is no channel input-cost constraint ($\mathbf{b}^n(x^n) = 0$ for all $x^n \in \mathcal{A}^n$), we will simplify the notation and write $D_1(n, \epsilon)$ for $D_1(n, \epsilon, \alpha)$ and \mathcal{W}_1 for $\mathcal{W}_1(\alpha)$.

In addition to restrictions stationarity and memorylessness assumptions (i)–(v) in Sections 2.2 and Section 3.5, we assume that the channel and the source are probabilistically matched in the following sense (cf. [12]).

(vi) There exist α , $P_{X^*|S}$ and $P_{Z^*|Y}$ such that P_{X^*} and $P_{Z^*|S}$ generated by the joint distribution $P_S P_{X^*|S} P_{Y|X} P_{Z^*|Y}$ achieve the capacity-cost function $C(\alpha)$ and the distortion-rate function $D(C(\alpha))$, respectively.

Condition (vi) ensures that symbol-by-symbol transmission attains the minimum average (over source realizations) distortion achievable among all codes of any blocklength. The following results pertain to the full distribution of the distortion incurred at the receiver output and not just its mean.

Theorem 3.20 (Achievability, symbol-by-symbol code). *Under restrictions (i)-(vi), if*

$$\mathbb{P} \left[\sum_{i=1}^n d(S_i, Z_i^*) > nd \right] \leq \epsilon \quad (3.163)$$

where $P_{Z^{n*}|S^n} = P_{Z^*|S} \times \dots \times P_{Z^*|S}$, and $P_{Z^*|S}$ achieves $D(C(\alpha))$, then there exists an (n, d, ϵ, α) symbol-by-symbol code (average cost constraint).

Proof. If (vi) holds, then there exist a symbol-by-symbol encoder and decoder such that the conditional distribution of the output of the decoder given the source outcome coincides with distribution $P_{Z^*|S}$, so the excess-distortion probability of this symbol-by-symbol code is given by the left side of (3.163). \square

Theorem 3.21 (Converse, symbol-by-symbol code). *Under restriction (v) and separable distortion measure, the parameters of any (n, d, ϵ, α) symbol-by-symbol code (average cost constraint) must satisfy*

$$\epsilon \geq \inf_{\substack{P_{Z|S}: \\ I(S;Z) \leq C(\alpha)}} \mathbb{P} \left[\sum_{i=1}^n d(S_i, Z_i) > nd \right] \quad (3.164)$$

where $P_{Z^n|S^n} = P_{Z|S} \times \dots \times P_{Z|S}$.

Proof. The excess-distortion probability at blocklength n , distortion d and cost α achievable among all single-letter codes $(P_{X|S}, P_{Z|Y})$ must satisfy

$$\epsilon \geq \inf_{\substack{P_{X|S}, P_{Z|Y}: \\ S-X-Y-Z \\ \mathbb{E}[b(X)] \leq \alpha}} \mathbb{P} [d_n(S^n, Z^n) > d] \quad (3.165)$$

$$\geq \inf_{\substack{P_{X|S}, P_{Z|Y}: \\ \mathbb{E}[b(X)] \leq \alpha \\ I(S;Z) \leq I(X;Y)}} \mathbb{P} [d_n(S^n, Z^n) > d] \quad (3.166)$$

where (3.166) holds since $S - X - Y - Z$ implies $I(S;Z) \leq I(X;Y)$ by the data processing inequality.

The right side of (3.166) is lower bounded by the right side of (3.164) because $I(\mathbf{X}; \mathbf{Y}) \leq C(\alpha)$ holds for all $P_{\mathbf{X}}$ with $\mathbb{E}[\mathbf{b}(\mathbf{X})] \leq \alpha$, and the distortion measure is separable. \square

Theorem 3.22 (Gaussian approximation, optimal symbol-by-symbol code). *Assume $\mathbb{E}[\mathbf{d}^3(\mathbf{S}, \mathbf{Z}^*)] < \infty$. Under restrictions (i)-(vi),*

$$D_1(n, \epsilon, \alpha) = D(C(\alpha)) + \sqrt{\frac{\mathcal{W}_1(\alpha)}{n}} Q^{-1}(\epsilon) + \frac{\theta_1(n)}{n} \quad (3.167)$$

$$\mathcal{W}_1(\alpha) = \text{Var}[\mathbf{d}(\mathbf{S}, \mathbf{Z}^*)] \quad (3.168)$$

where

$$\theta_1(n) \leq O(1) \quad (3.169)$$

Moreover, if there is no power constraint,

$$\theta_1(n) \geq \frac{D'(R)}{R^2} \theta(n) \quad (3.170)$$

$$\mathcal{W}_1 = \mathcal{W}(1) \quad (3.171)$$

where $\theta(n)$ is that in Theorem 3.10.

If $\text{Var}[\mathbf{d}(\mathbf{S}, \mathbf{Z}^*)] > 0$ and $\mathcal{S}, \hat{\mathcal{S}}$ are finite, then

$$\theta_1(n) \geq O(1) \quad (3.172)$$

Proof. Since the third absolute moment of $\mathbf{d}(S_i, Z_i^*)$ is finite, the achievability part of the result, namely, (3.167) with the remainder satisfying (3.169), follows by a straightforward application of the Berry-Esséen bound to (3.163), provided that $\text{Var}[\mathbf{d}(S_i, Z_i^*)] > 0$. If $\text{Var}[\mathbf{d}(S_i, Z_i^*)] = 0$, it follows trivially from (3.163).

To show the converse in (3.170), observe that since the set of all (n, n, d, ϵ) codes includes all (n, d, ϵ) symbol-by-symbol codes, we have $D(n, n, \epsilon) \leq D_1(n, \epsilon)$. Since $Q^{-1}(\epsilon)$ is positive or negative depending on whether $\epsilon < \frac{1}{2}$ or $\epsilon > \frac{1}{2}$, using (3.102) we conclude that we must necessarily have (3.171), which is, in fact, a consequence of conditions (b) in Section 2.2 and (c) in Section 3.2 and (vi). Now, (3.170) is simply the converse part of (3.101).

The proof of the refined converse in (3.172) is relegated to Appendix C.3. \square

In the absence of a cost constraint, Theorem 3.22 shows that if the source and the channel are

probabilistically matched in the sense of [12], then not only does symbol-by-symbol transmission achieve the minimum average distortion, but also the dispersion of JSCC (see (3.171)). In other words, not only do such symbol-by-symbol codes attain the minimum average distortion but also the variance of distortions at the decoder's output is the minimum achievable among all codes operating at that average distortion. In contrast, if there is an average cost constraint, the symbol-by-symbol codes considered in Theorem 3.22 probably do not attain the minimum excess distortion achievable among all blocklength- n codes, not even asymptotically. Indeed, as observed in [4], for the transmission of an equiprobable source over an AWGN channel under the average power constraint and the average block error probability performance criterion, the strong converse does not hold and the second-order term is of order $n^{-\frac{1}{3}}$, not $n^{-\frac{1}{2}}$, as in (3.167).

Two conspicuous examples that satisfy the probabilistic matching condition (vi), so that symbol-by-symbol coding is optimal in terms of average distortion, are the transmission of a binary equiprobable source over a binary-symmetric channel provided the desired bit error rate is equal to the crossover probability of the channel [70, Sec.11.8], [46, Problem 7.16], and the transmission of a Gaussian source over an additive white Gaussian noise channel under the mean-square error distortion criterion, provided that the tolerable source signal-to-noise ratio attainable by an estimator is equal to the signal-to-noise ratio at the output of the channel [71]. We dissect these two examples next. After that, we will discuss two additional examples where uncoded transmission is optimal.

3.8.2 Uncoded transmission of a BMS over a BSC

In the setup of Section 3.6, if the binary source is unbiased ($p = \frac{1}{2}$), then $C = 1 - h(\delta)$, $R(d) = 1 - h(d)$, and $D(C) = \delta$. If the encoder and the decoder are both identity mappings (uncoded transmission), the resulting joint distribution satisfies condition (vi). As is well known, regardless of the blocklength, the uncoded symbol-by-symbol scheme achieves the minimum bit error rate (averaged over source and channel). Here, we are interested instead in examining the excess distortion probability criterion. For example, consider an application where, if the fraction of erroneously received bits exceeds a certain threshold, then the entire output packet is useless.

Using (3.102) and (3.168), it is easy to verify that

$$\mathcal{W}(1) = \mathcal{W}_1 = \delta(1 - \delta) \quad (3.173)$$

that is, uncoded transmission is optimal in terms of dispersion, as anticipated in (3.171). Moreover, uncoded transmission attains the minimum bit error rate threshold $D(n, n, \epsilon)$ achievable among all

codes operating at blocklength n , regardless of the allowed ϵ , as the following result demonstrates.

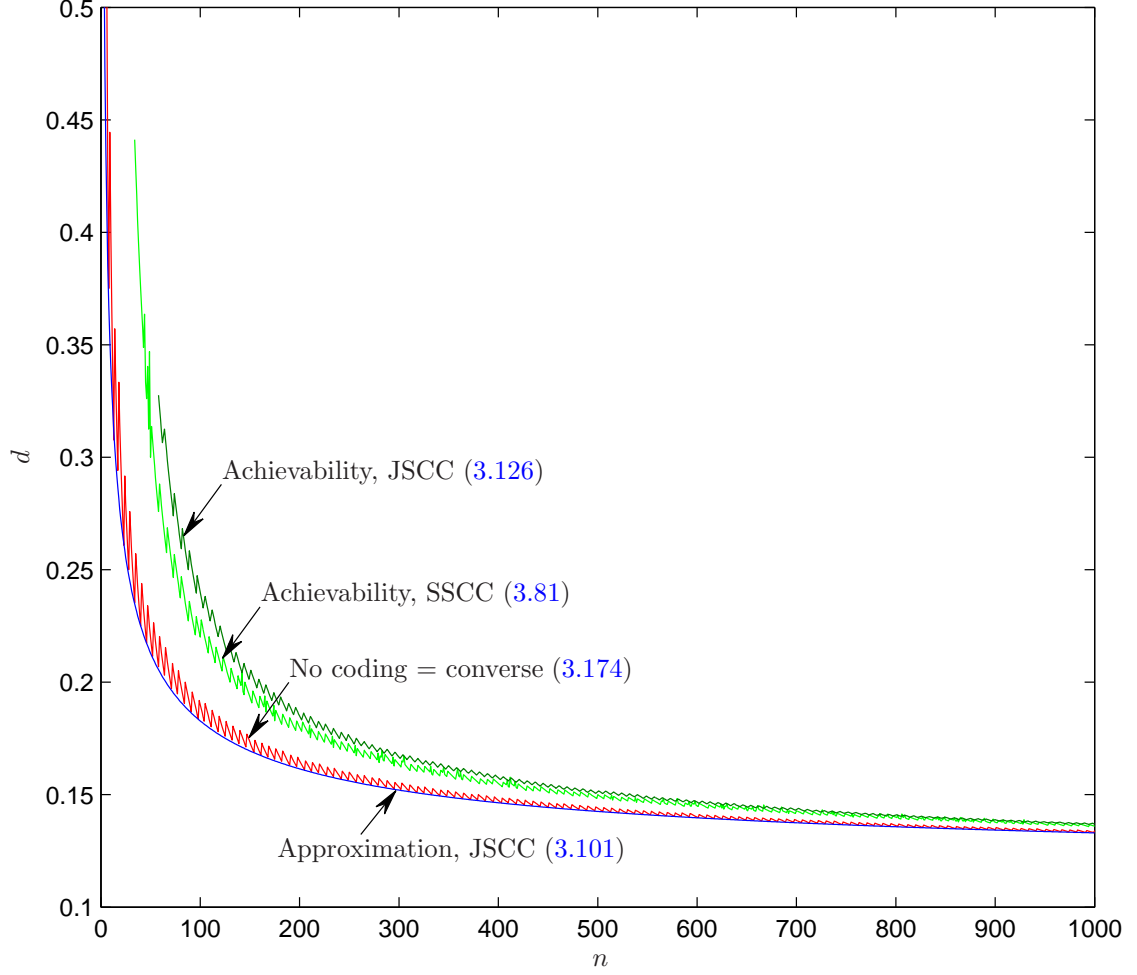


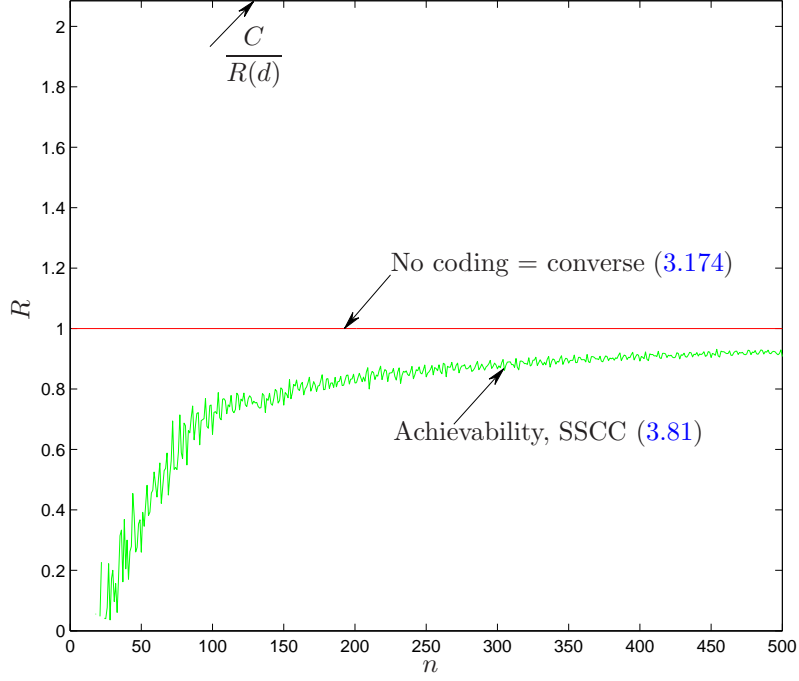
Figure 3.7: Distortion-blocklength tradeoff for the transmission of a fair BMS over a BSC with crossover probability $\delta = 0.11$ and $R = 1$, $\epsilon = 10^{-2}$.

Theorem 3.23 (BMS-BSC, symbol-by-symbol code). *Consider the the symbol-by-symbol scheme which is uncoded if $p \geq \delta$ and whose decoder always outputs the all-zero vector if $p < \delta$. It achieves, at blocklength n and excess distortion probability ϵ , regardless of $0 \leq p \leq \frac{1}{2}$, $\delta \leq \frac{1}{2}$,*

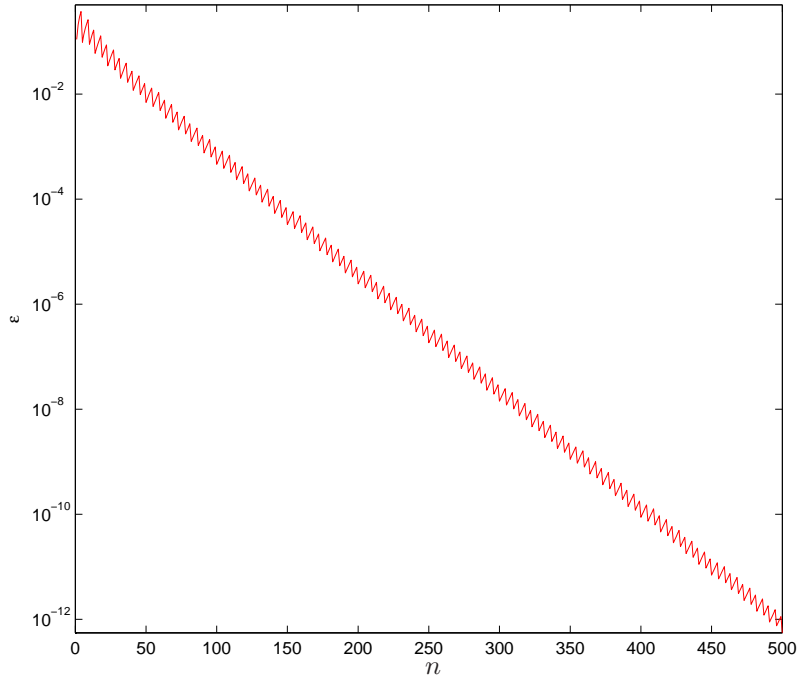
$$D_1(n, \epsilon) = \min \left\{ d: \sum_{t=0}^{\lfloor nd \rfloor} \binom{n}{t} \min\{p, \delta\}^t (1 - \min\{p, \delta\})^{n-t} \geq 1 - \epsilon \right\} \quad (3.174)$$

Moreover, if the source is equiprobable ($p = \frac{1}{2}$),

$$D_1(n, \epsilon) = D(n, n, \epsilon) \quad (3.175)$$



(a)



(b)

Figure 3.8: Rate-blocklength tradeoff (a) for the transmission of a fair BMS over a BSC with crossover probability $\delta = 0.11$ and $d = 0.22$. The excess-distortion probability ϵ is set to be the one achieved by the uncoded scheme (b).

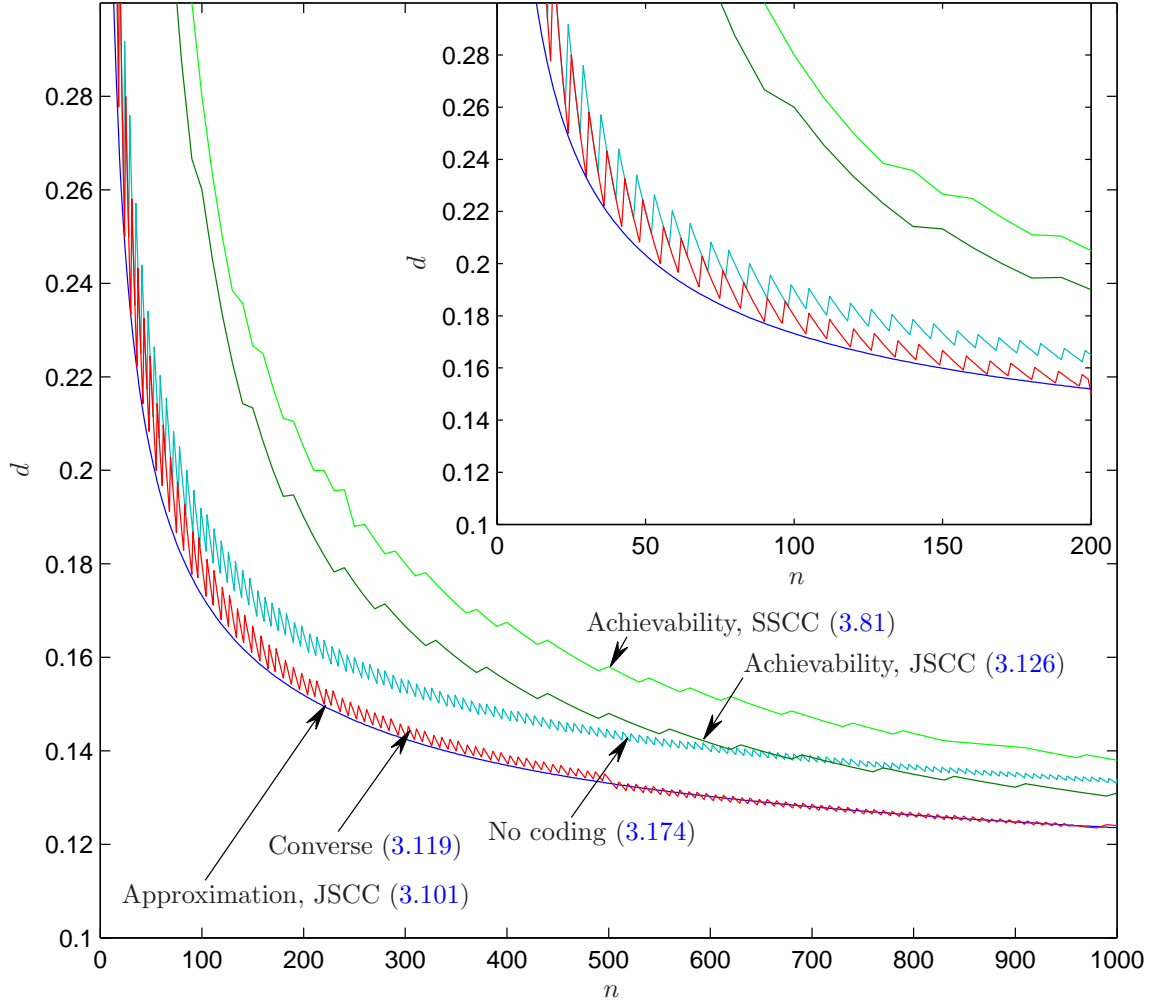


Figure 3.9: Distortion-blocklength tradeoff for the transmission of a BMS with $p = \frac{2}{5}$ over a BSC with crossover probability $\delta = 0.11$ and $R = 1$, $\epsilon = 10^{-2}$.

Proof. Direct calculation yields (3.174). To show (3.175), let us compare $d^* = D_1(n, \epsilon)$ with the conditions imposed on d by Theorem 3.13. Comparing (3.174) to (3.124), we see that either

- (a) equality in (3.174) is achieved, $r^* = nd^*$, $\lambda = 0$, and (plugging $k = n$ into (3.123))

$$\left\langle \frac{n}{nd^*} \right\rangle \leq \left\langle \frac{n}{\lfloor nd \rfloor} \right\rangle \quad (3.176)$$

thereby implying that $d \geq d^*$, or

- (b) $r^* = nd^* - 1$, $\lambda > 0$, and (3.123) becomes

$$\lambda \left\langle \frac{n}{nd^*} \right\rangle + \left\langle \frac{n}{nd^* - 1} \right\rangle \leq \left\langle \frac{n}{\lfloor nd \rfloor} \right\rangle \quad (3.177)$$

which also implies $d \geq d^*$. To see this, note that $d < d^*$ would imply $\lfloor nd \rfloor \leq nd^* - 1$ since nd^* is an integer, which in turn would require (according to (3.177)) that $\lambda \leq 0$, which is impossible.

□

For the transmission of the fair binary source over a BSC, Fig. 3.7 shows the distortion achieved by the uncoded scheme, the separated scheme and the JSCC scheme of Theorem 3.14 versus n for a fixed excess-distortion probability $\epsilon = 0.01$. The no coding / converse curve in Fig. 3.7 depicts one of those singular cases where the non-asymptotic fundamental limit can be computed precisely. As evidenced by this curve, the fundamental limits need not be monotonic with blocklength.

Figure 3.8(a) shows the rate achieved by separate coding when $d > \delta$ is fixed, and the excess-distortion probability ϵ , shown in Fig. 3.8(b), is set to be the one achieved by uncoded transmission, namely, (3.174). Figure 3.8(a) highlights the fact that at short blocklengths (say $n \leq 100$) separate source/channel coding is vastly suboptimal. As the blocklength increases, the performance of the separated scheme approaches that of the no-coding scheme, but according to Theorem 3.23 it can never outperform it. Had we allowed the excess distortion probability to vanish sufficiently slowly, the JSCC curve would have approached the Shannon limit as $n \rightarrow \infty$. However, in Figure 3.8(a), the exponential decay in ϵ is such that there is indeed an asymptotic rate penalty as predicted in [62].

For the biased binary source with $p = \frac{2}{5}$ and BSC with crossover probability 0.11, Figure 3.9 plots the maximum distortion achieved with probability 0.99 by the uncoded scheme, which in this case is asymptotically suboptimal. Nevertheless, uncoded transmission performs remarkably well in the displayed range of blocklengths, achieving the converse almost exactly at blocklengths less

than 100, and outperforming the JSCC achievability result in Theorem 3.14 at blocklengths as long as 700. This example substantiates that even in the absence of a probabilistic match between the source and the channel, symbol-by-symbol transmission, though asymptotically suboptimal, might outperform SSCC and even our random JSCC achievability bound in the finite blocklength regime.

3.8.3 Symbol-by-symbol coding for lossy transmission of a GMS over an AWGN channel

In the setup of Section 3.7, using (3.137) and (3.138), we find that

$$D(C(P)) = \frac{\sigma_S^2}{1+P} \quad (3.178)$$

The next result characterizes the distribution of the distortion incurred by the symbol-by-symbol scheme that attains the minimum average distortion.

Theorem 3.24 (GMS-AWGN, symbol-by-symbol code). *The following symbol-by-symbol transmission scheme in which the encoder and the decoder are the amplifiers:*

$$f_1(s) = as, \quad a^2 = \frac{P\sigma_N^2}{\sigma_S^2} \quad (3.179)$$

$$g_1(y) = by, \quad b = \frac{a\sigma_S^2}{a^2\sigma_S^2 + \sigma_N^2} \quad (3.180)$$

is an (n, d, ϵ, P) symbol-by-symbol code (with average cost constraint) such that

$$\mathbb{P}[W_0^n D(C(P)) > nd] = \epsilon \quad (3.181)$$

where W_0^n is chi-square distributed with n degrees of freedom.

Note that (3.181) is a particularization of (3.163). Using (3.181), we find that

$$\mathcal{W}_1(P) = 2 \frac{\sigma_S^4}{(1+P)^2} \quad (3.182)$$

On the other hand, using (3.102), we compute

$$\mathcal{W}(1, P) = 2 \frac{\sigma_S^4}{(1+P)^2} \left(2 - \frac{1}{(1+P)^2} \right) \quad (3.183)$$

$$> \mathcal{W}_1(P) \quad (3.184)$$

The difference between (3.184) and (3.171) is due to the fact that the optimal symbol-by-symbol code in Theorem 3.24 obeys an average power constraint, rather than the more stringent maximal power constraint of Theorem 3.10, so it is not surprising that for the practically interesting case $\epsilon < \frac{1}{2}$ the symbol-by-symbol code can outperform the best code obeying the maximal power constraint. Indeed, in the range of blocklengths displayed in Figure 3.10, the symbol-by-symbol code even outperforms the converse for codes operating under a maximal power constraint.

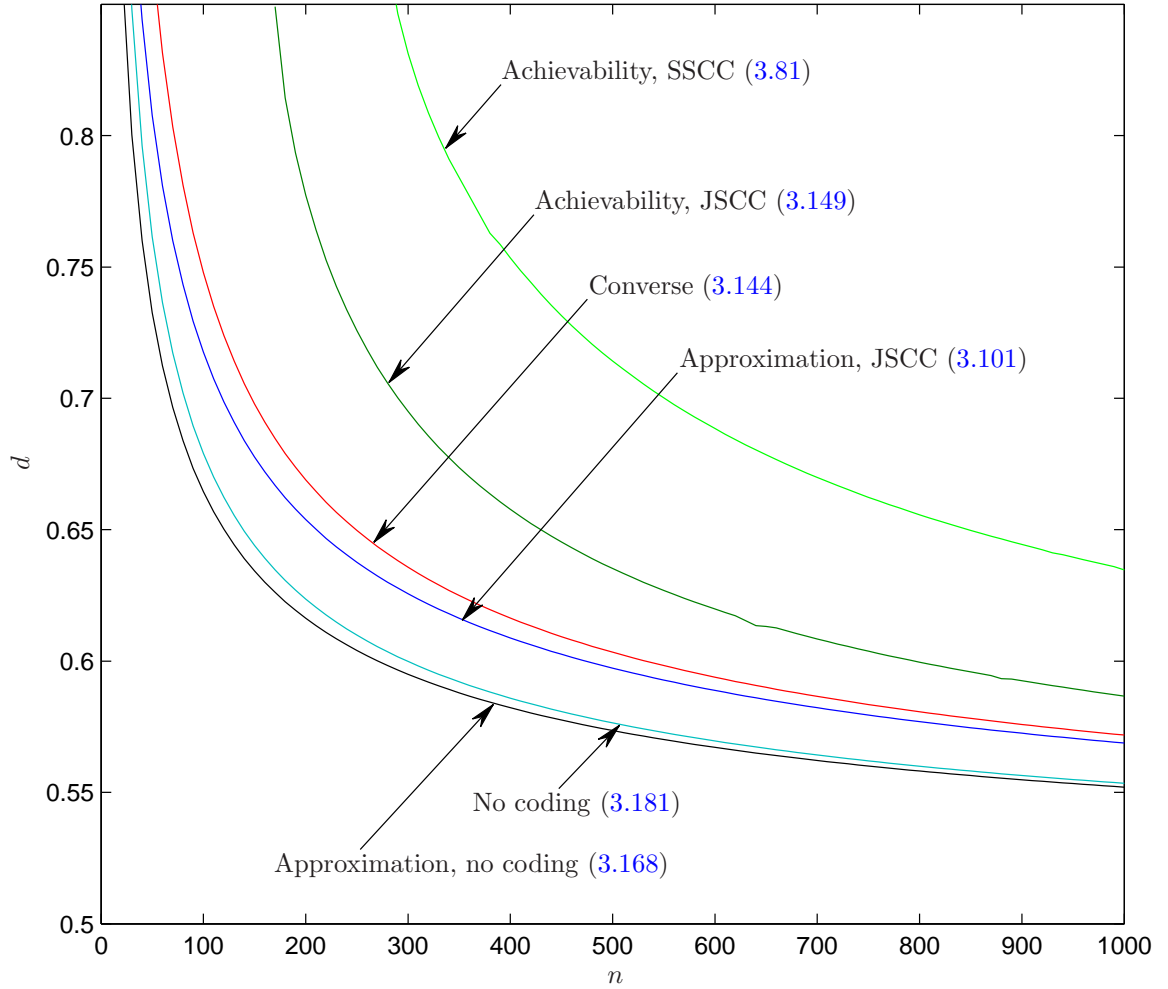


Figure 3.10: Distortion-blocklength tradeoff for the transmission of a GMS over an AWGN channel with $\frac{P}{\sigma_N^2} = 1$ and $R = 1$, $\epsilon = 10^{-2}$.

3.8.4 Uncoded transmission of a discrete memoryless source (DMS) over a discrete erasure channel (DEC) under erasure distortion measure

For a discrete source equiprobable on \mathcal{S} , the single-letter erasure distortion measure is defined as the following mapping $\mathbf{d}: \mathcal{S} \times \{\mathcal{S}, \mathbf{e}\} \mapsto [0, \infty]$:⁶

$$\mathbf{d}(\mathbf{s}, \mathbf{z}) = \begin{cases} 0 & \mathbf{z} = \mathbf{s} \\ \log |\mathcal{S}| & \mathbf{z} = \mathbf{e} \\ \infty & \text{otherwise} \end{cases} \quad (3.185)$$

For any $0 \leq d \leq \log |\mathcal{S}|$, the rate-distortion function is achieved by

$$P_{Z^*|S=s}(\mathbf{z}) = \begin{cases} 1 - \frac{d}{\log |\mathcal{S}|} & \mathbf{z} = \mathbf{s} \\ \frac{d}{\log |\mathcal{S}|} & \mathbf{z} = \mathbf{e} \end{cases} \quad (3.186)$$

The rate-distortion function and the \mathbf{d} -tilted information are given by, respectively,

$$R(d) = \log |\mathcal{S}| - d \quad (3.187)$$

$$j_{\mathcal{S}}(\mathbf{S}, d) = \log |\mathcal{S}| - d \quad (3.188)$$

The channel that is matched to the equiprobable DMS with the erasure distortion measure is the DEC, whose single-letter transition probability kernel $P_{Y|X}: \mathcal{A} \mapsto \{\mathcal{A}, \mathbf{e}\}$ is

$$P_{Y|X=x}(y) = \begin{cases} 1 - \delta & y = x \\ \delta & y = \mathbf{e} \end{cases} \quad (3.189)$$

and whose capacity is given by $C = \log |\mathcal{A}| - \delta$, achieved by equiprobable P_{X^*} . If $\mathcal{S} = \mathcal{A}$, we find that $D(C) = \delta \log |\mathcal{S}|$, and

$$\mathcal{W}_1 = \delta (1 - \delta) \log^2 |\mathcal{S}| \quad (3.190)$$

⁶The distortion measure in (3.185) is a scaled version of the erasure distortion measure found in literature, e.g. [7].

3.8.5 Symbol-by-symbol transmission of a DMS over a DEC under logarithmic loss

Let the source alphabet \mathcal{S} be finite, and let the reproduction alphabet $\hat{\mathcal{S}}$ be the set of all probability distributions on \mathcal{S} . The single-letter logarithmic loss distortion measure $\mathbf{d}: \mathcal{S} \times \hat{\mathcal{S}} \mapsto \mathbb{R}^+$ is defined by [72, 73]

$$\mathbf{d}(s, P_Z) = \imath_Z(s) \quad (3.191)$$

Curiously, for any $0 \leq d \leq H(\mathbf{S})$, the rate-distortion function and the \mathbf{d} -tilted information are given respectively by

$$R(d) = H(\mathbf{S}) - d \quad (3.192)$$

$$J_{\mathbf{S}}(\mathbf{s}, d) = \imath_{\mathbf{S}}(\mathbf{s}) - d \quad (3.193)$$

which coincides with (3.192) and (3.193) if the source is not equiprobable. In fact, the rate-distortion function is achieved by,

$$P_{P_Z^* | \mathbf{S}=\mathbf{s}}(P_Z) = \begin{cases} \frac{d}{H(\mathbf{S})} & P_Z = P_{\mathbf{S}} \\ 1 - \frac{d}{H(\mathbf{S})} & P_Z = 1_{\mathbf{S}}(\mathbf{s}) \end{cases} \quad (3.194)$$

and the channel that is matched to the equiprobable source under logarithmic loss is exactly the DEC in (3.189). Of course, unlike Section 3.8.4, the decoder we need is a simple one-to-one function that outputs $P_{\mathbf{S}}$ if the channel output is \mathbf{e} , and $1_{\mathbf{S}}(\mathbf{y})$ otherwise, where $\mathbf{y} \neq \mathbf{e}$ is the output of the DEC. Finally, it is easy to verify that the distortion-dispersion function of symbol-by-symbol coding under logarithmic loss is the same as that under erasure distortion and is given by (3.190).

3.9 Conclusion

In this chapter we gave a non-asymptotic analysis of joint source-channel coding including several achievability and converse bounds, which hold in wide generality and are tight enough to determine the dispersion of joint source-channel coding for the transmission of an abstract memoryless source over either a DMC or a Gaussian channel, under an arbitrary fidelity measure. We also investigated the penalty incurred by separate source-channel coding using both the source-channel dispersion and the particularization of our new bounds to (i) the binary source and the binary symmetric channel with bit error rate fidelity criterion and (ii) the Gaussian source and Gaussian channel under mean-square error distortion. Finally, we showed cases where symbol-by-symbol (uncoded)

transmission beats any other known scheme in the finite blocklength regime even when the source-channel matching condition is not satisfied.

The approach taken in this chapter to analyze the non-asymptotic fundamental limits of lossy joint source-channel coding is two-fold. Our new achievability and converse bounds apply to abstract sources and channels and allow for memory, while the asymptotic analysis of the new bounds leading to the dispersion of JSCC is focused on the most basic scenario of transmitting a stationary memoryless source over a stationary memoryless channel.

The major results and conclusions are the following.

- 1) Leveraging the concept of \mathbf{d} -tilted information (Definition 2.1), a general new converse bound (Theorem 3.1) generalizes the source coding bound in Theorem 2.12 to the joint source-channel coding setup.
- 2) The converse result in Theorem 3.4 capitalizes on two simple observations, namely, that any (d, ϵ) lossy code can be converted to a list code with list error probability ϵ , and that a binary hypothesis test between P_{SXY} and an auxiliary distribution on the same space can be constructed by choosing P_{SXY} when there is no list error. We have generalized the conventional notion of list, to allow the decoder to output a possibly uncountable set of source realizations.
- 3) As evidenced by our numerical results, the converse result in Theorem 3.5, which applies to those channels satisfying a certain symmetry condition and which is a consequence of the hypothesis testing converse in Theorem 3.4, can outperform the \mathbf{d} -tilted information converse in Corollary 3.3. Nevertheless, it is Corollary 3.3 that lends itself to analysis more easily and that leads to the JSCC dispersion for the general DMC.
- 4) Our random-coding-based achievability bound (Theorem 3.7) provides insights into the degree of separation between the source and the channel codes required for optimal performance in the finite blocklength regime. More precisely, it reveals that the dispersion of JSCC can be achieved in the class of (M, d, ϵ) JSCC codes (Definition 3.7). As in separate source/channel coding, in (M, d, ϵ) coding the inner channel coding block is connected to the outer source coding block by a noiseless link of capacity $\log M$, but unlike SSCC, the channel (resp. source) code can be chosen based on the knowledge of the source (resp. channel). The conventional SSCC in which the source code is chosen without knowledge of the channel and the channel code is chosen without knowledge of the source, although known to achieve the asymptotic fundamental limit of joint source-channel coding under certain quite weak conditions, is in general suboptimal in the finite

blocklength regime.

- 5) For the transmission of a stationary memoryless source over a stationary memoryless channel, the Gaussian approximation in Theorem 3.10 (neglecting the remainder $\theta(n)$) provides a simple estimate of the maximal non-asymptotically achievable joint source-channel coding rate. Appealingly, the dispersion of joint source-channel coding decomposes into two terms, the channel dispersion and the source dispersion. Thus, only two channel attributes, the capacity and dispersion, and two source attributes, the rate-distortion and rate-dispersion functions, are required to compute the Gaussian approximation to the maximal JSCC rate.
- 6) In those curious cases where the source and the channel are probabilistically matched so that symbol-by-symbol coding attains the minimum possible average distortion, Theorem 3.22 ensures that it also attains the dispersion of joint source-channel coding, that is, symbol-by-symbol coding results in the minimum variance of distortions among all codes operating at that average distortion.
- 7) Even in the absence of a probabilistic match between the source and the channel, symbol-by-symbol transmission, though asymptotically suboptimal, might outperform separate source-channel coding and joint source-channel random coding in the finite blocklength regime.

Chapter 4

Noisy lossy source coding

4.1 Introduction

The noisy source coding setting where the encoder has access only to a noise-corrupted version X of a source S , while the distortion is measured with respect to the true source (see Fig. 1.4), was first discussed by Dobrushin and Tsybakov [74], who showed that when the goal is to minimize the average distortion, the noisy source coding problem is asymptotically equivalent to a particular noiseless source coding problem. More precisely, for stationary memoryless sources observed through a stationary memoryless channel under a separable distortion measure, the noisy rate-distortion function is given by

$$R(d) = \min_{\substack{P_{Z|X}: \\ \mathbb{E}[d(S,Z)] \leq d \\ S-X-Z}} I(X;Z) \quad (4.1)$$

$$= \min_{\substack{P_{Z|X}: \\ \mathbb{E}[\bar{d}(X,Z)] \leq d}} I(X;Z) \quad (4.2)$$

where

$$\bar{d}(a,b) = \mathbb{E}[d(S,b)|X=a] \quad (4.3)$$

i.e. in the limit of infinite blocklengths, the problem is equivalent to the classical lossy source coding problem where the distortion measure is the conditional average of the original distortion measure given the noisy observation of the source. Berger [53, p.79] used the modified distortion measure (4.3) to streamline the proof of (4.2). Witsenhausen [75] explored the strength of distortion measures defined through conditional expectations such as in (4.3) to treat various so-called indirect

rate distortion problems.

Sakrison [76] showed that if both the source and its noise-corrupted version take values in a separable Hilbert space and the fidelity criterion is mean squared error, then asymptotically, an optimal code can be constructed by first creating a minimum mean-square estimate of the source outcome based on its noisy observation, and then quantizing this estimate as if it were noise-free. Wolf and Ziv [77] showed that Sakrison's result holds even nonasymptotically, namely, that the minimum average distortion achievable in one-shot noisy compression of the object S can be written as

$$D^*(M) = \mathbb{E} [|S - \mathbb{E}[S|X]|^2] + \inf_{f,c} \mathbb{E} [|c(f(X)) - \mathbb{E}[S|X]|^2] \quad (4.4)$$

where the infimum is over all encoders $f: \mathcal{X} \mapsto \{1, \dots, M\}$ and all decoders $c: \{1, \dots, M\} \mapsto \widehat{\mathcal{M}}$, and \mathcal{X} and $\widehat{\mathcal{M}}$ are the alphabets of the channel output and the decoder output, respectively. It is important to note that (4.4) is a direct consequence of the choice of the mean squared error distortion and does not hold in general. For vector quantization of a Gaussian signal corrupted by an additive independent Gaussian noise under weighted squared error distortion measure, Ayanoglu [78] found explicit expressions for the optimum quantizer values and the optimum quantization rule. Wolf and Ziv's result was extended to waveform vector quantization under weighted quadratic distortion measures and to autoregressive vector quantization under the Itakura-Saito distortion measure by Ephraim and Gray [79], as well as to a model in which the encoder and decoder have access to the history of their past inputs and outputs, allowing exploitation of inter-block dependence, by Fisher, Gibson and Koo [80]. Thus, the cascade of the optimal estimator followed by the optimal quantizer achieves the minimum average distortion in those settings as well.

Under the logarithmic loss distortion measure [73], the noisy source coding problem reduces to the information bottleneck problem [81].

In this chapter, we give new nonasymptotic achievability and converse bounds for the noisy source coding problem, which generalize the noiseless source coding bounds in Chapter 2. We observe that at finite blocklengths, the noisy coding problem is in general not equivalent to the noiseless coding problem with the modified distortion measure in (4.3). Essentially, the reason is that taking the expectation in (4.3) dismisses the randomness introduced by the noisy channel in Fig. 1.4, which nonasymptotically cannot be neglected. That additional randomness slows down the rate of approach to the asymptotic fundamental limit in the noisy source coding problem compared to the asymptotically equivalent noiseless problem. Specifically, we show that for noisy source coding of a discrete stationary memoryless source over a discrete stationary memoryless channel under a

separable distortion measure, $\mathcal{V}(d)$ in (1.9) is replaced by the noisy rate-dispersion function $\tilde{\mathcal{V}}(d)$, which can be expressed as

$$\tilde{\mathcal{V}}(d) = \mathcal{V}(d) + \lambda^{*2} \text{Var} [\mathbb{E} [\mathbf{d}(S, Z^*) - \bar{\mathbf{d}}(X, Z^*) | S, X]] \quad (4.5)$$

where $\lambda^* = -R'(d)$, and Z^* denotes the reproduction random variable that achieves the rate-distortion function (4.2).

The rest of the chapter is organized as follows. After introducing the basic definitions in Section 4.2, we proceed to show new general nonasymptotic converse and achievability bounds in Sections 4.3 and 4.4, respectively, along with their asymptotic analysis in Section 4.5. Finally, the example of a binary source observed through an erasure channel is considered in Section 4.6.

Parts of this chapter are presented in [19, 82].

4.2 Definitions

Consider the setup in Fig. 1.4 where we are given the distribution P_S on the alphabet \mathcal{M} and the transition probability kernel $P_{X|S}: \mathcal{M} \rightarrow \mathcal{X}$. We are also given the distortion measure $\mathbf{d}: \mathcal{M} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]$, where $\widehat{\mathcal{M}}$ is the representation alphabet. An (M, d, ϵ) code is a pair of mappings $P_{U|X}: \mathcal{X} \mapsto \{1, \dots, M\}$ and $P_{Z|U}: \{1, \dots, M\} \mapsto \widehat{\mathcal{M}}$ such that $\mathbb{P}[\mathbf{d}(S, Z) > d] \leq \epsilon$.

Define

$$\mathbb{R}_{S,X}(d) \triangleq \inf_{\substack{P_{Z|X}: \\ \mathbb{E}[\bar{\mathbf{d}}(X,Z)] \leq d}} I(X; Z) \quad (4.6)$$

where $\bar{\mathbf{d}}: \mathcal{X} \times \widehat{\mathcal{M}} \mapsto [0, +\infty]$ is given by

$$\bar{\mathbf{d}}(x, z) \triangleq \mathbb{E}[\mathbf{d}(S, z) | X = x] \quad (4.7)$$

and, as in Section 2.2, assume that the infimum is achieved by some $P_{Z^*|X}$ such that the constraint is satisfied with equality. Noting that this assumption guarantees differentiability of $\mathbb{R}_{S,X}(d)$, denote

$$\lambda^* = -\mathbb{R}'_{S,X}(d) \quad (4.8)$$

Furthermore, define, for an arbitrary $P_{Z|X}$

$$\bar{\mathbf{d}}_Z(s|x) \triangleq \mathbb{E}[\mathbf{d}(S, Z) | X = x, S = s] \quad (4.9)$$

where the expectation is with respect $P_{Z|XS} = P_{Z|X}$.

Definition 4.1 (noisy \mathbf{d} -tilted information). *For $d > d_{\min}$, the noisy \mathbf{d} -tilted information in $s \in \mathcal{M}$ given observation $x \in \mathcal{X}$ is defined as*

$$\tilde{j}_{S,X}(s, x, d) \triangleq D(P_{Z^*|X=x} \| P_{Z^*}) + \lambda^* \bar{\mathbf{d}}_{Z^*}(s|x) - \lambda^* d \quad (4.10)$$

where $P_{Z^*|X}$ achieves the infimum in (4.6).

As we will see, the intuitive meaning of the noisy \mathbf{d} -tilted information is the number of bits required to represent s within distortion d given observation x .

For the asymptotically equivalent noiseless source coding problem, we know that almost surely (Theorem 2.1)

$$j_X(x, d) = \iota_{X;Z^*}(x; Z^*) + \lambda^* \bar{\mathbf{d}}(x, Z^*) - \lambda^* d \quad (4.11)$$

$$= D(P_{Z^*|X=x} \| P_{Z^*}) + \lambda^* \mathbb{E} [\bar{\mathbf{d}}(x, Z^*) | X = x] - \lambda^* d \quad (4.12)$$

where $j_X(x, d)$ is the $\bar{\mathbf{d}}$ -tilted information in $x \in \mathcal{X}$ (defined in (2.6)). Therefore

$$\tilde{j}_{S,X}(s, x, d) = j_X(x, d) + \lambda^* \bar{\mathbf{d}}_{Z^*}(s|x) - \lambda^* \mathbb{E} [\bar{\mathbf{d}}_{Z^*}(S|x) | X = x] \quad (4.13)$$

Trivially

$$\mathbb{R}_{S,X}(d) = \mathbb{E} [\tilde{j}_{S,X}(S, X, d)] \quad (4.14)$$

$$= \mathbb{E} [j_X(X, d)] \quad (4.15)$$

For a given distribution $P_{\bar{Z}}$ on $\widehat{\mathcal{M}}$ and $\lambda > 0$ define the transition probability kernel (cf. (2.23))

$$dP_{\bar{Z}^*|X=x}(z) = \frac{dP_{\bar{Z}}(z) \exp(-\lambda \bar{\mathbf{d}}(x, z))}{\mathbb{E} [\exp(-\lambda \bar{\mathbf{d}}(x, \bar{Z}))]} \quad (4.16)$$

and define the function

$$\tilde{J}_{\bar{Z}}(s, x, \lambda) \triangleq D(P_{\bar{Z}^*|X=x} \| P_{\bar{Z}}) + \lambda \bar{\mathbf{d}}_{\bar{Z}^*}(s|x) \quad (4.17)$$

$$= J_{\bar{Z}}(x, \lambda) + \lambda \bar{\mathbf{d}}_{\bar{Z}^*}(s|x) - \lambda \mathbb{E} [\bar{\mathbf{d}}_{\bar{Z}^*}(S|x) | X = x] \quad (4.18)$$

where $J_{\bar{Z}}(x, \lambda)$ is defined in (2.24). Similar to (2.26), we refer to the function

$$\tilde{J}_{\bar{Z}}(s, x, \lambda) - \lambda d \quad (4.19)$$

the *generalized noisy \mathbf{d} -tilted information*.

4.3 Converse bounds

Theorem 4.1 (Converse). *Any (M, d, ϵ) code must satisfy*

$$\epsilon \geq \inf_{P_{Z|X}} \sup_{\gamma \geq 0, P_{\bar{X}|\bar{Z}}} \left\{ \mathbb{P} \left[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(\mathbf{d}(S, Z) - d) \geq \log M + \gamma \right] - \exp(-\gamma) \right\} \quad (4.20)$$

where (recall notation (2.22))

$$\iota_{\bar{X}|\bar{Z}\|X}(x; z) \triangleq \log \frac{dP_{\bar{X}|\bar{Z}=z}}{dP_X}(x) \quad (4.21)$$

Proof. Let the encoder and decoder be the random transformations $P_{U|X}$ and $P_{Z|U}$, where U takes values in $\{1, \dots, M\}$. Recall notation (2.33):

$$B_d(s) \triangleq \left\{ z \in \widehat{\mathcal{M}} : \mathbf{d}(s, z) \leq d \right\} \quad (4.22)$$

We have, for any $\gamma \geq 0$

$$\begin{aligned} & \mathbb{P} \left[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(d(S, Z) - d) \geq \log M + \gamma \right] \\ &= \mathbb{P} \left[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(d(S, Z) - d) \geq \log M + \gamma, d(S, Z) > d \right] \\ &+ \mathbb{P} \left[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(d(S, Z) - d) \geq \log M + \gamma, d(S, Z) \leq d \right] \end{aligned} \quad (4.23)$$

$$= \mathbb{P}[d(S, Z) > d] + \mathbb{P}[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) \geq \log M + \gamma, d(S, Z) \leq d] \quad (4.24)$$

$$\leq \epsilon + \mathbb{P}[\iota_{\bar{X}|\bar{Z}\|X}(X; Z) \geq \log M + \gamma] \quad (4.25)$$

$$\leq \epsilon + \frac{\exp(-\gamma)}{M} \mathbb{E}[\exp(\iota_{\bar{X}|\bar{Z}\|X}(X; Z))] \quad (4.26)$$

$$\leq \epsilon + \frac{\exp(-\gamma)}{M} \sum_{u=1}^M \sum_{z \in \widehat{\mathcal{M}}} P_{Z|U}(z|u) \sum_{x \in \mathcal{X}} P_X(x) \exp(\iota_{\bar{X}|\bar{Z}\|X}(x; z)) \quad (4.27)$$

$$= \epsilon + \frac{\exp(-\gamma)}{M} \sum_{u=1}^M \sum_{z \in \widehat{\mathcal{M}}} P_{Z|U}(z|u) \sum_{x \in \mathcal{X}} P_{\bar{X}|\bar{Z}}(x|z) \quad (4.28)$$

$$= \epsilon + \exp(-\gamma) \quad (4.29)$$

where

- (4.23) is by direct solution for the supremum;
- (4.26) is by Markov's inequality;
- (4.27) follows by upper-bounding

$$P_{U|X}(u|x) \leq 1 \quad (4.30)$$

for every $(x, u) \in \mathcal{M} \times \{1, \dots, M\}$.

Finally, (4.20) follows by choosing γ and $P_{\bar{X}|\bar{Z}}$ that give the tightest bound and $P_{Z|X}$ that gives the weakest in order to obtain a code-independent converse. \square

In our asymptotic analysis, we will use the following bound with suboptimal choices of λ and $P_{\bar{X}|\bar{Z}}$.

Corollary 4.2. *Any (M, d, ϵ) code must satisfy*

$$\epsilon \geq \sup_{\gamma \geq 0, P_{\bar{X}|\bar{Z}}} \left\{ \mathbb{E} \left[\inf_{z \in \widehat{\mathcal{M}}} \mathbb{P} \left[\iota_{\bar{X}|\bar{Z}\|X}(X; z) + \sup_{\lambda \geq 0} \lambda(d(S, z) - d) \geq \log M + \gamma | X \right] \right] - \exp(-\gamma) \right\} \quad (4.31)$$

Proof. We weaken (4.20) using

$$\begin{aligned} & \inf_{P_{Z|X}} \mathbb{P} \left[\imath_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(\mathbf{d}(S, Z) - d) \geq \log M + \gamma \right] \\ &= \mathbb{E} \left[\inf_{P_{Z|X}} \mathbb{P} \left[\imath_{\bar{X}|\bar{Z}\|X}(X; Z) + \sup_{\lambda \geq 0} \lambda(\mathbf{d}(S, Z) - d) \geq \log M + \gamma | X \right] \right] \end{aligned} \quad (4.32)$$

$$= \mathbb{E} \left[\inf_{z \in \widehat{\mathcal{M}}} \mathbb{P} \left[\imath_{\bar{X}|\bar{Z}\|X}(X; z) + \sup_{\lambda \geq 0} \lambda(\mathbf{d}(S, z) - d) \geq \log M + \gamma | X \right] \right] \quad (4.33)$$

where we used $S - X - Z$. \square

Remark 4.1. If $\text{supp}(P_{Z^*}) = \widehat{\mathcal{M}}$ and $P_{X|S}$ is the identity mapping so that $\mathbf{d}(S, z) = \bar{\mathbf{d}}(X, z)$ almost surely, for every z , then Corollary 4.2 reduces to the noiseless converse in Theorem 2.12 by using (4.11) after weakening (4.31) with $P_{\bar{X}|\bar{Z}} = P_{\bar{X}|Z^*}$ and $\lambda = \lambda^*$.

4.4 Achievability bounds

Theorem 4.3 (Achievability). *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{P_Z} \int_0^1 \mathbb{E} [\mathbb{P}^M [\pi(X, \bar{Z}) > t | X]] dt \quad (4.34)$$

where $P_{XZ} = P_X P_Z$, and

$$\pi(x, z) = \mathbb{P}[\mathbf{d}(S, z) > d | X = x] \quad (4.35)$$

Proof. The proof appeals to a random coding argument. Given M codewords (c_1, \dots, c_M) , the encoder \mathbf{f} and decoder \mathbf{c} achieving minimum excess distortion probability attainable with the given codebook operate as follows. Having observed $x \in \mathcal{X}$, the optimum encoder chooses

$$i^* \in \arg \min_i \pi(x, c_i) \quad (4.36)$$

with ties broken arbitrarily, so $\mathbf{f}(x) = i^*$ and the decoder simply outputs c_{i^*} , so $\mathbf{c}(\mathbf{f}(x)) = c_{i^*}$.

The excess distortion probability achieved by the scheme is given by

$$\mathbb{P}[d(S, c(f(X))) > d] = \mathbb{E}[\pi(X, c(f(X)))] \quad (4.37)$$

$$= \int_0^1 \mathbb{P}[\pi(X, c(f(X))) > t] dt \quad (4.38)$$

$$= \int_0^1 \mathbb{E}[\mathbb{P}[\pi(X, c(f(X))) > t | X]] dt \quad (4.39)$$

Now, we notice that

$$1 \{ \pi(x, c(f(x))) > t \} = 1 \left\{ \min_{i \in 1, \dots, M} \pi(x, c_i) > t \right\} \quad (4.40)$$

$$= \prod_{i=1}^M 1 \{ \pi(x, c_i) > t \} \quad (4.41)$$

and we average (4.39) with respect to the codewords Z_1, \dots, Z_M drawn i.i.d. from P_Z , independently of any other random variable, so that $P_{XZ_1 \dots Z_M} = P_X \times P_Z \times \dots \times P_Z$, to obtain

$$\int_0^1 \mathbb{E} \left[\prod_{i=1}^M \mathbb{P}[\pi(X, Z_M) > t | X] \right] dt = \int_0^1 \mathbb{E} [\mathbb{P}^M [\pi(X, \bar{Z}) > t | X]] dt \quad (4.42)$$

Since there must exist a codebook achieving the excess distortion probability below or equal to the average over codebooks, (4.34) follows. \square

Remark 4.2. Notice that we have actually shown that the right-hand side of (4.34) gives the exact minimum excess distortion probability of random coding, averaged over codebooks drawn i.i.d. from P_Z .

Remark 4.3. In the noiseless case, $S = X$, for all $t \in [0, 1)$,

$$\pi(x, z) = 1 \{ d(x, z) > d \} \quad (4.43)$$

and the bound in Theorem 4.3 reduces to the noiseless random coding bound in Theorem 2.16.

The bound in (4.34) can be weakened to obtain the following result, which generalizes Shannon's bound for noiseless lossy compression (see e.g. Theorem 2.4).

Corollary 4.4. *There exists an (M, d, ϵ) code with*

$$\epsilon \leq \inf_{\gamma \geq 0, P_{Z|X}} \left\{ \mathbb{P}[d(S, Z) > d] + \mathbb{P}[\iota_{X;Z}(X; Z) > \log M - \gamma] + e^{-\exp(\gamma)} \right\} \quad (4.44)$$

where $P_{SXZ} = P_S P_{X|S} P_{Z|X}$.

Proof. Fix $\gamma \geq 0$ and transition probability kernel $P_{Z|X}$. Let $P_X \rightarrow P_{Z|X} \rightarrow P_Z$ (i.e. P_Z is the marginal of $P_X P_{Z|X}$), and let $P_{X\bar{Z}} = P_X P_Z$. We use the nonasymptotic covering lemma [44, Lemma 5] (also derived in (2.114)) to establish

$$\mathbb{E} [\mathbb{P}^M [\pi(X, \bar{Z}) > t|X]] \leq \mathbb{P} [\pi(X, Z) > t] + \mathbb{P} [\iota_{X;Z}(X; Z) > \log M - \gamma] + e^{-\exp(\gamma)} \quad (4.45)$$

Applying (4.45) to (4.42) and noticing that

$$\int_0^1 \mathbb{P} [\pi(X, Z) > t] dt = \mathbb{E} [\pi(X, Z)] \quad (4.46)$$

$$= \mathbb{P} [\mathbf{d}(S, Z) > d] \quad (4.47)$$

we obtain (4.44). \square

The following weakening of Theorem 4.3 is tighter than that in Corollary 4.4. It uses the generalized \mathbf{d} -tilted information and is amenable to an accurate second-order analysis. See Theorem 2.19 for a noiseless lossy compression counterpart.

Theorem 4.5 (Achievability, generalized \mathbf{d} -tilted information). *Suppose that $P_{Z|X}$ is such that almost surely*

$$\mathbf{d}(S, Z) = \bar{\mathbf{d}}_Z(S|X) \quad (4.48)$$

Then there exists an (M, d, ϵ) code with

$$\begin{aligned} \epsilon \leq \inf_{\gamma, \beta, \delta, P_Z} \Big\{ & \mathbb{E} \Big[\inf_{\lambda > 0} \Big\{ \mathbb{P} [D(P_{Z|X=x} \| P_Z) + \lambda \bar{\mathbf{d}}_Z(S|x) - \lambda(d - \delta) > \log \gamma - \log \beta | X] \\ & + \mathbb{P} [\bar{\mathbf{d}}_Z(S|X) > d | X] \\ & + |1 - \beta \mathbb{P} [d - \delta \leq \bar{\mathbf{d}}_Z(S|X) \leq d | X]|^+ \Big\} \Big] + e^{-\frac{M}{\gamma}} \Big\} \end{aligned} \quad (4.49)$$

Proof. The bound in (4.34) implies that for an arbitrary P_Z , there exists an (M, d, ϵ) code with

$$\begin{aligned} \epsilon &\leq \int_0^1 \mathbb{E} [\mathbb{P}^M [\pi(X, \bar{Z}) > t|X]] dt \\ &\leq e^{-\frac{M}{\gamma}} \mathbb{E} \left[\min \left\{ 1, \gamma \int_0^1 \mathbb{P} [\pi(X, \bar{Z}) \leq t|X] dt \right\} \right] + \int_0^1 \mathbb{E} [|1 - \gamma \mathbb{P} [\pi(X, \bar{Z}) \leq t|X] dt|^+] \end{aligned} \quad (4.50)$$

$$\leq e^{-\frac{M}{\gamma}} + \int_0^1 \mathbb{E} [|1 - \gamma \mathbb{P} [\pi(X, \bar{Z}) \leq t|X] dt|^+] \quad (4.51)$$

where to obtain (4.50) we applied [45]

$$(1-p)^M \leq e^{-Mp} \leq e^{-\frac{M}{\gamma}} \min(1, \gamma p) + |1 - \gamma p|^+ \quad (4.52)$$

The first term in the right side of (4.50) is upper bounded using the following chain of inequalities.

$$\begin{aligned} &\int_0^1 |1 - \gamma \mathbb{P} [\pi(X, \bar{Z}) \leq t|X = x]|^+ \\ &\leq \int_0^1 |1 - \gamma \mathbb{E} [\exp(-\iota_{Z|X} \bar{Z}(x; Z)) \mathbf{1} \{\pi(x, Z) \leq t\} |X = x]|^+ \end{aligned} \quad (4.53)$$

$$\leq \int_0^1 |1 - \gamma \mathbf{1} \{\pi(x) \leq t\} \mathbb{E} [\exp(-\iota_{Z|X} \bar{Z}(x; Z))]|^+ \quad (4.54)$$

$$= \pi(x) + (1 - \pi(x)) |1 - \gamma \mathbb{E} [\exp(-\iota_{Z|X} \bar{Z}(x; Z))]|^+ \quad (4.55)$$

$$\leq \pi(x) + (1 - \pi(x)) |1 - \gamma \exp(-D(P_{Z|X=x} \| P_{\bar{Z}}))|^+ \quad (4.56)$$

$$\leq \pi(x) + |1 - \gamma \exp(-D(P_{Z|X=x} \| P_{\bar{Z}})) \mathbb{P} [\bar{d}_Z(S|x) \leq d]|^+ \quad (4.57)$$

$$\leq \pi(x) + |1 - \gamma \exp(-D(P_{Z|X=x} \| P_{\bar{Z}})) \mathbb{P} [d - \delta \leq \bar{d}_Z(S|x) \leq d]|^+ \quad (4.58)$$

$$\leq \pi(x) + |1 - \gamma \mathbb{E} [\exp(-g(S, x) - \lambda \delta)] \mathbb{P} [d - \delta \leq \bar{d}_Z(S|x) \leq d]|^+ \quad (4.59)$$

$$\leq \pi(x) + \mathbb{P} [g(S, x) > \log \gamma - \log \beta - \lambda \delta] + |1 - \beta \mathbb{P} [d - \delta \leq \bar{d}_Z(S|x) \leq d]|^+ \quad (4.60)$$

where

- in (4.54) we denoted

$$\pi(x) \triangleq \mathbb{P} [\bar{d}_Z(S|x) > d] \quad (4.61)$$

where the probability is evaluated with respect to $P_{S|X=x}$, and observed using (4.48) that almost surely

$$\pi(X, Z) = \pi(X) \quad (4.62)$$

- (4.56) is by Jensen's inequality;
- in (4.56), we denoted

$$g(s, x) = D(P_{Z|X=x} \| P_{\bar{Z}}) + \lambda \bar{d}_Z(S|x) - \lambda d \quad (4.63)$$

- to obtain (4.60), we bounded

$$\gamma \exp(-g(S, x)) \geq \begin{cases} \beta & \text{if } g(S, x) \leq \log \gamma - \log \beta - \lambda \delta \\ 0 & \text{otherwise} \end{cases} \quad (4.64)$$

Taking the expectation of (4.60) and recalling (4.50), (4.49) follows. \square

4.5 Asymptotic analysis

In this section, we pass from the single shot setup of Sections 4.3 and 4.4 to a block setting by letting the alphabets be Cartesian products $\mathcal{M} = \mathcal{S}^k$, $\mathcal{X} = \mathcal{A}^k$, $\widehat{\mathcal{M}} = \hat{\mathcal{S}}^k$, and we study the second order asymptotics in k of $M^*(k, d, \epsilon)$, the minimum achievable number of representation points compatible with the excess distortion constraint $\mathbb{P} [d(S^k, Z^k) > d] \leq \epsilon$. We make the following assumptions.

- (i) $P_{S^k X^k} = P_S P_{X|S} \times \dots \times P_S P_{X|S}$ and

$$d(s^k, z^k) = \frac{1}{k} \sum_{i=1}^k d(s_i, z_i) \quad (4.65)$$

- (ii) The alphabets \mathcal{S} , \mathcal{A} , $\hat{\mathcal{S}}$ are finite sets.

- (iii) The distortion level satisfies $d_{\min} < d < d_{\max}$, where

$$d_{\min} = \inf \{d: \mathbb{R}_{\mathcal{S}, \mathcal{X}}(d) < \infty\} \quad (4.66)$$

and $d_{\max} = \inf_{z \in \hat{\mathcal{S}}} \mathbb{E} [d(\mathbf{S}, \mathbf{z})]$, where the expectation is with respect to the unconditional distribution of \mathbf{S} .

- (iv) The function $\mathbb{R}_{\mathcal{X}, \mathcal{Z}^*}(d)$ in (2.28) is twice continuously differentiable in a neighborhood of $P_{\mathcal{X}}$.

The following result is obtained via a technical second order analysis of Corollary 4.2 and Theorem 4.5.

Theorem 4.6 (Gaussian approximation). *For $0 < \epsilon < 1$,*

$$\log M^*(k, d, \epsilon) = kR(d) + \sqrt{k\tilde{\mathcal{V}}(d)}Q^{-1}(\epsilon) + O(\log k) \quad (4.67)$$

$$\tilde{\mathcal{V}}(d) = \text{Var} [\tilde{j}_{S;X}(S, X, d)] \quad (4.68)$$

Remark 4.4. The rate-dispersion function of the asymptotically equivalent noiseless problem is given by (see (2.143))

$$\mathcal{V}(d) = \text{Var} [j_X(X, d)] \quad (4.69)$$

where $j_X(X, d)$ is defined in (4.11). To verify that the decomposition (4.5) indeed holds, which implies that $\tilde{\mathcal{V}}(d) > \mathcal{V}(d)$ unless there is no noise, write

$$\tilde{\mathcal{V}}(d) = \text{Var} [j_X(X, d) + \lambda^* \bar{d}_{Z^*}(S|X) - \lambda^* \mathbb{E} [\bar{d}_{Z^*}(S|X)|X]] \quad (4.70)$$

$$\begin{aligned} &= \text{Var} [j_X(X, d)] + \lambda^{*2} \text{Var} [\bar{d}_{Z^*}(S|X) - \mathbb{E} [\bar{d}_{Z^*}(S|X)|X]] \\ &\quad + 2\lambda^* \text{Cov} (j_X(X, d), \bar{d}_{Z^*}(S|X) - \mathbb{E} [\bar{d}_{Z^*}(S|X)|X]) \end{aligned} \quad (4.71)$$

where the covariance is zero:

$$\begin{aligned} &\mathbb{E} [(j_X(X, d) - R(d)) (\bar{d}_{Z^*}(S|X) - \mathbb{E} [\bar{d}_{Z^*}(S|X)|X])] \\ &= \mathbb{E} [(j_X(X, d) - R(d)) \mathbb{E} [\bar{d}_{Z^*}(S|X) - \mathbb{E} [\bar{d}_{Z^*}(S|X)|X]]] \end{aligned} \quad (4.72)$$

$$= 0 \quad (4.73)$$

4.6 Erased fair coin flips

Let $S^k \in \{0, 1\}^k$ be the output of the binary equiprobable source, X^k be the output of the binary erasure channel with erasure rate δ driven by S^k . The compressor only observes X^k , and the goal is to minimize the bit error rate with respect to S^k . For $d = \frac{\delta}{2}$, codes with rate approaching the rate-distortion function were constructed in [83]. For $\frac{\delta}{2} \leq d \leq \frac{1}{2}$, the rate-distortion function is given by

$$R(d) = (1 - \delta) \left(\log 2 - h \left(\frac{d - \frac{\delta}{2}}{1 - \delta} \right) \right) \quad (4.74)$$

Throughout the section, we assume $\frac{\delta}{2} < d < \frac{1}{2}$ and $0 < \epsilon < 1$. We call this problem the binary erased source (BES) problem.

The rate-distortion function in (4.74) is achieved by $P_{Z^*}(0) = P_{Z^*}(1) = \frac{1}{2}$ and

$$P_{X|Z^*}(a|b) = \begin{cases} 1 - d - \frac{\delta}{2} & b = a \\ d - \frac{\delta}{2} & b \neq a \neq ? \\ \delta & a = ? \end{cases} \quad (4.75)$$

where $a \in \{0, 1, ?\}$ and $b \in \{0, 1\}$, so

$$\tilde{J}_{S,X}(S, X, b, d) = \imath_{X;Z^*}(X; b) + \lambda^* d(S, b) - \lambda^* d \quad (4.76)$$

$$= -\lambda^* d + \begin{cases} \log \frac{2}{1 + \exp(-\lambda^*)} & \text{w.p. } 1 - \delta \\ \lambda^* & \text{w.p. } \frac{\delta}{2} \\ 0 & \text{w.p. } \frac{\delta}{2} \end{cases} \quad (4.77)$$

The rate-dispersion function is given by the variance of (4.77):

$$\tilde{V}(d) = \delta(1 - \delta) \log^2 \cosh \left(\frac{\lambda^*}{2 \log e} \right) + \frac{\delta}{4} \lambda^{*2} \quad (4.78)$$

$$\lambda^* = -R'(d) = \log \frac{1 - \frac{\delta}{2} - d}{d - \frac{\delta}{2}} \quad (4.79)$$

The rate-dispersion function in (4.78) and the blocklength required to sustain a given excess distortion are plotted in Fig. 4.1. Note that as d approaches $\frac{\delta}{2}$, the rate-dispersion function grows without limit. This is to be expected, because for $d = \frac{\delta}{2}$, even in the hypothetical case where the decoder knows X^k , the number of erroneously represented bits is binomially distributed with mean $k \frac{\delta}{2}$, so no code can achieve probability of distortion exceeding $d = \frac{\delta}{2}$ lower than $\frac{1}{2}$. Therefore, the validity of (4.67) for $\epsilon < \frac{1}{2}$ requires $\tilde{V}(\delta/2) = \infty$.

The following converse strengthens Corollary 4.2 by exploiting the symmetry of the erased coin flips setting.

Theorem 4.7 (Converse, BES). *Any (k, M, d, ϵ) code must satisfy*

$$\epsilon \geq \sum_{j=0}^k \binom{k}{j} \delta^j (1 - \delta)^{k-j} \sum_{i=0}^j 2^{-j} \binom{j}{i} \left[1 - M 2^{-(k-j)} \left\langle \frac{k-j}{\lfloor kd - i \rfloor} \right\rangle \right]^+ \quad (4.80)$$

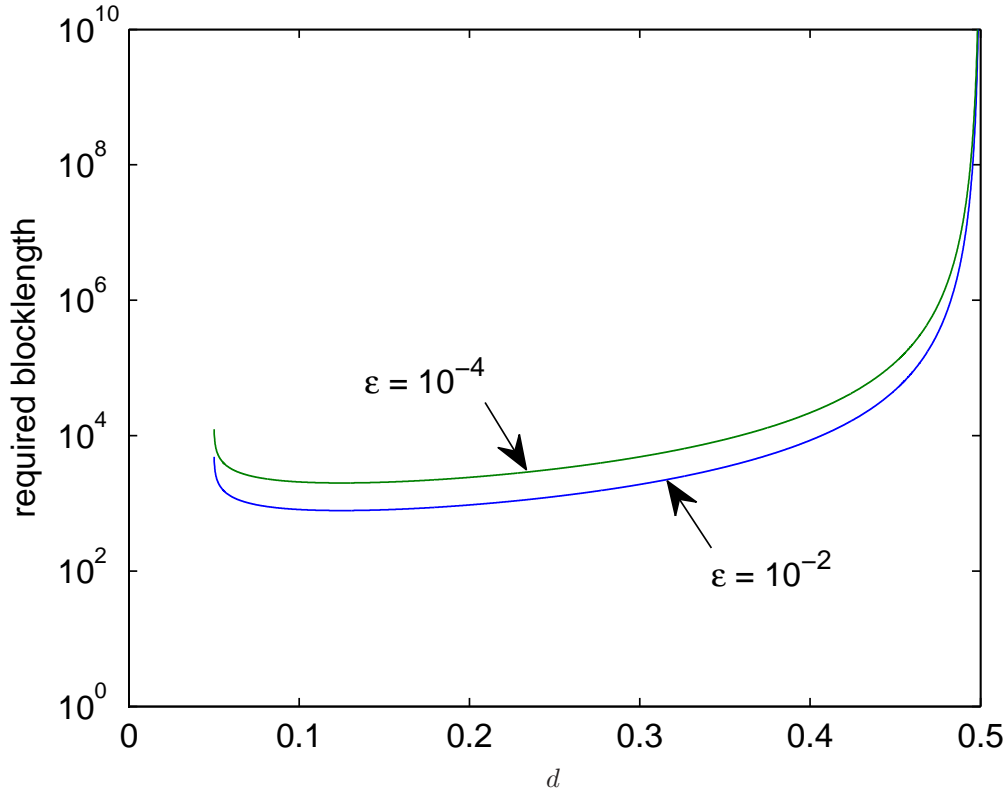
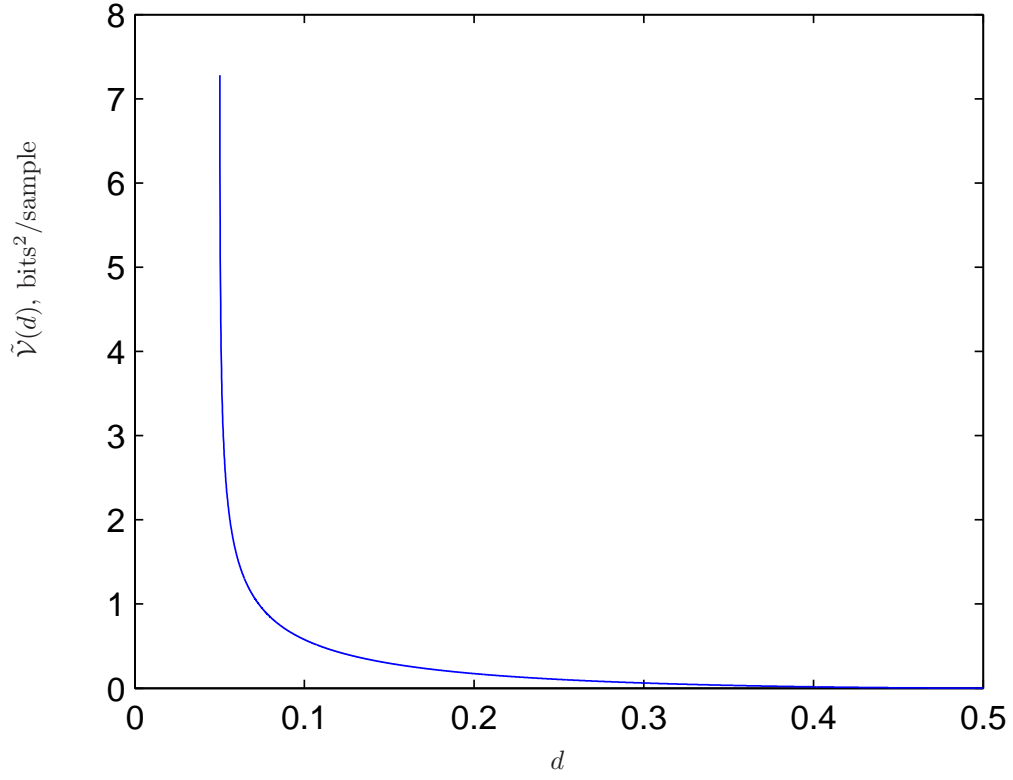


Figure 4.1: Rate-dispersion function (bits) of the fair binary source observed through an erasure channel with erasure rate $\delta = 0.1$.

Proof. Fix a (k, M, d, ϵ) code. Even if the decompressor knows erasure locations, the probability that j erased bits are at Hamming distance ℓ from their representation is

$$\mathbb{P}[j \text{ } d(S^j, Z^j) = \ell \mid X^j = (? \dots ?)] = 2^{-j} \binom{j}{\ell} \quad (4.81)$$

because given $X^j = (? \dots ?)$, S_i 's are i.i.d. binary independent of Y^j .

The probability that $k-j$ nonerased bits lie within Hamming distance ℓ from their representation can be upper bounded using Theorem 2.26:

$$\mathbb{P}[(k-j)d(S^{k-j}, Y^{k-j}) \leq \ell \mid X^{k-j} = S^{k-j}] \leq M 2^{-k+j} \left\langle \begin{matrix} k-j \\ \ell \end{matrix} \right\rangle \quad (4.82)$$

Since the errors in the erased symbols are independent of the errors in the nonerased ones,

$$\begin{aligned} & \mathbb{P}[d(S^k, Z^k) \leq d] \\ &= \sum_{j=0}^k \mathbb{P}[j \text{ erasures in } S^k] \sum_{i=0}^j \mathbb{P}[j \text{ } d(S^j, Z^j) = i \mid X^j = (? \dots ?)] \\ & \quad \cdot \mathbb{P}[(k-j)d(S^{k-j}, Z^{k-j}) \leq nd - i \mid X^{k-j} = S^{k-j}] \\ &\leq \sum_{j=0}^k \binom{k}{j} \delta^j (1-\delta)^{k-j} \sum_{i=0}^j 2^{-j} \binom{j}{i} \min \left\{ 1, M 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor nd - i \rfloor \end{matrix} \right\rangle \right\} \end{aligned} \quad (4.83)$$

□

The following achievability bound is a particularization of Theorem 4.3.

Theorem 4.8 (Achievability, BES). *There exists a (k, M, d, ϵ) code such that*

$$\epsilon \leq \sum_{j=0}^k \binom{k}{j} \delta^j (1-\delta)^{k-j} \sum_{i=0}^j 2^{-j} \binom{j}{i} \left(1 - 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor nd - i \rfloor \end{matrix} \right\rangle \right)^M \quad (4.84)$$

Proof. Consider the ensemble of codes with M codewords drawn i.i.d. from the equiprobable distribution on $\{0, 1\}^k$. As discussed in the proof of Theorem 4.7, the distortion in the erased symbols does not depend on the codebook and is given by (4.81). The probability that the Hamming distance between the nonerased symbols and their representation exceeds ℓ , averaged over the code ensemble is found as in Theorem 2.28:

$$\mathbb{P}[(k-j)d(S^{k-j}, \mathbf{C}(\mathbf{f}(X^{k-j}))) > \ell \mid S^{k-j} = X^{k-j}] = \left(1 - 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \ell \end{matrix} \right\rangle \right)^M \quad (4.85)$$

where $C(m)$, $m = 1, \dots, M$ are i.i.d on $\{0, 1\}^{k-j}$. Averaging over the erasure channel, we have

$$\begin{aligned}
& \mathbb{P} [d(S^k, C(f(X^k))) > d] \\
&= \sum_{j=0}^k \mathbb{P}[j \text{ erasures in } S^k] \sum_{i=0}^j \mathbb{P} [j \ d(S^j, C(f(X^j))) = i | X^j = ? \dots ?] \\
&\quad \cdot \mathbb{P} [(k-j)d(S^{k-j}, C(f(X^{k-j}))) > nd - i | X^{k-j} = S^{k-j}] \\
&= \sum_{j=0}^k \binom{k}{j} \delta^j (1-\delta)^{k-j} \sum_{i=0}^j 2^{-j} \binom{j}{i} \left(1 - 2^{-(k-j)} \left\langle \frac{k-j}{\lfloor nd-i \rfloor} \right\rangle \right)^M
\end{aligned} \tag{4.86}$$

Since there must exist at least one code whose excess-distortion probability is no larger than the average over the ensemble, there exists a code satisfying (4.84). \square

Theorem 4.9 (Gaussian approximation, BES). *The minimum achievable rate at blocklength k satisfies*

$$\log M^*(k, d, \epsilon) = kR(d) + \sqrt{k\tilde{\mathcal{V}}(d)Q^{-1}(\epsilon)} + \theta(k) \tag{4.87}$$

where $R(d)$ and $\tilde{\mathcal{V}}(d)$ are given by (4.74) and (4.78), respectively, and the remainder term satisfies

$$O(1) \leq \theta(k) \leq \frac{1}{2} \log k + \log \log k + O(1) \tag{4.88}$$

Proof. Appendix D.3. \square

4.7 Erased fair coin flips: asymptotically equivalent problem

According to (4.7), the distortion measure of the asymptotically equivalent noiseless problem is given by

$$\bar{d}(1, 1) = \bar{d}(0, 0) = 0 \tag{4.89}$$

$$\bar{d}(1, 0) = \bar{d}(0, 1) = 1 \tag{4.90}$$

$$\bar{d}(?, 1) = \bar{d}(?, 0) = \frac{1}{2} \tag{4.91}$$

The \mathbf{d} -tilted information is given by taking the expectation of (4.77) with respect to \mathbf{S} :

$$j_X(X, d) = -\lambda^* d + \begin{cases} \log \frac{2}{1+\exp(-\lambda^*)} & \text{w.p. } 1-\delta \\ \frac{\lambda^*}{2} & \text{w.p. } \delta \end{cases} \quad (4.92)$$

Its variance is equal to

$$\mathcal{V}(d) = \delta(1-\delta) \log^2 \cosh \left(\frac{\lambda^*}{2 \log e} \right) \quad (4.93)$$

$$= \tilde{\mathcal{V}}(d) - \frac{\delta}{4} \lambda^{*2} \quad (4.94)$$

Tight achievability and converse bounds for the ternary source with binary representation alphabet and the distortion measure in (4.89)–(4.91) are obtained as follows.

Theorem 4.10 (Converse, asymptotically equivalent BES). *Any (k, M, d, ϵ) code must satisfy*

$$\epsilon \geq \sum_{j=0}^{\lfloor 2kd \rfloor} \binom{k}{j} \delta^j (1-\delta)^{k-j} \left[1 - M 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor kd - \frac{1}{2}j \rfloor \end{matrix} \right\rangle \right]^+ \quad (4.95)$$

Proof. Fix a (k, M, d, ϵ) code. While j erased bits contribute $\frac{1}{2} \frac{j}{k}$ to the total distortion regardless of the code, the probability that $k-j$ nonerased bits lie within Hamming distance ℓ from their representation can be upper bounded using Theorem 2.26:

$$\mathbb{P} \left[(k-j) \bar{\mathbf{d}}(X^{k-j}, Z^{k-j}) \leq \ell \mid \text{no erasures in } X^{k-j} \right] \leq M 2^{-k+j} \left\langle \begin{matrix} k-j \\ \ell \end{matrix} \right\rangle \quad (4.96)$$

We have

$$\begin{aligned} & \mathbb{P} [\bar{\mathbf{d}}(X^k, Z^k) \leq d] \\ &= \sum_{j=0}^{\lfloor 2kd \rfloor} \mathbb{P}[j \text{ erasures in } X^k] \mathbb{P} \left[(k-j) \bar{\mathbf{d}}(X^{k-j}, Z^{k-j}) \leq kd - \frac{1}{2}j \mid \text{no erasures in } X^{k-j} \right] \end{aligned} \quad (4.97)$$

$$\leq \sum_{j=0}^{\lfloor 2kd \rfloor} \binom{k}{j} \delta^j (1-\delta)^{k-j} \min \left\{ 1, M 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor kd - \frac{1}{2}j \rfloor \end{matrix} \right\rangle \right\} \quad (4.98)$$

□

Theorem 4.11 (Achievability, asymptotically equivalent BES). *There exists a (k, M, d, ϵ) code such*

that

$$\epsilon \leq \sum_{j=0}^k \binom{k}{j} \delta^j (1-\delta)^{k-j} \left(1 - 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor kd - \frac{1}{2}j \rfloor \end{matrix} \right\rangle \right)^M \quad (4.99)$$

Proof. Consider the ensemble of codes with M codewords drawn i.i.d. from the equiprobable distribution on $\{0, 1\}^k$. Every erased symbol contributes $\frac{1}{2k}$ to the total distortion. The probability that the Hamming distance between the nonerased symbols and their representation exceeds ℓ , averaged over the code ensemble is found as in Theorem 2.28:

$$\mathbb{P}[(k-j)\bar{d}(X^{k-j}, \mathbf{C}(\mathbf{f}(X^{k-j}))) > \ell | \text{no erasures in } X^{k-j}] = \left(1 - 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \ell \end{matrix} \right\rangle \right)^M \quad (4.100)$$

where $\mathbf{C}(m)$, $m = 1, \dots, M$ are i.i.d on $\{0, 1\}^{k-j}$. Averaging over the erasure channel, we have

$$\begin{aligned} & \mathbb{P}[\bar{d}(S^k, \mathbf{C}(\mathbf{f}(X^k))) > d] \\ &= \sum_{j=0}^k \mathbb{P}[j \text{ erasures in } X^k] \mathbb{P}\left[(k-j)\bar{d}(S^{k-j}, \mathbf{C}(\mathbf{f}(X^{k-j}))) > kd - \frac{1}{2}j | \text{no erasures in } X^{k-j}\right] \end{aligned} \quad (4.101)$$

$$= \sum_{j=0}^k \binom{k}{j} \delta^j (1-\delta)^{k-j} \left(1 - 2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor kd - \frac{1}{2}j \rfloor \end{matrix} \right\rangle \right)^M \quad (4.102)$$

Since there must exist at least one code whose excess-distortion probability is no larger than the average over the ensemble, there exists a code satisfying (4.99). \square

The bounds in Theorems 4.7 and 4.8 and the approximation in Theorem 4.6 (with the remainder term equal to 0 and $\frac{\log k}{2k}$ - these choices are justified by (4.88)), as well as the bounds in Theorems 4.10 and 4.11 for the asymptotically equivalent problem together with their Gaussian approximation, are plotted in Fig. 4.2. We note the following.

- The achievability and converse bounds are extremely tight, even at short blocklengths, as evidenced by Fig. 4.3 where we magnified the short blocklength region;
- The dispersion for both problems is small enough that the third-order term matters.
- Despite the fact that the asymptotically achievable rate in the two problems is the same, there is a very noticeable difference between their nonasymptotically achievable rates in the displayed region of blocklengths. For example, at blocklength 1000, the penalty over the rate-distortion function is 9% for erased coin flips and only 4% for the asymptotically equivalent problem.

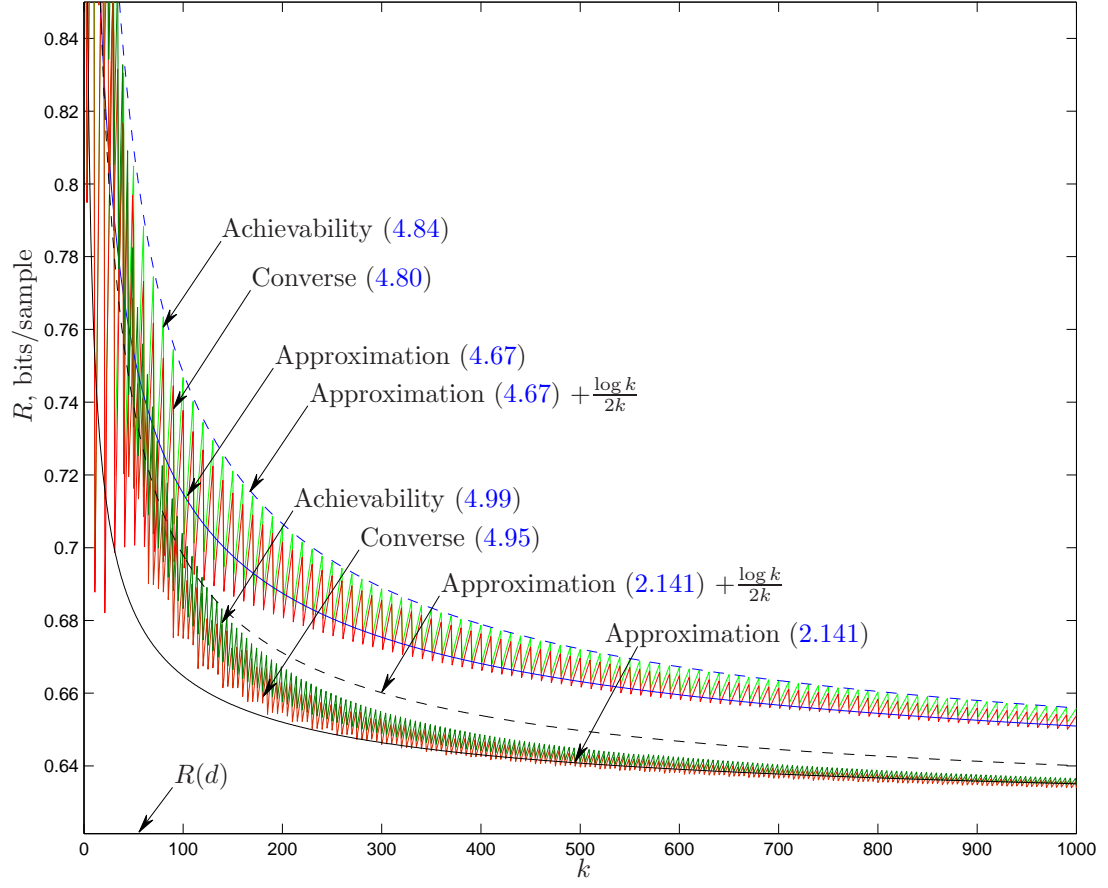


Figure 4.2: Rate-blocklength tradeoff for the fair binary source observed through an erasure channel, as well as that for the asymptotically equivalent problem, with $\delta = 0.1$, $d = 0.1$, $\epsilon = 0.1$.

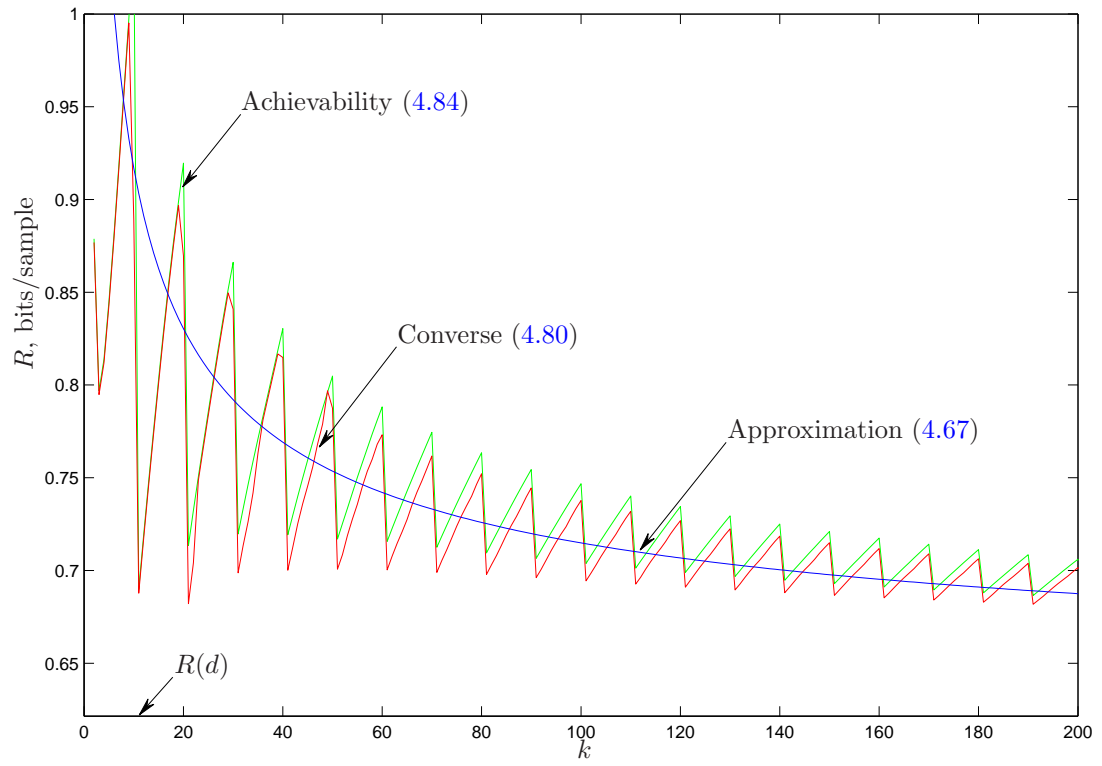


Figure 4.3: Rate-blocklength tradeoff for the fair binary source observed through an erasure channel with $\delta = 0.1$, $d = 0.1$, $\epsilon = 0.1$ at short blocklengths.

Chapter 5

Channels with cost constraints

5.1 Introduction

This chapter is concerned with the maximum channel coding rate achievable at average error probability $\epsilon > 0$ where the cost of each codeword is constrained. The material in this chapter was presented in part in [84].

The capacity-cost function $C(\beta)$ of a channel specifies the maximum achievable channel coding rate compatible with vanishing error probability and with codeword cost not exceeding β in the limit of large blocklengths. We consider stationary memoryless channels with separable cost function, i.e.

- (i) $P_{Y^n|X^n} = P_{Y|X} \times \dots \times P_{Y|X}$, with $P_{Y|X}: \mathcal{A} \rightarrow \mathcal{B}$;
- (ii) $b_n(x^n) = \frac{1}{n} \sum_{i=1}^n b(x_i)$ where $b: \mathcal{A} \rightarrow [0, \infty]$.

In this case,

$$C(\beta) = \sup_{\mathbb{E}[b(X)] \leq \beta} I(X; Y) \quad (5.1)$$

A channel is said to satisfy the strong converse if $\epsilon \rightarrow 1$ as $n \rightarrow \infty$ for any code operating at a rate above the capacity. For memoryless channels without cost constraints, the strong converse was first shown by Wolfowitz: [49] treats the discrete memoryless channel (DMC), while [85] generalizes the result to memoryless channels whose input alphabet is finite while the output alphabet is the real line. Arimoto [86] showed a new converse bound stated in terms of Gallager's random coding exponent, which also leads to the strong converse for the DMC. Kemperman [87] showed that the strong converse holds for a DMC with feedback. For a particular discrete channel with finite memory, the strong converse was shown by Wolfowitz [88] and independently by Feinstein [89], a result soon

generalized to a more general stationary discrete channel with finite memory [90]. In a more general setting not requiring the assumption of stationarity or finite memory, Verdú and Han [91] showed a necessary and sufficient condition for a channel without cost constraints to satisfy the strong converse. In the special case of finite-input channels, that necessary and sufficient condition boils down to the capacity being equal to the limit of maximal normalized mutual informations. In turn, that condition is implied by the information stability of the channel [92], a condition which in general is not easy to verify.

As far as channel coding with input cost constraints, a general necessary and sufficient condition for a channel with cost constraints to satisfy the strong converse was shown by Han [43, Theorem 3.7.1]. The strong converse for DMC with separable cost was shown by Csiszár and Körner [46, Theorem 6.11] and by Han [43, Theorem 3.7.2]. Regarding analog channels, the strong converse has only been studied in the context of additive Gaussian noise channels with the cost function being the power of the channel input block, $\mathbf{b}_n(x^n) = \frac{1}{n}|x^n|^2$. In the most basic case of the memoryless additive white Gaussian noise (AWGN) channel, the strong converse was shown by Shannon [10] (contemporaneously with Wolfowitz’s finite alphabet strong converse). Yoshihara [93] proved the strong converse for the time-continuous channel with additive Gaussian noise having an arbitrary spectrum and also gave a simple proof of Shannon’s strong converse result. Under the requirement that the power of each message converges stochastically to a given constant β , the strong converse for the AWGN channel with feedback was shown by Wolfowitz [94]. Note that in all those analyses of the power-constrained AWGN channel the cost constraint is meant on a per-codeword basis. In fact, as was observed by Polyanskiy [4, Theorem 77] in the context of the AWGN channel, the strong converse ceases to hold if the cost constraint is averaged over the codebook.

For a survey on existing results in joint source-channel coding see Section 3.1.

As we mentioned in Section 1.2, channel dispersion quantifies the backoff from capacity, unescapable at finite blocklengths due to the random nature of the channel coming into play, as opposed to the asymptotic representation of the channel as a deterministic bit pipe of a given capacity. Polyanskiy et al. [3] found the dispersion of the DMC without cost constraints as well as that of the AWGN channel with a power constraint. Hayashi [95, Theorem 3] showed the dispersion of the DMC with and without cost constraints (with the loose estimate of $o(\sqrt{n})$ for the third order term). For constant composition codes over the DMC, Polyanskiy [4, Sec. 3.4.6] found the dispersion of constant composition codes over the DMC invoking the $\kappa\beta$ bound [3, Theorem 25] to prove the achievability part, while Moulin [96] refined the third order term in the expansion of the maximum achievable code rate, under regularity conditions.

In this chapter, we show a new non-asymptotic converse bound for general channels with input cost constraints in terms of a random variable we refer to as the \mathbf{b} -tilted information density, which parallels the notion of \mathbf{d} -tilted information for lossy compression in Section 2.2. Not only does the new bound lead to a general strong converse result but it is also tight enough to find the channel dispersion-cost function and the third order term equal to $\frac{1}{2} \log n$ when coupled with the corresponding achievability bound. More specifically, we show that for the DMC, $M^*(n, \epsilon, \beta)$, the maximum achievable code size at blocklength n , error probability ϵ and cost β , is given by, under mild regularity assumptions,

$$\log M^*(n, \epsilon, \beta) = nC(\beta) - \sqrt{nV(\beta)}Q^{-1}(\epsilon) + \frac{1}{2} \log n + O(1) \quad (5.2)$$

where $V(\beta)$ is the dispersion-cost function, and $Q^{-1}(\cdot)$ is the inverse of the Gaussian complementary cdf, thereby refining Hayashi's result [95] and providing a matching converse to the result of Moulin [96]. We observe that the capacity-cost and the dispersion-cost functions are given by the mean and the variance of the \mathbf{b} -tilted information density. This novel interpretation juxtaposes nicely with the corresponding results in Chapter 2 (\mathbf{d} -tilted information in rate-distortion theory).

Section 5.2 introduces the \mathbf{b} -tilted information density. Section 5.3 states the new non-asymptotic converse bound which holds for a general channel with cost constraints, without making any assumptions on the channel (e.g. alphabets, stationarity, memorylessness). An asymptotic analysis of the converse and achievability bounds, including the proof of the strong converse and the expression for the channel dispersion-cost function, is presented in Section 5.4. Section 5.5 generalizes the results in Sections 5.3 and 5.4 to the lossy joint source-channel coding setup.

5.2 \mathbf{b} -tilted information density

In this section, we introduce the concept of \mathbf{b} -tilted information density and several relevant properties.

Fix the transition probability kernel $P_{Y|X}: \mathcal{X} \rightarrow \mathcal{Y}$ and the cost function $\mathbf{b}: \mathcal{X} \mapsto [0, \infty]$. In the application of this single-shot approach in Section 5.4, \mathcal{X} , \mathcal{Y} , $P_{Y|X}$ and \mathbf{b} will become \mathcal{A}^n , \mathcal{B}^n ,

$P_{Y^n|X^n}$ in (i) and \mathbf{b}^n in (ii), respectively. As in (3.4), denote: ¹

$$\mathbb{C}(\beta) = \sup_{\substack{P_X: \\ \mathbb{E}[\mathbf{b}(X)] \leq \beta}} I(X; Y) \quad (5.3)$$

$$\lambda^* = \mathbb{C}'(\beta) \quad (5.4)$$

Further, recalling notation (2.22), define the function

$$J_{Y|X\|\bar{Y}}(x; y, \beta) \triangleq \iota_{Y|X\|\bar{Y}}(x; y) - \lambda^* (\mathbf{b}(x) - \beta) \quad (5.5)$$

As before, if $P_X \rightarrow P_{Y|X} \rightarrow P_Y$, instead of writing $Y|X\|Y$ in the subscripts we write $X; Y$.

The special case of (5.5) with $P_{\bar{Y}} = P_{Y^*}$, where P_{Y^*} is the unique output distribution that achieves the supremum in (5.3), defines \mathbf{b} -tilted information density:

Definition 5.1 (\mathbf{b} -tilted information density). *The \mathbf{b} -tilted information density between $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ is*

$$J_{X;Y}^*(x; y, \beta) \triangleq \iota_{X;Y}^*(x; y) - \lambda^* (\mathbf{b}(x) - \beta) \quad (5.6)$$

where, as in (3.6), we abbreviated

$$\iota_{X;Y}^*(x; y) \triangleq \iota_{Y|X\|Y^*}(x; y) \quad (5.7)$$

Since P_{Y^*} is unique even if there are several (or none) input distributions P_{X^*} that achieve supremum in (5.3), there is no ambiguity in Definition 5.1. If there are no cost constraints (i.e. $\mathbf{b}(x) = 0 \forall x \in \mathcal{X}$), then $\mathbb{C}'(\beta) = 0$ regardless of β , and

$$J_{X;Y}^*(x; y, \beta) = \iota_{X;Y}^*(x; y) \quad (5.8)$$

The counterpart of the \mathbf{b} -tilted information density in rate-distortion theory is the \mathbf{d} -tilted information in Section 2.2.

¹The difference in notation in (5.1) and (5.3) is intentional. While $C(\beta)$ in (5.1) has the operational interpretation of being the capacity-cost function of the stationary memoryless channel with single-letter transition probability kernel $P_{Y|X}$, $\mathbb{C}(\beta)$ simply denotes the optimum of the maximization problem in the right side of (5.3); its operational meaning does not, at this point, concern us.

Denote

$$\beta_{\min} = \inf_{x \in \mathcal{X}} \mathbf{b}(x) \quad (5.9)$$

$$\beta_{\max} = \sup \{ \beta \geq 0 : \mathbb{C}(\beta) < \mathbb{C}(\infty) \} \quad (5.10)$$

A nontrivial generalization of the well-known properties of information density in the case of no cost constraints, the following result highlights the importance of \mathbf{b} -tilted information density in the optimization problem (5.3). It will be of key significance in the asymptotic analysis in Section 5.4.

Theorem 5.1. *Fix $\beta_{\min} < \beta < \beta_{\max}$. Assume that P_{X^*} achieving (5.3) is such that*

$$\mathbb{E} [\mathbf{b}(X^*)] = \beta \quad (5.11)$$

Let $P_X \rightarrow P_{Y|X} \rightarrow P_Y$, $P_{X^} \rightarrow P_{Y|X} \rightarrow P_{Y^*}$. It holds that*

$$\mathbb{C}(\beta) = \sup_{P_X} \mathbb{E} [j_{X;Y}(X; Y, \beta)] \quad (5.12)$$

$$= \sup_{P_X} \mathbb{E} [j_{X;Y}^*(X; Y, \beta)] \quad (5.13)$$

$$= \mathbb{E} [j_{X;Y}^*(X^*; Y^*, \beta)] \quad (5.14)$$

$$= \mathbb{E} [j_{X;Y}^*(X^*; Y^*, \beta) | X^*] \quad (5.15)$$

where (5.15) holds P_{X^*} -a.s.

Proof. Appendix E.1. □

Throughout Chapter 5, we assume that the assumptions of Theorem 5.1 hold.

Corollary 5.2.

$$\text{Var} [j_{X;Y}^*(X^*; Y^*, \beta)] = \mathbb{E} [\text{Var} [j_{X;Y}^*(X^*; Y^*, \beta) | X^*]] \quad (5.16)$$

$$= \mathbb{E} [\text{Var} [\iota_{X;Y}^*(X^*; Y^*) | X^*]] \quad (5.17)$$

Proof. Appendix E.2. □

Example 5.1. For n uses of a memoryless AWGN channel with unit noise power and total power not exceeding nP , $\mathbb{C}(P) = \frac{n}{2} \log(1 + P)$, and the output distribution that achieves (5.3) is $Y^{n*} \sim$

$\mathcal{N}(0, (1+P)\mathbf{I})$. Therefore

$$J_{X^n; Y^n}^*(x^n; y^n, P) = \frac{n}{2} \log(1+P) - \frac{\log e}{2} |y^n - x^n|^2 + \frac{\log e}{2(1+P)} (|y^n|^2 - |x^n|^2 + nP) \quad (5.18)$$

It is easy to check that under $P_{Y^n|X^n=x^n}$, the distribution of $J_{X^n; Y^n}^*(x^n; Y^n, P)$ is the same as that of (by ‘ \sim ’ we mean equality in distribution)

$$J_{X^n; Y^n}^*(x^n; Y^n, P) \sim \frac{n}{2} \log(1+P) - \frac{P \log e}{2(1+P)} \left[W_{n \frac{|x^n|^2}{P^2}}^n - n - \frac{|x^n|^2}{P^2} \right] \quad (5.19)$$

where W_λ^ℓ denotes a non central chi-square distributed random variable with ℓ degrees of freedom and non-centrality parameter λ . The mean of (5.19) is $\frac{n}{2} \log(1+P)$, in accordance with (5.15), while its variance is $\frac{1}{2} \frac{(nP^2 + 2|x^n|^2)}{(1+P)^2} \log^2 e$ which becomes $nV(P)$ after averaging with respect to X^{n*} distributed according to $P_{X^{n*}} \sim \mathcal{N}(0, P\mathbf{I})$, as we will see in Section 5.4.3 (cf. [3]).

Example 5.2. Consider the memoryless binary symmetric channel (BSC) with crossover probability δ and Hamming per-symbol cost, $\mathbf{b}(\mathbf{x}) = \mathbf{x}$. The capacity-cost function is given by

$$C(\beta) = \begin{cases} h(\beta \star \delta) - h(\delta) & \beta \leq \frac{1}{2} \\ 1 - h(\delta) & \beta > \frac{1}{2} \end{cases} \quad (5.20)$$

where $\beta \star \delta = (1-\beta)\delta + \beta(1-\delta)$. The capacity-cost function is achieved by $P_{X^*}(1) = \min\{\beta, \frac{1}{2}\}$, and $C'(\beta) = (1-2\delta) \log \frac{1-\beta\star\delta}{\beta\star\delta}$ for $\beta \leq \frac{1}{2}$, and

$$J_{X^*; Y^*}^*(X^*; Y^*, \beta) = h(\beta \star \delta) - h(\delta) + \begin{cases} \delta \log \frac{1-\delta}{\delta} \frac{\beta\star\delta}{1-\beta\star\delta} & \text{w.p. } (1-\delta)(1-\beta) \\ -(1-\delta) \log \frac{1-\delta}{\delta} \frac{\beta\star\delta}{1-\beta\star\delta} & \text{w.p. } \delta(1-\beta) \\ \delta \log \frac{1-\delta}{\delta} \frac{1-\beta\star\delta}{\beta\star\delta} & \text{w.p. } (1-\delta)\beta \\ -(1-\delta) \log \frac{1-\delta}{\delta} \frac{1-\beta\star\delta}{\beta\star\delta} & \text{w.p. } \delta\beta \end{cases} \quad (5.21)$$

The capacity-cost function (the mean of $J_{X^*; Y^*}^*(X^*; Y^*, \beta)$) and the dispersion-cost function (the variance of $J_{X^*; Y^*}^*(X^*; Y^*, \beta)$) are plotted in Fig. 5.1.

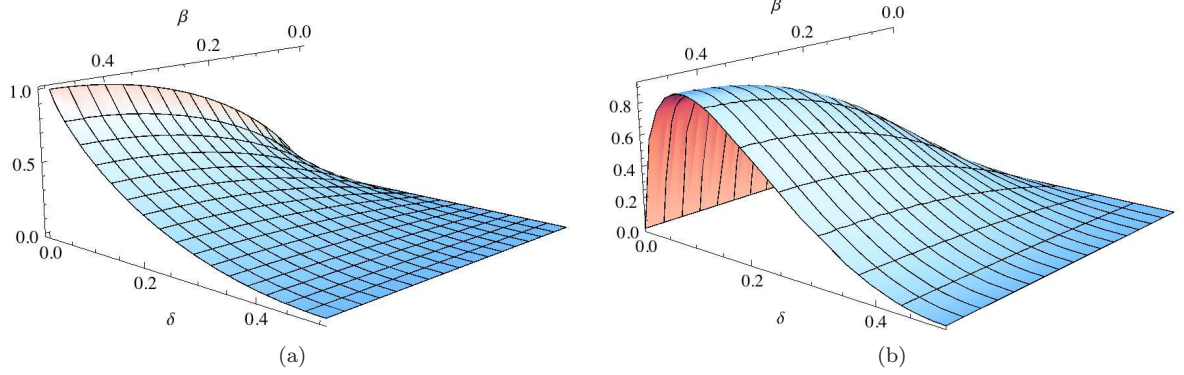


Figure 5.1: Capacity-cost function (a) and the dispersion-cost function (b) for BSC with crossover probability δ where the normalized Hamming weight of codewords is constrained not to exceed β .

5.3 New converse bound

Converse and achievability bounds give necessary and sufficient conditions, respectively, on (M, ϵ, β) in order for a code to exist with M codewords and average error probability not exceeding ϵ and β , respectively. Such codes (allowing stochastic encoders and decoders) are rigorously defined next.

Definition 5.2 ((M, ϵ, β) code). *An (M, ϵ, β) code for $\{P_{Y|X}, \mathbf{b}\}$ is a pair of random transformations $P_{X|S}$ (encoder) and $P_{Z|Y}$ (decoder) such that $\mathbb{P}[S \neq Z] \leq \epsilon$, where the probability is evaluated with S equiprobable on an alphabet of cardinality M , $S - X - Y - Z$, and the codewords satisfy the maximal cost constraint (a.s.)*

$$\mathbf{b}(X) \leq \beta \quad (5.22)$$

The non-asymptotic quantity of principal interest is $M^*(\epsilon, \beta)$, the maximum code size achievable at error probability ϵ and cost β . Blocklength will enter into the picture later when we consider (M, d, ϵ) codes for $\{P_{Y^n|X^n}, \mathbf{b}_n\}$, where $P_{Y^n|X^n}: \mathcal{A}^n \mapsto \mathcal{B}^n$ and $\mathbf{b}_n: \mathcal{A}^n \mapsto [0, \infty]$. We will call such codes (n, M, d, ϵ) codes, and denote the corresponding non-asymptotically achievable maximum code size by $M^*(n, \epsilon, \beta)$. For now, though, blocklength n is immaterial, as the converse and achievability bounds do not call for any Cartesian product structure of the channel input and output alphabets. Accordingly, forgoing n , just as we did in Chapters 2–4, we state the converse for a generic pair $\{P_{Y|X}, \mathbf{b}\}$, rather than the less general $\{P_{Y^n|X^n}, \mathbf{b}_n\}$.

Theorem 5.3 (Converse). *The existence of an (M, ϵ, β) code for $\{P_{Y|X}, \mathbf{b}\}$ requires that*

$$\epsilon \geq \inf_X \max_{\gamma > 0} \left\{ \sup_{\bar{Y}} \mathbb{P} [J_{Y|X|\bar{Y}}(X; Y, \beta) \leq \log M - \gamma] - \exp(-\gamma) \right\} \quad (5.23)$$

$$\geq \max_{\gamma > 0} \left\{ \sup_{\bar{Y}} \inf_{x \in \mathcal{X}} \mathbb{P} [J_{Y|X|\bar{Y}}(x; Y, \beta) \leq \log M - \gamma | X = x] - \exp(-\gamma) \right\} \quad (5.24)$$

Proof. Fix an (M, ϵ) code $\{P_{X|S}, P_{Z|Y}\}$, $\gamma > 0$, and an auxiliary probability distribution $P_{\bar{Y}}$ on \mathcal{Y} .

Since $\mathbf{b}(X) \leq \beta$, we have

$$\begin{aligned} & \mathbb{P} [J_{Y|X|\bar{Y}}(X; Y, \beta) \leq \log M - \gamma] \\ & \leq \mathbb{P} [i_{Y|X|\bar{Y}}(X; Y) - \lambda^*(\mathbf{b}(X) - \beta) \leq \log M - \gamma] \end{aligned} \quad (5.25)$$

$$\leq \mathbb{P} [i_{Y|X|\bar{Y}}(X; Y) \leq \log M - \gamma] \quad (5.26)$$

$$= \mathbb{P} [i_{Y|X|\bar{Y}}(X; Y) \leq \log M - \gamma, Z \neq S] + \mathbb{P} [i_{Y|X|\bar{Y}}(X; Y) \leq \log M - \gamma, Z = S] \quad (5.27)$$

$$\begin{aligned} & \leq \mathbb{P} [Z \neq S] + \frac{1}{M} \sum_{m=1}^M \sum_{x \in \mathcal{X}} P_{X|S}(x|m) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) P_{Z|Y}(m|y) 1 \{P_{Y|X}(y|x) \leq P_{\bar{Y}}(y) M \exp(-\gamma)\} \\ & \leq \epsilon + \exp(-\gamma) \sum_{y \in \mathcal{Y}} P_{\bar{Y}}(y) \sum_{m=1}^M P_{Z|Y}(m|y) \sum_{x \in \mathcal{X}} P_{X|S}(x|m) \end{aligned} \quad (5.28)$$

$$\leq \epsilon + \exp(-\gamma) \sum_{y \in \mathcal{Y}} P_{\bar{Y}}(y) \sum_{m=1}^M P_{Z|Y}(m|y) \sum_{x \in \mathcal{X}} P_{X|S}(x|m) \quad (5.29)$$

$$\leq \epsilon + \exp(-\gamma) \quad (5.30)$$

Optimizing over $\gamma > 0$ and the distribution of the auxiliary random variable \bar{Y} , we obtain the best possible bound for a given P_X , which is generated by the encoder $P_{X|S}$. Choosing P_X that gives the weakest bound to remove the dependence on the code, (5.23) follows.

To show (5.24), we weaken (5.23) by moving \inf_X inside $\sup_{\bar{Y}}$, and write

$$\inf_X \mathbb{P} [J_{Y|X|\bar{Y}}(X; Y, \beta) \leq \log M - \gamma] = \inf_X \sum_{x \in \mathcal{X}} P_X(x) \mathbb{P} [J_{Y|X|\bar{Y}}(x; Y, \beta) \leq \log M - \gamma | X = x] \quad (5.31)$$

$$= \inf_{x \in \mathcal{X}} \mathbb{P} [J_{Y|X|\bar{Y}}(x; Y, \beta) \leq \log M - \gamma | X = x] \quad (5.32)$$

□

Remark 5.1. At short blocklengths, it is possible to get a better bound by giving more freedom in (5.5) not restricting λ^* to be (5.4).

Achievability bounds for channels with cost constraints can be obtained from the random coding bounds in [3, 58] by restricting the distribution from which the codewords are drawn to satisfy $\mathbf{b}(X) \leq \beta$ a.s. In particular, for the DMC, we may choose P_{X^n} to be equiprobable on the set of codewords of type which is closest to the input distribution P_{X^*} that achieves the capacity-cost function. As we will see in Section 5.4.3, owing to (5.17), such constant composition codes achieve the dispersion of channel coding under input cost constraints.

5.4 Asymptotic analysis

In this section, we reintroduce the blocklength n into the non-asymptotic converse of Section 5.3, i.e. let X and Y therein turn into X^n and Y^n , and perform its analysis, asymptotic in n .

5.4.1 Assumptions

The following basic assumptions hold throughout Section 5.4.

- (i) The channel is stationary and memoryless, $P_{Y^n|X^n} = P_{Y|X} \times \dots \times P_{Y|X}$.
- (ii) The cost function is separable, $\mathbf{b}_n(x^n) = \frac{1}{n} \sum_{i=1}^n \mathbf{b}(x_i)$, where $\mathbf{b}: \mathcal{A} \mapsto [0, \infty]$.
- (iii) The codewords are constrained to satisfy the maximal power constraint (5.22).
- (iv) $\sup_{\mathbf{x} \in \mathcal{A}} \text{Var} \left[J_{X;Y}^*(\mathbf{x}; Y, \beta) | X = x \right] = V_{\max} < \infty$.

Under these assumptions, the capacity-cost function $C(\beta) = \mathbb{C}(\beta)$ is given by (5.1). Observe that in view of assumption (i), as long as $P_{\bar{Y}^n}$ is a product distribution, $P_{\bar{Y}^n} = P_{\bar{Y}} \times \dots \times P_{\bar{Y}}$,

$$J_{Y^n|X^n||\bar{Y}^n}(x^n; y^n, \beta) = \sum_{i=1}^n J_{Y|X||\bar{Y}}(x_i; y_i, \beta) \quad (5.33)$$

5.4.2 Strong converse

We show that if transmission occurs at a rate greater than the capacity-cost function, the error probability must converge to 1, regardless of the specifics of the code. Toward this end, we fix some $\alpha > 0$, we choose $\log M \geq nC(\beta) + 2n\alpha$, and we weaken the bound (5.24) in Theorem 5.3 by fixing $\gamma = n\alpha$ and $P_{\bar{Y}^n} = P_{Y^*} \times \dots \times P_{Y^*}$, where Y^* is the output distribution that achieves $C(\beta)$, to

obtain

$$\epsilon \geq \inf_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^n J_{\mathbf{X}; \mathbf{Y}}^*(x_i; Y_i, \beta) \leq nC(\beta) + n\alpha \right] - \exp(-n\alpha) \quad (5.34)$$

$$\geq \inf_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^n J_{\mathbf{X}; \mathbf{Y}}^*(x_i; Y_i, \beta) \leq \sum_{i=1}^n c(x_i) + n\alpha \right] - \exp(-n\alpha) \quad (5.35)$$

where for notational convenience we have abbreviated $c(\mathbf{x}) = \mathbb{E} \left[J_{\mathbf{X}; \mathbf{Y}}^*(\mathbf{x}; \mathbf{Y}, \beta) | \mathbf{X} = \mathbf{x} \right]$, and (5.35) employs (5.13).

To show that the right side of (5.35) converges to 1, we invoke the following law of large numbers for non-identically distributed random variables.

Theorem 5.4 (e.g. [97]). *Suppose that W_i are uncorrelated and $\sum_{i=1}^{\infty} \text{Var} \left[\frac{W_i}{b_i} \right] < \infty$ for some strictly positive sequence (b_n) increasing to $+\infty$. Then,*

$$\frac{1}{b_n} \left(\sum_{i=1}^n W_i - \mathbb{E} \left[\sum_{i=1}^n W_i \right] \right) \rightarrow 0 \text{ in } L^2 \quad (5.36)$$

Let $W_i = J_{\mathbf{X}; \mathbf{Y}}^*(x_i; Y_i, \beta)$ and $b_i = i$. Since (recall (iv))

$$\sum_{i=1}^{\infty} \text{Var} \left[\frac{1}{i} J_{\mathbf{X}; \mathbf{Y}}^*(x_i; Y_i, \beta) | X_i = x_i \right] \leq V_{\max} \sum_{i=1}^{\infty} \frac{1}{i^2} \quad (5.37)$$

$$< \infty \quad (5.38)$$

by virtue of Theorem 5.4 the right side of (5.35) converges to 1, so any channel satisfying (i)–(iv) also satisfies the strong converse.

As noted in [4, Theorem 77] in the context of the AWGN channel, the strong converse does not hold if the power constraint is averaged over the codebook, i.e. if, in lieu of (5.22), the cost requirement is

$$\frac{1}{M} \sum_{m=1}^M \mathbb{E} [\mathbf{b}(X) | S = m] \leq \beta \quad (5.39)$$

To see why, fix a code of rate $C(\beta) < R < C(2\beta)$ none of whose codewords costs more than 2β and whose error probability vanishes as n increases, $\epsilon \rightarrow 0$. Since $R < C(2\beta)$, such a code exists. Now, replace half of the codewords with the all-zero codeword (assuming $\mathbf{b}(\mathbf{0}) = 0$) while leaving the decision regions of the remaining codewords untouched. The average cost of the new code satisfies (5.39), its rate is greater than the capacity-cost function, $R > C(\beta)$, yet its average error probability

does not exceed $\epsilon + \frac{1}{2} \rightarrow \frac{1}{2}$.

5.4.3 Dispersion

First, we give the operational definition of the dispersion-cost function of any channel.

Definition 5.3 (Dispersion-cost function). *The channel dispersion-cost function, measured in squared information units per channel use, is defined by*

$$V(\beta) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \frac{(nC(\beta) - \log M^*(n, \epsilon, \beta))^2}{2 \log_e \frac{1}{\epsilon}} \quad (5.40)$$

An explicit expression for the dispersion-cost function of a memoryless channel is given in the next result.

Theorem 5.5. *In addition to assumptions (i)–(iv), assume that the capacity-achieving input distribution P_{X^*} is unique and that the channel has finite input and output alphabets.*

$$\log M^*(n, \epsilon, \beta) = nC(\beta) - \sqrt{nV(\beta)}Q^{-1}(\epsilon) + \theta(n) \quad (5.41)$$

$$C(\beta) = \mathbb{E} [j_{X,Y}^*(X^*; Y^*, \beta)] \quad (5.42)$$

$$V(\beta) = \text{Var} [j_{X,Y}^*(X^*; Y^*, \beta)] \quad (5.43)$$

where $P_{X^*Y^*} = P_{X^*}P_{Y|X}$, and the remainder term $\theta(n)$ satisfies:

a) If $V(\beta) > 0$,

$$-\frac{1}{2}(|\text{supp}(P_{X^*})| - 1) \log n + O(1) \leq \theta(n) \quad (5.44)$$

$$\leq \frac{1}{2} \log n + O(1) \quad (5.45)$$

b) If $V(\beta) = 0$, (5.44) holds, and (5.45) is replaced by

$$\theta(n) \leq O\left(n^{\frac{1}{3}}\right) \quad (5.46)$$

Proof. Converse. Full details are given in Appendix E.3. The main steps of the refined asymptotic analysis of the bound in Theorem 5.3 are as follows. First, building on the ideas of [98, 99], we weaken the bound in (5.24) by a careful choice of a non-product auxiliary distribution $P_{\tilde{Y}^n}$. Second,

using Theorem 5.1 and the technical tools developed in Appendix A.3, we show that the minimum in the right side of (5.24) is lower bounded by ϵ for the choice of M in (5.41).

Achievability. Full details are given in Appendix E.4, which provides an asymptotic analysis of the Dependence Testing bound of [3] in which the random codewords are of type closest to P_{X^*} , rather than drawn from the product distribution $P_X \times \dots \times P_X$, as in achievability proofs for channel coding without cost constraints. We use Corollary 5.2 to establish that such constant composition codes achieve the dispersion-cost function. \square

Remark 5.2. According to a recent result of Moulin [96], the achievability bound on the remainder term in (5.45) can be tightened to match the converse bound in (5.45), thereby establishing that

$$\theta(n) = \frac{1}{2} \log n + O(1) \quad (5.47)$$

provided that the following regularity assumptions hold:

- The random variable $i_{X,Y}^*(X^*; Y^*)$ is of nonlattice type;
- $\text{supp}(P_{X^*}) = \mathcal{A}$;
- $\text{Cov} \left[i_{X,Y}^*(X^*; Y^*), i_{X,Y}^*(\bar{X}^*; Y^*) \right] < \text{Var} \left[i_{X,Y}^*(X^*; Y^*) \right]$ where $P_{\bar{X}^* X^* Y^*}(\bar{x}, x, y) = \frac{1}{P_{Y^*}(y)} P_{X^*}(\bar{x}) P_{Y|X}(y|\bar{x}) P_{Y|X}(y|x) P_{X^*}(x)$.

Remark 5.3. Theorem 5.5 applies to channels with abstract alphabets provided that a certain symmetry assumption is satisfied. More precisely, for all $x \in \mathcal{A}$ such that $\mathbf{b}(x) = \beta$, (5.41) with

$$C(\beta) = D(P_{Y|X=x} \| P_{Y^*}) \quad (5.48)$$

$$V(\beta) = \text{Var} \left[i_{X,Y}^*(x; Y) | X = x \right] \quad (5.49)$$

and the remainder satisfying

$$-f_n + O(1) \leq \theta(n) \leq \frac{1}{2} \log n + O(1) \quad (5.50)$$

where $f_n = o(\sqrt{n})$ in specified in (d) below, holds for those channels and cost functions that in addition to (i)–(iii), meet the following criteria.

- (a) The cost function $\mathbf{b}: \mathcal{A} \rightarrow [0, \infty]$ is such that for all $\gamma \in [\beta, \infty)$, $\mathbf{b}^{-1}(\gamma)$ is nonempty. In particular, this condition is satisfied if the channel input alphabet \mathcal{A} is a metric space, and \mathbf{b} is continuous and unbounded with $\mathbf{b}(0) = 0$.

- (b) The distribution of $\iota_{X^n; Y^n}^*(x^n; Y^n)$, where $P_{Y^{n*}} = P_{Y^*} \times \dots \times P_{Y^*}$ does not depend on the choice of $x^n \in \mathcal{F}$, where $\mathcal{F} = \{x^n \in \mathcal{A}^n : \mathbf{b}(x^n) = \beta\}$.
- (c) For all \mathbf{x} in the projection of \mathcal{F} onto \mathcal{A} ,

$$\mathbb{E} \left[\left| j_{X; Y}^*(\mathbf{X}; \mathbf{Y}, \beta) - C(\beta) \right|^3 \mid \mathbf{X} = \mathbf{x} \right] < \infty \quad (5.51)$$

- (d) There exists a distribution P_{X^n} supported on \mathcal{F} such that $\iota_{Y^n \| Y^{n*}}(Y^n)$, where $P_{X^n} \rightarrow P_{Y^n | X^n} \rightarrow P_{Y^n}$, is almost surely upper-bounded by $f_n = o(\sqrt{n})$.

The proof is explained in Appendix E.5.

Remark 5.4. Theorem 5 with the remainder in (5.50) (with $f_n = O(1)$) also holds for the AWGN channel with maximal signal-to-noise ratio P , offering a novel interpretation of the expression

$$V(P) = \frac{1}{2} \left(1 - \frac{1}{(1+P)^2} \right) \log^2 e \quad (5.52)$$

found in [3], as the variance of the \mathbf{b} -tilted information density. Note that the AWGN channel satisfies the conditions of Remark 5.3 with P_{X^n} uniform on the power sphere and $f_n = O(1)$.

Remark 5.5. If the capacity-achieving distribution is not unique,

$$V(\beta) = \begin{cases} \min \text{Var} \left[j_{X; Y}^*(\mathbf{X}^*; \mathbf{Y}^*, \beta) \right] & 0 < \epsilon \leq \frac{1}{2} \\ \max \text{Var} \left[j_{X; Y}^*(\mathbf{X}^*; \mathbf{Y}^*, \beta) \right] & \frac{1}{2} < \epsilon < 1 \end{cases} \quad (5.53)$$

where the optimization is performed over all P_{X^*} that achieve $C(\beta)$.

The converse bound in Theorem 5.3, the matching achievability bound in [3, Theorem 17], and the Gaussian approximation in Theorem 5.5 in which the remainder is approximated by $\theta(n) \approx \frac{1}{2} \log n$ are plotted in Figures 5.2, 5.3, 5.4, 5.5 for the BSC with Hamming cost discussed in Example 5.2. As evidenced by the plots, although the minimum over the channel inputs in (5.24) may be difficult to analyze, it is not difficult to compute (in polynomial time), at least for the DMC.

5.5 Joint source-channel coding

In this section we state the counterparts of Theorems 5.3 and 5.5 in the lossy joint source-channel coding setting. Proofs of the results in this section are obtained by fusing the proofs in Sections 5.3 and 5.4 and those in Chapter 3.

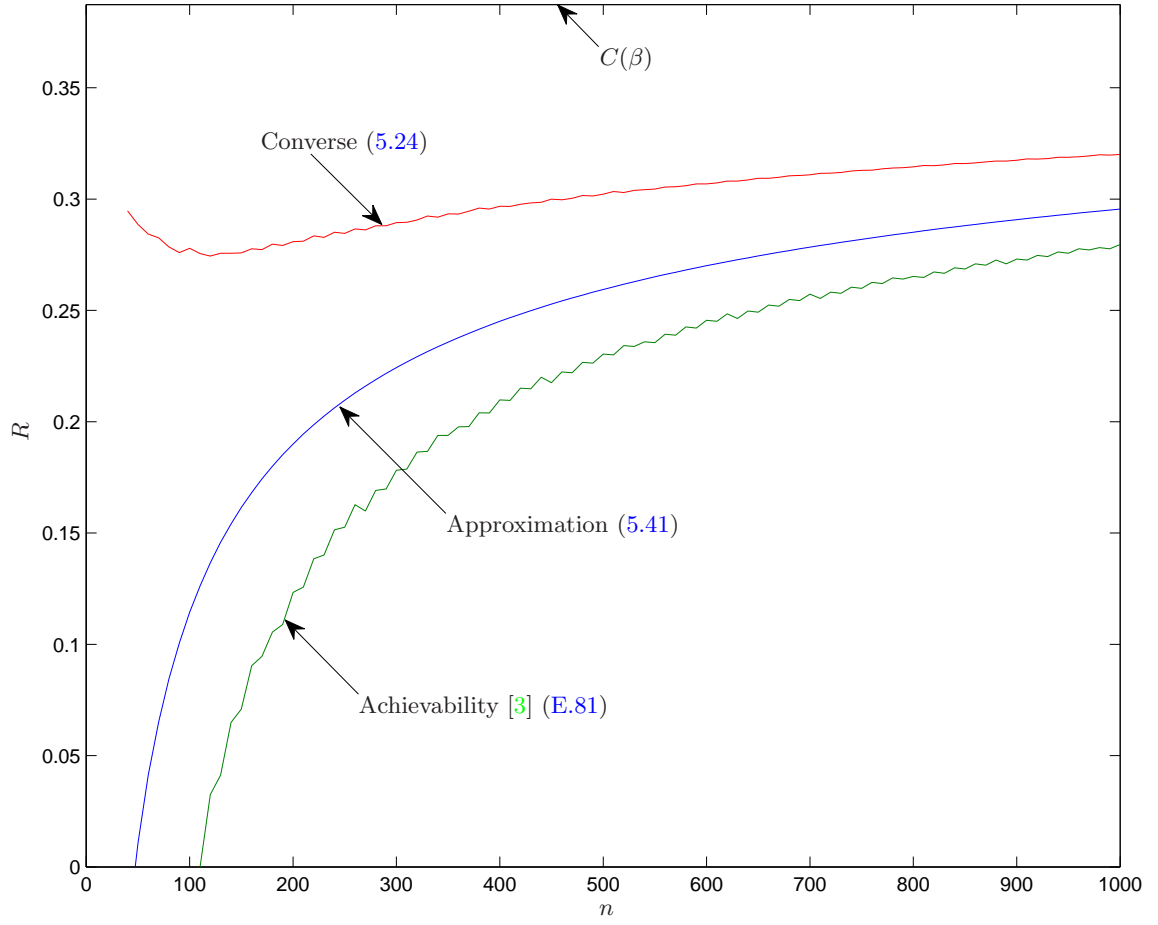


Figure 5.2: Rate-blocklength tradeoff for BSC with crossover probability $\delta = 0.11$ where the normalized Hamming weight of codewords is constrained not to exceed $\beta = 0.25$ and the tolerated error probability is $\epsilon = 10^{-4}$.

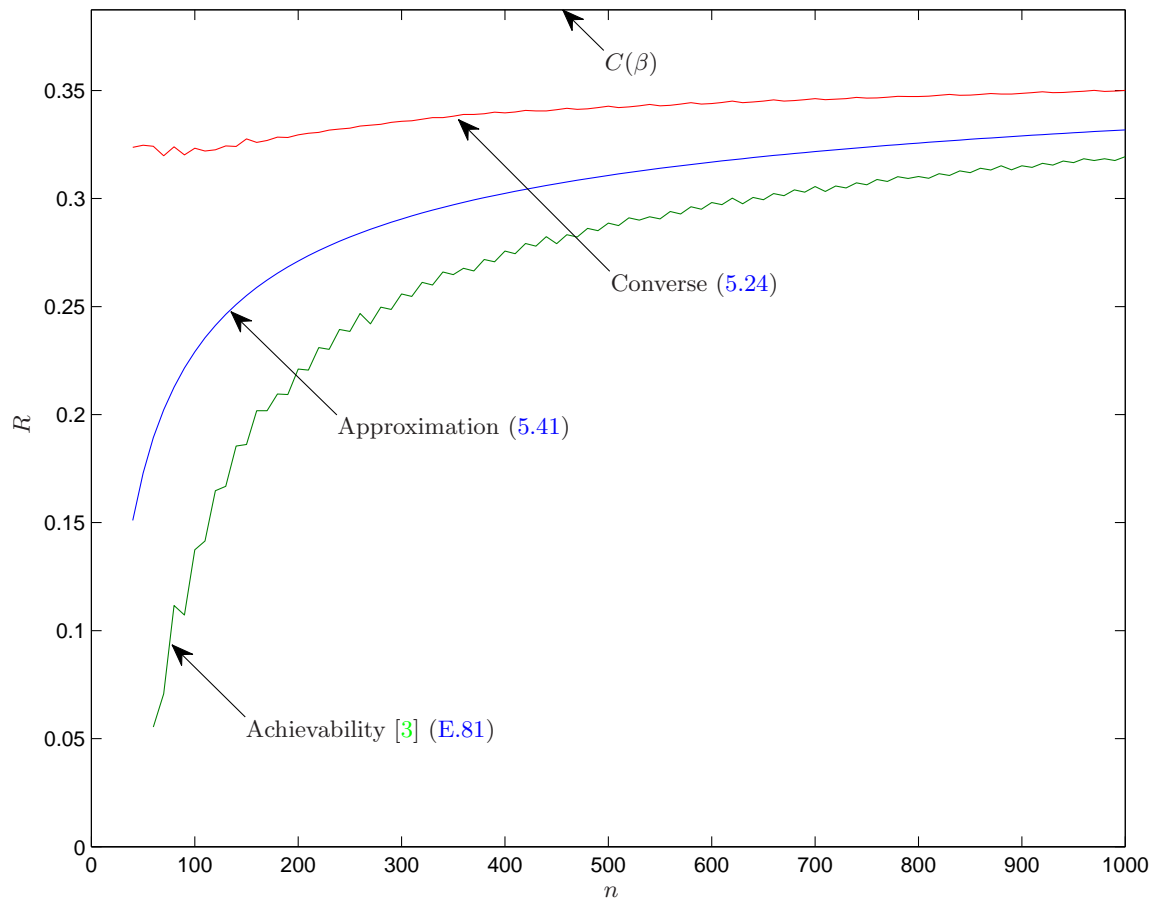


Figure 5.3: Rate-blocklength tradeoff for BSC with crossover probability $\delta = 0.11$ where the normalized Hamming weight of codewords is constrained not to exceed $\beta = 0.25$ and the tolerated error probability is $\epsilon = 10^{-2}$.

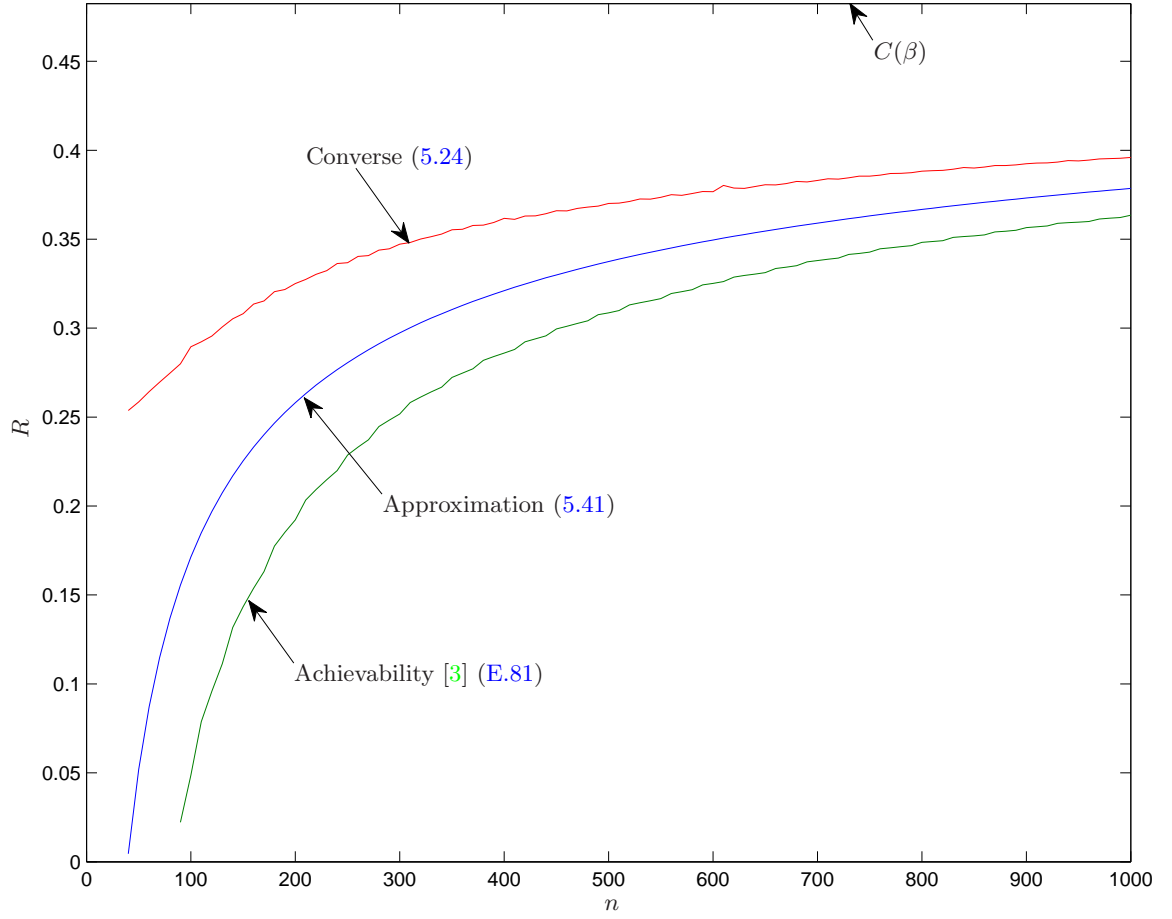


Figure 5.4: Rate-blocklength tradeoff for BSC with crossover probability $\delta = 0.11$ where the normalized Hamming weight of codewords is constrained not to exceed $\beta = 0.4$ and the tolerated error probability is $\epsilon = 10^{-4}$.

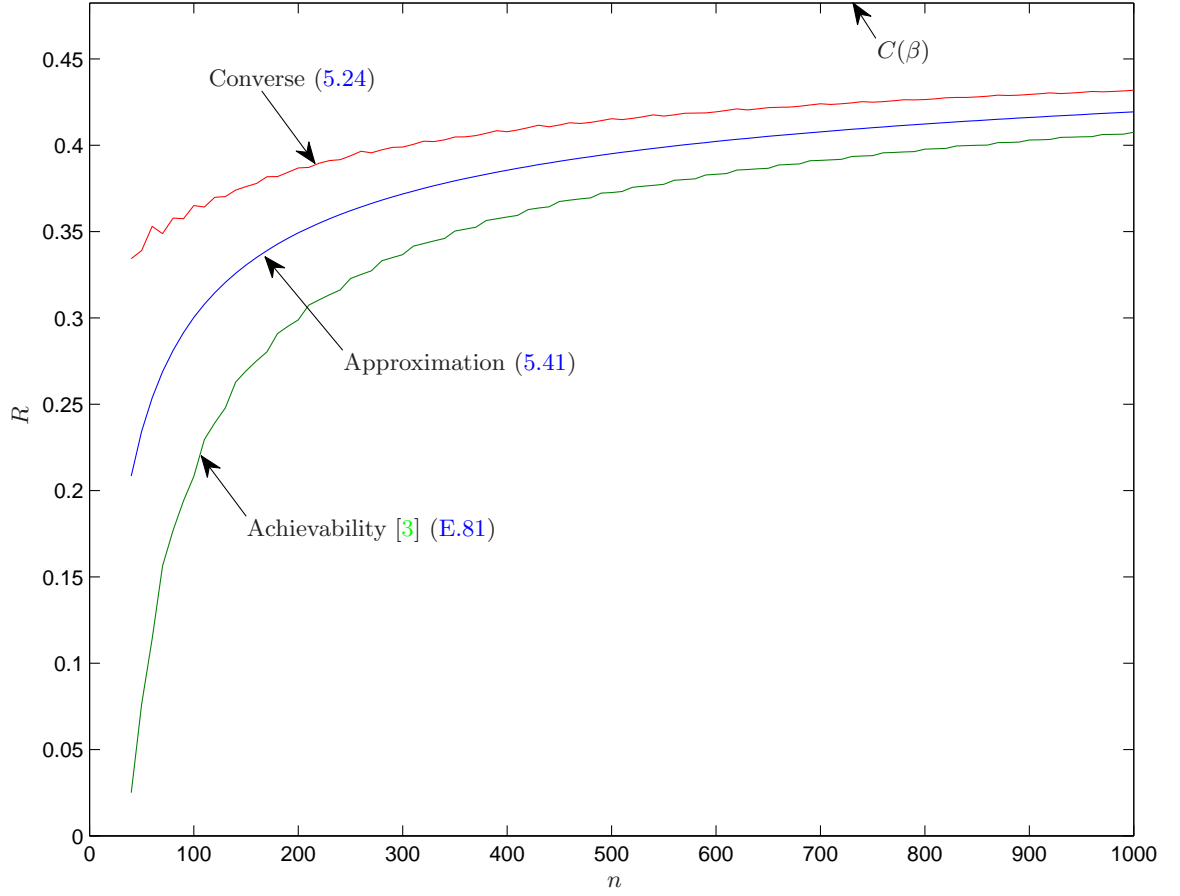


Figure 5.5: Rate-blocklength tradeoff for BSC with crossover probability $\delta = 0.11$ where the normalized Hamming weight of codewords is constrained not to exceed $\beta = 0.4$ and the tolerated error probability is $\epsilon = 10^{-2}$.

As discussed in 1.5, in the joint source-channel coding problem setup the source is no longer equiprobable on an alphabet of cardinality M , as in Definition 5.2, but is rather arbitrarily distributed on an abstract alphabet \mathcal{M} . Further, instead of reproducing the transmitted S under a probability of error criterion, we might be interested in approximating S within a certain distortion, so that a decoding failure occurs if the distortion between the source and its reproduction exceeds a given distortion level d , i.e. if $\mathbf{d}(S, Z) > d$. A (d, ϵ, β) code is a code for a fixed source-channel pair such that the probability of exceeding distortion d is no larger than ϵ and no channel codeword costs more than β . A (d, ϵ, β) code in a block coding setting, when a source block of length k is mapped to a channel block of length n , is called a $(k, n, d, \epsilon, \beta)$ code. The counterpart of the \mathbf{b} -tilted information density in lossy compression is the \mathbf{d} -tilted information, $j_S(s, d)$, which, in a certain sense, quantifies the number of bits required to reproduce the source outcome $s \in \mathcal{M}$ within distortion d . For rigorous definitions and further details we refer the reader back to Chapter 3.

Theorem 5.6 (Converse). *The existence of a (d, ϵ, β) code for S and $P_{Y|X}$ requires that*

$$\epsilon \geq \inf_{P_{X|S}} \max_{\gamma > 0} \left\{ \sup_{\bar{Y}} \mathbb{P} [j_S(S, d) - j_{Y|X\|\bar{Y}}(X; Y, \beta) \geq \gamma] - \exp(-\gamma) \right\} \quad (5.54)$$

$$\geq \max_{\gamma > 0} \left\{ \sup_{\bar{Y}} \mathbb{E} \left[\inf_{x \in \mathcal{X}} \mathbb{P} [j_S(S, d) - j_{Y|X\|\bar{Y}}(x; Y, \beta) \geq \gamma \mid S] \right] - \exp(-\gamma) \right\} \quad (5.55)$$

where the probabilities in (5.54) and (5.55) are with respect to $P_S P_{X|S} P_{Y|X}$ and $P_{Y|X=x}$, respectively.

Under the usual memorylessness assumptions, applying Theorem 5.4 to the bound in (5.55), it is easy to show that the strong converse holds for lossy joint source-channel coding over channels with input cost constraints. A more refined analysis leads to the following result.

Theorem 5.7 (Gaussian approximation). *Assume the channel has finite input and output alphabets. Under restrictions (i)–(iv) of Section 2.6.2 and (ii)–(iv) of Section 5.4.1, the parameters of the optimal (k, n, d, ϵ) code satisfy*

$$nC(\beta) - kR(d) = \sqrt{nV(\beta) + k\mathcal{V}(d)} Q^{-1}(\epsilon) + \theta(n) \quad (5.56)$$

where $\mathcal{V}(d) = \text{Var}[j_S(S, d)]$, $V(\beta)$ is given in (5.43), and the remainder $\theta(n)$ satisfies, if $V(\beta) > 0$,

$$-\frac{1}{2} \log n + O(\sqrt{\log n}) \leq \theta(n) \quad (5.57)$$

$$\leq \bar{\theta}(n) + \left(\frac{1}{2} |\text{supp}(P_{X^*})| - 1 \right) \log n \quad (5.58)$$

where $\bar{\theta}(n)$ denotes the upper bound on $\theta(n)$ given in Theorem 3.10. If $V(\beta) = 0$, the upper bound on $\theta(n)$ stays the same, and the lower one becomes (5.46).

5.6 Conclusion

We introduced the concept of \mathbf{b} -tilted information density (Definition 5.1), a random variable whose distribution plays the key role in the analysis of optimal channel coding under input cost constraints. We showed a new converse bound (Theorem 5.3), which gives a lower bound on the average error probability in terms of the cdf of the \mathbf{b} -tilted information density. The properties of \mathbf{b} -tilted information density listed in Theorem 5.1 play a key role in the asymptotic analysis of the bound in Theorem 5.3 in Section 5.4, which does not only lead to the strong converse and the dispersion-cost function when coupled with the corresponding achievability bound, but it also proves that the third order term in the asymptotic expansion (5.2) is upper bounded (in the most common case of $V(\beta) > 0$) by $\frac{1}{2} \log n + O(1)$. In addition, we showed in Section 5.5 that the results of Chapter 3 generalize to coding over channels with cost constraints and also tightened the estimate of the third order term in Chapter 3. As propounded in [98, 99], the gateway to refined analysis of the third order term is an apt choice of a non-product distribution $P_{\tilde{\mathbf{Y}}_n}$ in the bounds in Theorems 5.3 and 5.6.

Chapter 6

Open problems

This concluding chapter provides a survey of open problems in the subject of finite blocklength lossy compression and identifies possible future research directions.

6.1 Lossy compression of sources with memory

As most real world sources, such as text, images, video and audio, are not memoryless, finite blocklength analysis of lossy compression for sources with memory has evident practical importance. While the core results of this thesis, namely, new tight achievability and converse bounds to the minimum achievable source coding rate as a function of blocklength and tolerable distortion, allow for memory, analysis and numerical computation of those bounds has been performed only in the most basic setting of lossy compression of a stationary memoryless source. It would be both analytically insightful and practically relevant to derive an analytical approximation to the minimum achievable finite blocklength coding rate of Markov sources similar in flavor to the Gaussian approximation in (1.9).

In the related scenarios of lossless data compression of a Markov source and data transmission over a binary symmetric channel in which the crossover probability evolves as a binary symmetric Markov chain, such approximations have been derived in [100] and [101], respectively. In rate-distortion theory, however, such a pursuit meets unique challenges, as even for the simplest model of a source with memory, namely, a binary symmetric Markov source with bit error rate distortion, the asymptotic fundamental limit is not known for all distortion allowances, let alone its finite blocklength refinements.

What is known in rate-distortion theory for sources with memory is sketched next. The coding

theorem for ergodic discrete-alphabet sources with memory [7] shows that the rate-distortion function, which gives the minimum asymptotically achievable rate, is expressed as a limit of a sequence of solutions of a certain convex optimization problem parameterized by blocklength k :

$$R(d) = \limsup_{k \rightarrow \infty} \inf_{\substack{P_{Z^k|S^k}: \\ \mathbb{E}[d(S^k, Z^k)] \leq d}} \frac{1}{k} I(S^k; Z^k) \quad (6.1)$$

For general (nonergodic and/or nonstationary) sources, a formula for the rate-distortion function in terms of limsup in probability of information rate was given by Han [43] and by Steinberg and Verdú [102].

In general the limsup in (6.1) is difficult to evaluate. In the simple case of a binary symmetric Markov source with bit error rate distortion, namely, a source with $P_{S^k} = \prod_{i=1}^k P_{S^i|S^{i-1}}$, where

$$P_{S_k|S_{k-1}}(s_k|s_{k-1}) = \begin{cases} p & s_k \neq s_{k-1} \\ 1-p & s_k = s_{k-1} \end{cases} \quad (6.2)$$

a partial answer is provided by Gray [103] who showed that

$$R(d) \geq h(p) - h(d) \quad (6.3)$$

with equality in the following small distortion region,

$$0 \leq d \leq \frac{1}{2} \left(1 - \left(\frac{p}{1-p} \right)^2 \right) \quad (6.4)$$

For higher distortions, upper and lower bounds allowing to compute the rate-distortion function in this case with desired accuracy have been recently shown in [104]. Gray [105] showed a lower bound to the rate-distortion function of finite-state finite-alphabet Markov sources with a balanced distortion measure and identified conditions under which it coincides with its corresponding upper bound.

For variable-length lossy compression of sources with memory, Kontoyiannis [32] presented upper and lower bounds to the minimum achievable encoded length as a function of a given source realization and required fidelity of reproduction, which eventually hold with probability 1 for a sufficiently large blocklength k . A major weakness of these bounds is that they have exponential computational complexity.

Notably, all the papers [32, 103–105] implicitly use the representation of the rate-distortion function via the \mathbf{d} -tilted information (see Theorem 2.1) to obtain their results.

6.2 Average distortion achievable at finite blocklengths

For reasons explained in Section 1.8, the fidelity criterion this thesis heeds is the probability of exceeding a given distortion. As evidenced by the simplicity of our bounds, not only excess distortion is the most basic and natural way to look at lossy compression problems at finite blocklengths, it also accounts for stochastic variability of the source in the finite blocklength regime in a way looking just at the average distortion does not. Indeed, the redundancy result [41] in (2.80) asserts that the minimum average distortion as a function of blocklength k approaches the distortion-rate function as $O\left(\frac{\log k}{k}\right)$, while our result in (2.177) states that the minimum excess distortion approaches the distortion-rate function as $O\left(\frac{1}{\sqrt{k}}\right)$, dwarfing the overhead of $O\left(\frac{\log k}{k}\right)$ measured in terms of average distortion.

Nevertheless, it would be enlightening to complement our finite blocklength results on excess distortion with those on average distortion, and, in particular, to contrast the approximation in (2.80) with corresponding finite blocklength bounds. Most of the existing bounds in vector quantization [106] are asymptotic. The most well-known is that for fixed analog vector dimension, a signal-to-noise-ratio achieved by fine quantization grows 6 dB for every increase of one bit [107–109]; furthermore, uniform quantization is suboptimal by at most 1.53 dB [110, 111], and scalar quantization of Gaussian sources suffers a penalty of 4.35 dB. Some progress in bounding the minimum average distortion achievable by a vector quantizer of a fixed dimension has recently been announced in [112], where a number of finite blocklength bounds on average distortion are shown.

As mentioned in Section 2.10, since the minimum achievable average distortion can be written as

$$D(k, R) = \inf_{\mathbf{f}, \mathbf{c}: |\mathbf{f}| \leq \exp(kR)} \int_0^\infty \mathbb{P}[\mathbf{d}(S^k, \mathbf{c}(\mathbf{f}(S^k))) > \xi] d\xi \quad (6.5)$$

finite blocklength bounds on average distortion can be obtained by integrating our bounds on excess distortion over all distortion thresholds. However, while one might expect an achievability bound obtained by an integration of the random coding bound in (2.16) (making the choice of the code, i.e. $P_{\bar{Z}}$, after the integration) to be reasonably tight, the same unfortunately cannot be said about integrating excess-distortion converse bounds. The reason is that in this way we compute a lower

bound to

$$\int_0^\infty \inf_{f, c: |f| \leq \exp(kR)} \mathbb{P} [d(S^k, c(f(S^k))) > \xi] d\xi \quad (6.6)$$

while in reality the choice of code cannot be matched to all distortion thresholds ξ . This difficulty cannot be circumvented with our \mathbf{d} -tilted information bound in Theorem 2.12 because it essentially has optimization over all codes built in; however the converse bound in Theorem 4.20 does not, thus one can expect to obtain a tighter average distortion bound from it by performing the optimization over the code (i.e. $P_{Z|X}$) after the integration.

The integrated bounds, the approximation [42] in (2.80), together with the performance of a sample of quantization schemes, are plotted in Figures 6.1 and 6.2, for the case of the Gaussian source under mean squared error distortion.

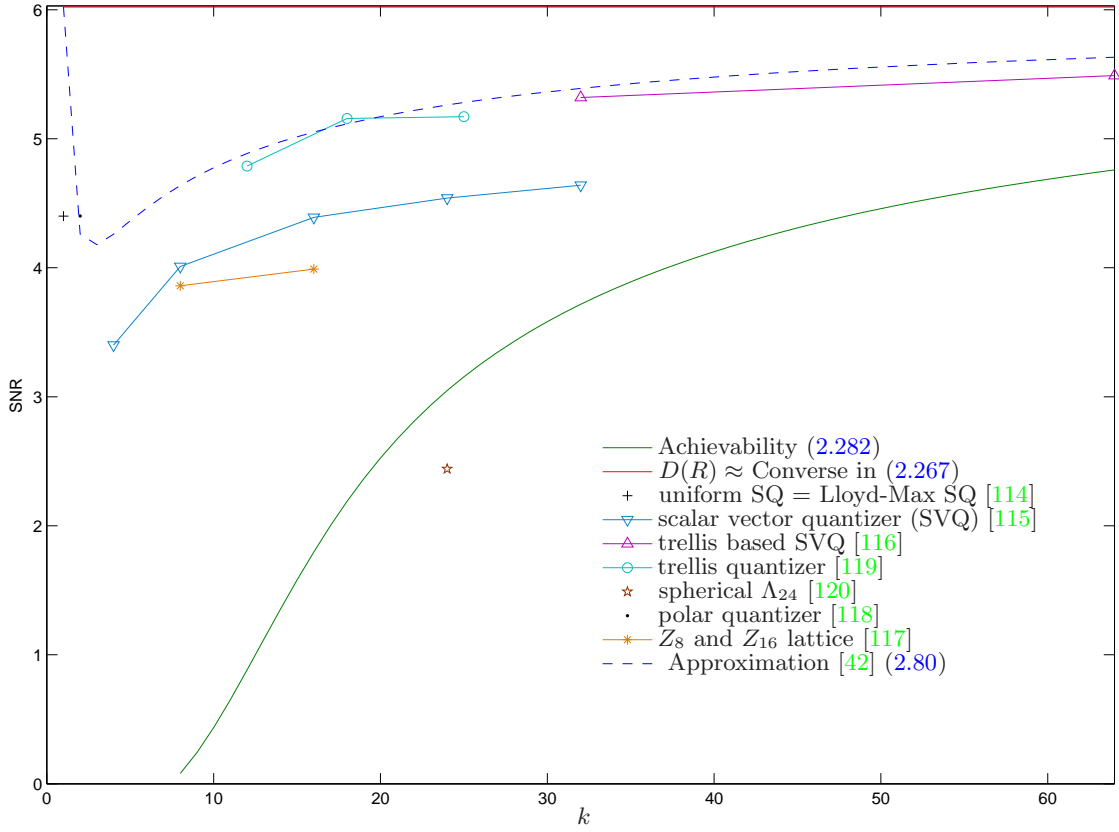


Figure 6.1: Comparison of various quantization schemes for GMS in terms of average SNR $= -10 \log_{10} \mathbb{E} [d(S^k, Z^k)]$. $R = 1$ bit/source sample.

Even more ground is left untouched in the realm of lossy joint source-channel coding under the average distortion constraint. Indeed, while the redundancy result [41] in (2.80) suggests that a similar beautiful expansion should hold in the joint source-channel coding setting, to date not even

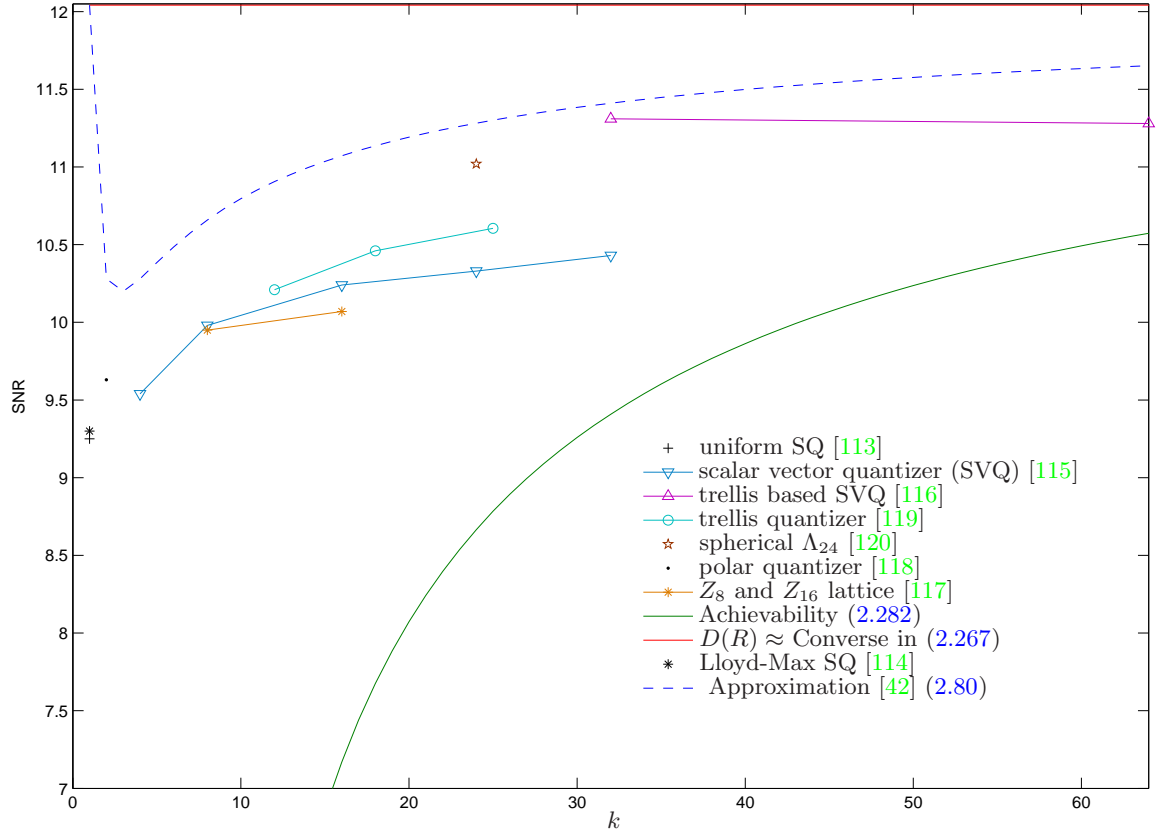


Figure 6.2: Comparison of various quantization schemes for GMS in terms of average SNR $= -10 \log_{10} \mathbb{E} [d(S^k, Z^k)]$. $R = 2$ bits/source sample.

a redundancy result of order $O\left(\frac{\log k}{k}\right)$ has been shown. Pilc’s achievability bound [40, 63] is based on a separated scheme and leads to a redundancy result of order $O\left(\sqrt{\frac{\log k}{k}}\right)$. This estimate can be tightened using our random coding joint achievability scheme in Theorem 3.8.

6.3 Nonasymptotic network information theory

Using nonasymptotic covering (Lemma 2.18) and packing lemmas, achievability bounds for several multiuser information theory problems have been shown in [44]. Unfortunately, as we saw in Chapter 2, in the single-user case the nonasymptotic covering lemma leads to Shannon’s bound (Theorem 2.4) which is rather loose for blocklengths of order 1000. Thus more refined bounding techniques (perhaps in the spirit of Theorem 2.19) are required to develop nonasymptotic achievability bounds for multiuser information theory that are tight.

New non-asymptotic achievability bounds for three side-information problems, namely, the Wyner-Ahlsvede-Körner problem of almost-lossless source coding with rate-limited side-information, the Wyner-Ziv [121] problem of lossy source coding with side-information at the decoder and the Gelfand-Pinsker problem of channel coding with noncausal state information available at the encoder have recently been proposed by Watanabe et al. [122].

A clever technique to show one-shot achievability bounds in network information theory has been recently developed by Yassaee et. al. [123]. It involves a stochastic decoder which draws a message randomly from a posterior probability distribution induced by the code and an application of Jensen’s inequality to bound the expectations of jointly convex functions of several random variables that result from the error probability analysis of that decoder. The technique leads to a number of novel achievability bounds for the multiuser settings of Gelfand-Pinsker, Berger-Tung, Heegard-Berger/Kaspi, broadcast channel, multiple description coding and joint source-channel coding over a MAC.

To judge the tightness of the nonasymptotic achievability bounds in [44, 122, 123], matching converse results are needed. The progress in this direction has however been more modest. Indeed, in e.g. the Wyner-Ziv setting not even a general strong converse result has been shown. Most previous work in that direction is focused on obtaining bounds to the reliability function using type counting arguments, which only apply to finite alphabet sources. Arutyunyan and Marutyan [124] appear to be the first who studied error exponents for the Wyner-Ziv problem, however, neither their upper nor lower bounds have been proven rigorously. Jayaraman and Berger [125] also studied error exponents for the Wyner-Ziv problem, although they restricted their attention to just one of

the possible error events, namely, the binning error.

The defining feature of the approach for proving nonasymptotic converses adopted in this thesis is that it, quite unexpectedly, relies on the properties of the extremal mutual information problem (which represents the *asymptotic* fundamental limit) to show a *nonasymptotic* converse. It would be interesting to see if this approach can be generalized to yield yet undiscovered converses in multiterminal information theory. In this direction, we have already shown that it succeeds for the lossy source coding problem with side information at both compressor and decompressor [22]. General converses for such considerably more challenging multiterminal source coding scenarios as the Wyner-Ziv and the Ahlswede-Körner settings should also be within our reach.

It is known that in compression of a memoryless Gaussian source under the mean-square error distortion when the decompressor has access to the AWGN-corrupted outputs of the source, asymptotically, there is no benefit of making this side information also available at the compressor. It would be interesting to see whether any benefit transpires at finite blocklengths.

6.4 Other problems in nonasymptotic information theory

Another open problem is finite-blocklength analysis of successive refinement [126], which consists of first approximating data using only a few bits of information, then iteratively improving the approximation as more and more bits are supplied. It is known that in a few instances, such as finite-alphabet sources with symbol error rate distortion and Gaussian sources with mean-square distortion, the requirement of successive refinement entails no penalty in rate. Is that true also in terms of the dispersion-distortion function? Or is it one of those instances where, as in separate vs. joint source-channel coding, the penalty materializes at finite blocklength?

For difference-distortion measures and in the region of rates for which the Shannon lower bound is tight, it would be interesting to see whether we can leverage non-asymptotic results in lossless compression to provide non-asymptotic results for lossy compression.

For those channel coding settings with cost constraints where, in lieu of the achievable rate, the figure of merit is the achievable rate per unit cost, an elegant expression for the fundamental limit provided that the number of degrees of freedom is allowed to grow indefinitely was found in [127]. What happens if the number of degrees of freedom is bounded is an open question. Can the nonasymptotic fundamental limit be expressed in terms of the ratio $\frac{J_{X,Y}^*(X,b)}{b(X)}$, where $J_{X,Y}^*(x,b)$ and $b(x)$ are the b -tilted information (see Chapter 5) and the cost in x , respectively?

Systematic codes are those where each codeword contains the uncoded information source string

plus a string of parity-check symbols. Shamai et al. [128] characterized the asymptotically achievable average distortion and found the necessary and sufficient conditions under which systematic transmission does not incur loss of optimality. It would be enlightening to quantify the penalty of systematic communication in the finite blocklength regime.

Appendix A

Tools

This appendix lists instrumental auxiliary results.

A.1 Properties of relative entropy

Lemma A.1 ([129]). *Let $0 \leq \alpha \leq 1$, and let $P_{\bar{X}}$ and P_{X^*} be two distributions on the same probability space. Then,*

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} D(\alpha P_{\bar{X}} + (1 - \alpha) P_{X^*} \| P_{X^*}) = 0 \quad (\text{A.1})$$

as long as the relative entropy in (A.1) is finite.

Lemma A.2 ([24]). *Let $g: \mathcal{X} \mapsto [-\infty, +\infty]$ and let \bar{X} be a random variable on \mathcal{X} such that $\mathbb{E} [\exp (g(\bar{X}))] < \infty$. Then,*

$$\mathbb{E} [g(X)] - D(X \| \bar{X}) \leq \log \mathbb{E} [\exp (g(\bar{X}))] \quad (\text{A.2})$$

with equality if and only if X has distribution P_{X^} such that*

$$\imath_{X^* \| \bar{X}}(x) = g(x) - \log \mathbb{E} [\exp (g(\bar{X}))] \quad (\text{A.3})$$

Proof. If the left side of (A.2) is not $-\infty$, we can write

$$\mathbb{E}[g(X)] - D(X\|\bar{X}) = \mathbb{E}[g(X) - \imath_{X\|X^*}(X) - \imath_{X^*\|\bar{X}}(X)] \quad (\text{A.4})$$

$$= \log \mathbb{E}[\exp(g(\bar{X}))] - D(X\|X^*) \quad (\text{A.5})$$

which is maximized by letting $P_X = P_{X^*}$. \square

Lemma A.3 (Pinsker's inequality and its reverse). *For two random variables X and \bar{X} defined on the same finite alphabet \mathcal{A}*

$$\frac{1}{2} |P_X - P_{\bar{X}}|^2 \log e \leq D(X\|\bar{X}) \quad (\text{A.6})$$

$$\leq \frac{\log e}{\min_{a \in \mathcal{A}} P_{\bar{X}}(a)} |P_X - P_{\bar{X}}|^2 \quad (\text{A.7})$$

where $|\cdot|$ denotes the Euclidean distance.

In fact, a stronger inequality than (A.6) in which $|\cdot|$ is the total variation distance also holds for random variables defined on the same abstract space. A proof of (A.6) can be found in e.g. [130, Lemma 11.6.1]. The proof of (A.7), also provided below, is given in e.g. [131, Lemma 6.3].

Proof of (A.7).

$$D(X\|\bar{X}) \leq \sum_{a \in \mathcal{A}} P_X(a) \left(\frac{P_X(a)}{P_{\bar{X}}(a)} - 1 \right) \log e \quad (\text{A.8})$$

$$= \sum_{b \in \mathcal{B}} \frac{1}{P_{\bar{X}}(b)} (P_X(b) - P_{\bar{X}}(b))^2 \log e \quad (\text{A.9})$$

$$\leq \frac{\log e}{\min_{b \in \mathcal{B}} P_{\bar{X}}(b)} |P_X - P_{\bar{X}}|^2 \quad (\text{A.10})$$

\square

A.2 Properties of sums of independent random variables

Lemma A.4. *Let zero-mean random variables $W_i, i = 1, 2, \dots$ be independent, and*

$$0 < V_{\min} \leq V_k \leq V_{\max} \quad (\text{A.11})$$

$$T_k \leq T_{\max} \quad (\text{A.12})$$

where V_k, T_k are defined in (2.157) and (2.158). Denote

$$B_{\max} = \frac{c_0 T_{\max}}{V_{\min}^{\frac{3}{2}}} \quad (\text{A.13})$$

1. For arbitrary $b > 0$ and all

$$\tau > \tau_0 \triangleq 2B_{\max} \sqrt{2\pi V_{\max}} + b \quad (\text{A.14})$$

$$k \geq \frac{1}{2V_{\min}} \frac{\tau^2}{\log \tau - \log \tau_0} \quad (\text{A.15})$$

where $c_0 > 0$ is the constant in Theorem 2.23, it holds that

$$\mathbb{P} \left[0 \leq \sum_{i=1}^k W_i < \tau \right] \geq \frac{b}{\sqrt{k}} \quad (\text{A.16})$$

$$\mathbb{P} \left[\sum_{i=1}^k W_i < \tau \right] \geq \frac{1}{2} + \frac{b}{\sqrt{k}} \quad (\text{A.17})$$

Inequality (A.17) would still hold if (A.14) is relaxed replacing $2c_0$ by c_0 .

2. For arbitrary $b > B_{\max}$ and

$$k \geq e^{\frac{1}{2\pi}(b-B_{\max})^2} \quad (\text{A.18})$$

it holds that

$$\mathbb{P} \left[\sum_{i=1}^k W_i > \sqrt{V_{\max} k \log k} \right] \leq \frac{b}{\sqrt{k}} \quad (\text{A.19})$$

3. For arbitrary $\tau > 0$ and all $k \geq 1$, it holds that

$$\mathbb{P} \left[\sum_{i=1}^k W_i > k\tau \right] \leq \frac{V_{\max}}{k\tau^2} \quad (\text{A.20})$$

Instead of both (A.11) and (A.12), just $V_k \leq V_{\max}$ is required.

4. [3, Lemma 47]. For $\tau > 0$

$$\mathbb{E} \left[\exp \left\{ - \sum_{i=1}^k W_i \right\} 1 \left\{ \sum_{i=1}^k W_i > \log \tau \right\} \right] \leq 2 \left(\frac{\log 2}{\sqrt{2\pi V_k}} + \frac{2B_k}{\sqrt{k}} \right) \frac{1}{\tau \sqrt{k}} \quad (\text{A.21})$$

where B_k is the Berry-Essen ratio (2.159).

Proof of Lemma A.4.1. By the Berry-Esséen Theorem 2.23

$$\mathbb{P} \left[0 \leq \sum_{i=1}^k W_i < \tau \right] \geq \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\tau}{\sqrt{k}V_k}} e^{-\frac{u^2}{2}} du - \frac{2c_0 T_k}{V_k^{\frac{3}{2}}} \frac{1}{\sqrt{k}} \quad (\text{A.22})$$

$$\geq \left(\frac{\tau}{\sqrt{2\pi}V_k} e^{-\frac{\tau^2}{2kV_k}} - \frac{2c_0 T_k}{V_k^{\frac{3}{2}}} \right) \frac{1}{\sqrt{k}} \quad (\text{A.23})$$

$$\geq \left(\frac{\tau}{\sqrt{2\pi}V_{\max}} e^{-\frac{\tau^2}{2kV_{\min}}} - \frac{2c_0 T_{\max}}{V_{\min}^{\frac{3}{2}}} \right) \frac{1}{\sqrt{k}} \quad (\text{A.24})$$

$$\geq \frac{b}{\sqrt{k}} \quad (\text{A.25})$$

where (A.24) holds as long as (A.14) and (A.15) are satisfied. Inequality (A.17) follows replacing 0 in the lower integration limit in (A.22) by $-\infty$ and $2c_0$ by c_0 in (A.22). \square

Proof of Lemma A.4.2. The Berry-Esséen inequality (2.155) implies

$$\mathbb{P} \left[\sum_{i=1}^k W_i > \sqrt{V_{\max} k \log_e k} \right] \leq \frac{B_{\max}}{\sqrt{k}} + Q \left(\sqrt{\log_e k} \right) \quad (\text{A.26})$$

$$< \left(B_{\max} + \frac{1}{\sqrt{2\pi \log k}} \right) \frac{1}{\sqrt{k}} \quad (\text{A.27})$$

where to get (A.27), we used

$$Q(t) < \frac{1}{\sqrt{2\pi}t} e^{-\frac{t^2}{2}} \quad (\text{A.28})$$

\square

Proof of Lemma A.4.3. By Chebyshev's inequality

$$\mathbb{P} \left[\sum_{i=1}^k W_i > k\tau \right] \leq \mathbb{P} \left[\left(\sum_{i=1}^k W_i \right)^2 > (k\tau)^2 \right] \quad (\text{A.29})$$

$$< \frac{V_{\max}}{k\tau^2} \quad (\text{A.30})$$

\square

Due to the Berry-Esséen theorem 2.23, properties of the Q -function play important role in analyzing the tails of distributions of sums of independent random variables. We list a few relevant properties next.

Lemma A.5. 1. For $x \geq 0, \xi \geq 0$

$$Q(x + \xi) \geq Q(x) - \frac{\xi}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (\text{A.31})$$

2. For arbitrary x and ξ ,

$$Q(x + \xi) \geq Q(x) - \frac{|\xi|^+}{\sqrt{2\pi}} \quad (\text{A.32})$$

3. Fix $\xi \geq 0$. Then, for all $x \geq -\frac{1}{2\xi}$,

$$Q(x(1 + \xi)) \geq Q(x) - \frac{8\xi}{\sqrt{2\pi}e} \quad (\text{A.33})$$

Proof of Lemmas A.5.1 and A.5.2. $Q(x)$ is convex for $x \geq 0$, and $Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, so (A.31) and (A.32) follow. \square

Proof of Lemma A.5.3. If $x \geq 0$, we use (A.31) to obtain

$$Q(x) - Q(x(1 + \xi)) \leq \frac{\xi x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (\text{A.34})$$

$$\leq \xi e^{-1} \sqrt{\frac{2}{\pi}} \quad (\text{A.35})$$

where (A.35) holds because the maximum of (A.34) is attained at $x^2 = 2$.

If $-\frac{1}{2\xi} \leq x \leq 0$, we use $Q(x) = 1 - Q(-x)$ to obtain

$$Q(x) - Q(x(1 + \xi)) = Q(|x|(1 - \xi|x|)) - Q(|x|) \quad (\text{A.36})$$

$$\leq \frac{\xi x^2}{\sqrt{2\pi}} e^{-\frac{x^2(1 - \xi|x|)^2}{2}} \quad (\text{A.37})$$

$$\leq \frac{\xi x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{8}} \quad (\text{A.38})$$

$$\leq \frac{8\xi}{\sqrt{2\pi}e} \quad (\text{A.39})$$

where (A.38) is due to $(1 - \xi|x|)^2 \geq \frac{1}{4}$ in $|x| \leq \frac{1}{2\xi}$, and (A.39) holds because the maximum of (A.38) is attained at $x^2 = 8$. \square

A.3 Minimization of the cdf of a sum of independent random variables

Let \mathcal{D} be a metric space with metric $d: \mathcal{D}^2 \mapsto \mathbb{R}^+$. Define the random variable Z on \mathcal{D} . Let $W_i, i = 1, \dots, n$ be independent conditioned on Z . Denote

$$\mu_n(z) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[W_i | Z = z] \quad (\text{A.40})$$

$$V_n(z) = \frac{1}{n} \sum_{i=1}^n \text{Var}[W_i | Z = z] \quad (\text{A.41})$$

$$T_n(z) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[|W_i - \mathbb{E}[W_i]|^3 | Z = z] \quad (\text{A.42})$$

Let $\ell_1, \ell_2, \ell_3, L_1, F_1, F_2, V_{\min}$ and T_{\max} be positive constants. We assume that there exist $z^* \in \mathcal{D}$ and sequences μ_n^*, V_n^* such that for all $z \in \mathcal{D}$,

$$\mu_n^* - \mu_n(z) \geq \ell_1 d^2(z, z^*) - \frac{\ell_2}{\sqrt{n}} d(z, z^*) - \frac{\ell_3}{n} \quad (\text{A.43})$$

$$\mu_n^* - \mu_n(z^*) \leq \frac{L_1}{n} \quad (\text{A.44})$$

$$|V_n(z) - V_n^*| \leq F_1 d(z, z^*) + \frac{F_2}{\sqrt{n}} \quad (\text{A.45})$$

$$V_{\min} \leq V_n(z) \quad (\text{A.46})$$

$$T_n(z) \leq T_{\max} \quad (\text{A.47})$$

Theorem A.6. *In the setup described above, under assumptions (A.43)–(A.47), for any $A > 0$, there exists a $K \geq 0$ such that, for all $|\Delta| \leq \delta_n$ (where δ_n is specified below) and all sufficiently large n :*

1. If $\delta_n = \frac{A}{\sqrt{n}}$,

$$\min_{z \in \mathcal{D}} \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* - \Delta) | Z = z \right] \geq Q \left(\Delta \sqrt{\frac{n}{V_n^*}} \right) - \frac{K}{\sqrt{n}} \quad (\text{A.48})$$

2. For $\delta_n = A \sqrt{\frac{\log n}{n}}$,

$$\min_{z \in \mathcal{D}} \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* - \Delta) | Z = z \right] \geq Q \left(\Delta \sqrt{\frac{n}{V_n^*}} \right) - K \sqrt{\frac{\log n}{n}} \quad (\text{A.49})$$

3. Fix $0 \leq \beta \leq \frac{1}{6}$. If in (A.45), $V_n^* = 0$ (which implies that $V_{\min} = 0$ in (A.46)), then there exists $K \geq 0$ such that for all $\Delta > \frac{A}{n^{\frac{1}{2}+\beta}}$, where $A > 0$ is arbitrary

$$\min_{z \in \mathcal{D}} \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* + \Delta) \mid Z = z \right] \geq 1 - \frac{K}{A^{\frac{3}{2}}} \frac{1}{n^{\frac{1}{4} - \frac{3}{2}\beta}} \quad (\text{A.50})$$

4. If the following tighter version of (A.43) and (A.45) holds:

$$\mu_n^* - \mu_n(z) \geq \ell_1 d^2(z, z^*) - \frac{\ell_3}{n} \quad (\text{A.51})$$

$$|V_n(z) - V_n^*| \leq F_1 d(z, z^*) + \frac{F_2}{n} \quad (\text{A.52})$$

and $\delta_n = 2\ell_1 T_{\max}^{\frac{1}{3}} V_{\min}^{\frac{5}{2}} F_1^{-2}$, then

$$\min_{z \in \mathcal{D}} \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* - \Delta) \mid Z = z \right] \geq Q \left(\Delta \sqrt{\frac{n}{V_n^*}} \right) - \frac{K}{\sqrt{n}} \quad (\text{A.53})$$

5. If in lieu of (A.43)–(A.45), only the following weaker conditions hold:

$$\mu_n^* - \mu_n(z) \geq 0 \quad (\text{A.54})$$

$$\mu_n^* - \mu_n(z^*) \leq o(1) \quad (\text{A.55})$$

$$|V_n(z^*) - V_n^*| \leq o(1) \quad (\text{A.56})$$

no other z satisfies (A.55), and $\delta_n = o(1)$, then

$$\min_{z \in \mathcal{D}} \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* - \Delta) \mid Z = z \right] \geq Q \left(\Delta \sqrt{\frac{n}{V_n^*}} \right) - \sqrt{n} o(\delta_n) \quad (\text{A.57})$$

In order to prove Theorem A.6, we first show three auxiliary lemmas. The first two deal with approximate optimization of functions.

If f and g approximate each other, and the minimum of f is approximately attained at x , then g is also approximately minimized at x , as the following lemma formalizes.

Lemma A.7. Fix $\eta > 0$, $\xi > 0$. Let \mathcal{D} be an arbitrary set, and let $f: \mathcal{D} \mapsto \mathbb{R}$ and $g: \mathcal{D} \mapsto \mathbb{R}$ be such that

$$\sup_{x \in \mathcal{D}} |f(x) - g(x)| \leq \eta \quad (\text{A.58})$$

Further, assume that f and g attain their minima. Then,

$$g(x) \leq \min_{y \in \mathcal{D}} g(y) + \xi + 2\eta \quad (\text{A.59})$$

as long as x satisfies

$$f(x) \leq \min_{y \in \mathcal{D}} f(y) + \xi \quad (\text{A.60})$$

(see Fig. A.1).

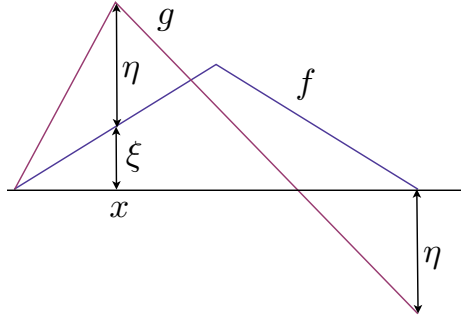


Figure A.1: An example where (A.59) holds with equality.

Proof of Lemma A.7. Let $x^* \in \mathcal{D}$ be such that $g(x^*) = \min_{y \in \mathcal{D}} g(y)$. Using (A.58) and (A.60), write

$$g(x) \leq \min_{y \in \mathcal{D}} f(y) + g(x) - f(x) + \xi \quad (\text{A.61})$$

$$\leq \min_{y \in \mathcal{D}} f(y) + \eta + \xi \quad (\text{A.62})$$

$$\leq f(x^*) + \eta + \xi \quad (\text{A.63})$$

$$= g(x^*) - g(x^*) + f(x^*) + \eta + \xi \quad (\text{A.64})$$

$$\leq g(x^*) + 2\eta + \xi \quad (\text{A.65})$$

□

The following lemma is reminiscent of [3, Lemma 64].

Lemma A.8. *Let \mathcal{D} be a compact metric space, and let $d: \mathcal{D}^2 \rightarrow \mathbb{R}^+$ be a metric. Fix $f: \mathcal{D} \mapsto \mathbb{R}$ and $g: \mathcal{D} \mapsto \mathbb{R}$. Let*

$$\mathcal{D}^\star = \left\{ x \in \mathcal{D}: f(x) = \max_{y \in \mathcal{D}} f(y) \right\} \quad (\text{A.66})$$

Suppose that for some constants $\ell > 0, L > 0$, we have, for all $(x, x^\star) \in \mathcal{D} \times \mathcal{D}^\star$,

$$f(x^\star) - f(x) \geq \ell d^2(x, x^\star) \quad (\text{A.67})$$

$$|g(x^\star) - g(x)| \leq Ld(x, x^\star) \quad (\text{A.68})$$

Then, for any positive scalars φ, ψ ,

$$\max_{x \in \mathcal{D}} \{\varphi f(x) \pm \psi g(x)\} \leq \varphi f(x^\star) \pm \psi g(x^\star) + \frac{L^2 \psi^2}{4\ell \varphi} \quad (\text{A.69})$$

Moreover, if, instead of (A.67), f satisfies

$$f(x^\star) - f(x) \geq \ell d(x, x^\star) \quad (\text{A.70})$$

then, for any positive scalars ψ, φ such that

$$L\psi \leq \ell\varphi \quad (\text{A.71})$$

we have

$$\max_{x \in \mathcal{D}} \{\varphi f(x) \pm \psi g(x)\} = \varphi f(x^\star) \pm \psi g(x^\star) \quad (\text{A.72})$$

Proof of Lemma A.8. Let x_0 achieve the maximum on the left side of (A.69). Using (A.67) and (A.68), we have, for all $x^\star \in \mathcal{D}^\star$,

$$0 \leq \varphi (f(x_0) - f(x^\star)) \pm \psi (g(x_0) - g(x^\star)) \quad (\text{A.73})$$

$$\leq -\ell \varphi d^2(x_0, x^\star) + L\psi d(x_0, x^\star) \quad (\text{A.74})$$

$$\leq \frac{L^2 \psi^2}{4\ell \varphi} \quad (\text{A.75})$$

where (A.75) follows because the maximum of (A.74) is achieved at $d(x_0, x^\star) = \frac{L\psi}{2\ell\varphi}$.

To show (A.72), observe using (A.70) and (A.68) that

$$0 \leq \varphi(f(x_0) - f(x^*)) \pm \psi(g(x_0) - g(x^*)) \quad (\text{A.76})$$

$$\leq (-\ell\varphi + L\psi) d(x_0, x^*) \quad (\text{A.77})$$

$$\leq 0 \quad (\text{A.78})$$

where (A.78) follows from (A.71). □

The following lemma deals with behavior of the Q -function.

We are now equipped to prove Theorem A.6.

Proof of Theorem A.6. To show (A.53), denote for brevity $\zeta = d(z, z^*)$ and write

$$\begin{aligned} & \mathbb{P} \left[\sum_{i=1}^n W_i > n(\mu_n^* + \Delta) \mid Z = z \right] \\ & \leq \mathbb{P} \left[\sum_{i=1}^n W_i > n \left(\mu_n(z) + \ell_1 \zeta^2 - \frac{\ell_2}{\sqrt{n}} \zeta - \frac{\ell_3}{n} + \frac{A}{n^{\frac{1}{2} + \beta}} \right) \mid Z = z \right] \end{aligned} \quad (\text{A.79})$$

$$\leq \frac{1}{n} \frac{F_1 \zeta + \frac{F_2}{\sqrt{n}}}{\left(\ell_1 \zeta^2 - \frac{\ell_2}{\sqrt{n}} \zeta - \frac{\ell_3}{n} + \frac{A}{n^{\frac{1}{2} + \beta}} \right)^2} \quad (\text{A.80})$$

$$\leq \frac{K}{A^{\frac{3}{2}}} \frac{1}{n^{\frac{1}{4} - \frac{3}{2}\beta}} \quad (\text{A.81})$$

where

- (A.79) uses (A.43) and the assumption on the range of Δ ;
- (A.80) is due to Chebyshev's inequality;
- (A.81) is by a straightforward algebraic exercise revealing that ζ that maximizes the left side of (A.81) is proportional to $\frac{A^{\frac{1}{2}}}{n^{\frac{1}{4} + \frac{1}{2}\beta}}$.

We proceed to show (A.48), (A.49), (A.50) and (A.57).

Denote

$$g_n(z) = \mathbb{P} \left[\sum_{i=1}^n W_i \leq n(\mu_n^* - \Delta) \mid Z = z \right] \quad (\text{A.82})$$

Using (A.46) and (A.47), observe

$$\frac{c_0 T_n(z)}{V_n^{\frac{3}{2}}(z)} \leq B = \frac{c_0 T_{\max}}{V_{\min}^{\frac{3}{2}}} < \infty \quad (\text{A.83})$$

Therefore the Berry-Esséen bound yields:

$$|g_n(z) - Q(\sqrt{n}\nu_n(z))| \leq \frac{B}{\sqrt{n}} \quad (\text{A.84})$$

where

$$\nu_n(z) = \frac{\mu_n(z) - \mu_n^* + \Delta}{\sqrt{V_n(z)}} \quad (\text{A.85})$$

Denote

$$\nu_n^* = \frac{\Delta}{\sqrt{V_n^*}} \quad (\text{A.86})$$

Since

$$g_n(z) = Q(\sqrt{n}\nu_n^*) + [g_n(z) - Q(\sqrt{n}\nu_n(z))] + [Q(\sqrt{n}\nu_n(z)) - Q(\sqrt{n}\nu_n^*)] \quad (\text{A.87})$$

$$\geq Q(\sqrt{n}\nu_n^*) - \frac{B}{\sqrt{n}} + [Q(\sqrt{n}\nu_n(z)) - Q(\sqrt{n}\nu_n^*)] \quad (\text{A.88})$$

so to show (A.48) and (A.50), it suffices to show that

$$Q(\sqrt{n}\nu_n^*) - \min_{z \in \mathcal{D}} Q(\sqrt{n}\nu_n(z)) \leq \frac{q}{\sqrt{n}} \quad (\text{A.89})$$

for some $q \geq 0$. Since Q is monotonically decreasing, to achieve the minimum in (A.89) we need to maximize $\sqrt{n}\nu_n(z)$.

To show (A.57), we just need $\sqrt{n}o(\delta_n)$ in the right side of (A.89), which is immediate from (A.32) and [3, Lemma 63], which implies

$$\max_{z \in \mathcal{D}} \nu_n(z) = \nu_n^* + o(\delta_n) \quad (\text{A.90})$$

To show (A.49), replacing q with $q\sqrt{\log n}$ in the right side of (A.89) would suffice. We proceed to show (A.48), (A.49) and (A.50).

As will be proven shortly, for appropriately chosen $a, b, c > 0$ we can write

$$\nu_n^* - \frac{a\delta_n}{\sqrt{n}} \leq \max_{z \in \mathcal{D}} \nu_n(z) \quad (\text{A.91})$$

$$\leq \nu_n^* + b\nu_n^{*2} + \frac{c\delta_n}{\sqrt{n}} \quad (\text{A.92})$$

for n large enough. If the stricter conditions of Theorem A.6.3 hold, $\frac{\delta_n}{\sqrt{n}}$ in (A.91) and (A.92) can be replaced by $\frac{\delta_n}{n}$. Further, if

$$\Delta \geq -\frac{\sqrt{V_{\min}}}{2b} = -A \quad (\text{A.93})$$

then $\nu_n^* \geq -\frac{1}{2b}$, and Lemma A.5.3 applies with $x \leftarrow \sqrt{n}\nu_n^*$ and $\xi \leftarrow \frac{b}{\sqrt{n}}$ therein. So, using (A.91), (A.92), the fact that $Q(\cdot)$ is monotonically decreasing and Lemma A.5.3, we conclude that there exists $q > 0$ such that

$$\begin{aligned} & Q(\sqrt{n}\nu_n^*) - \min_{z \in \mathcal{D}} Q(\sqrt{n}\nu_n(z)) \\ & \leq Q(\sqrt{n}\nu_n^*) - Q(\sqrt{n}\nu_n^* + \sqrt{n}b\nu_n^{*2} + c\delta_n) \end{aligned} \quad (\text{A.94})$$

$$\leq Q(\sqrt{n}\nu_n^*) - Q(\sqrt{n}\nu_n^* + \sqrt{n}b\nu_n^{*2}) + \frac{c}{\sqrt{2\pi}}\delta_n \quad (\text{A.95})$$

$$\leq \frac{q}{\sqrt{n}} + \frac{c}{\sqrt{2\pi}}\delta_n \quad (\text{A.96})$$

where

- (A.95) is due to (A.31);
- (A.96) holds by Lemma A.5.3 as long as $\nu_n^* \geq -\frac{1}{2b}$.

Under the stricter conditions of Theorem A.6.3, δ_n in (A.96) is replaced by $\frac{\delta_n}{\sqrt{n}}$. Thus, (A.96) establishes (A.48), (A.49) and (A.50).

It remains to prove (A.91) and (A.92). Observe that for $a, b > 0$

$$\left| \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right| \leq \frac{|a - b|}{2 \min\{a, b\}^{\frac{3}{2}}} \quad (\text{A.97})$$

so, using (A.45) and (A.46), we conclude

$$\left| \frac{1}{\sqrt{V_n(z)}} - \frac{1}{\sqrt{V_n^*}} \right| \leq F_1' d(z, z^*) + \frac{F_2'}{\sqrt{n}} \quad (\text{A.98})$$

where

$$F'_1 = \frac{1}{2} V_{\min}^{-\frac{3}{2}} F_1 \quad (\text{A.99})$$

$$F'_2 = \frac{1}{2} V_{\min}^{-\frac{3}{2}} F_2 \quad (\text{A.100})$$

Using (A.44) and (A.98), we lower-bound $\max_{z \in \mathcal{D}} \nu_n(z)$ as

$$\max_{z \in \mathcal{D}} \nu_n(z) \geq \nu_n(z^*) \quad (\text{A.101})$$

$$\geq \nu_n^* - \frac{a\delta_n}{\sqrt{n}} \quad (\text{A.102})$$

To upper-bound $\max_{z \in \mathcal{D}} \nu_n(z)$, denote for convenience

$$f_n(z) = \frac{\mu_n(z) - \mu_n^*}{\sqrt{V_n(z)}} \quad (\text{A.103})$$

$$g_n(z) = \frac{1}{\sqrt{V_n(z)}} \quad (\text{A.104})$$

and note, using (A.43), (A.44), (A.46), (A.47) and (by Hölder's inequality)

$$V_n(z) \leq T_{\max}^{\frac{2}{3}} \quad (\text{A.105})$$

that

$$f_n(z^*) - f_n(z) = \frac{\mu_n(z^*) - \mu_n^*}{\sqrt{V_n(z^*)}} - \frac{\mu_n(z) - \mu_n^*}{\sqrt{V_n(z)}} \quad (\text{A.106})$$

$$\geq \ell'_1 d^2(z, z^*) - \frac{\ell'_2}{\sqrt{n}} d(z, z^*) - \frac{\ell'_3}{n} \quad (\text{A.107})$$

where

$$\ell'_1 = T_{\max}^{-\frac{1}{3}} \ell_1 \quad (\text{A.108})$$

$$\ell'_2 = V_{\min}^{-\frac{1}{2}} \ell_2 \quad (\text{A.109})$$

$$\ell'_3 = V_{\min}^{-\frac{1}{2}} (L_1 + \ell_3) \quad (\text{A.110})$$

Let z_0 achieve the maximum $\max_{z \in \mathcal{D}} \nu_n(z)$, i.e.

$$\max_{z \in \mathcal{D}} \nu_n(z) = f_n(z_0) + \Delta g_n(z_0) \quad (\text{A.111})$$

Using (A.98) and (A.107), we have,

$$0 \leq (f_n(z_0) - f_n(z^*)) + \Delta (g_n(z_0) - g_n(z^*)) \quad (\text{A.112})$$

$$\leq -\ell'_1 d^2(z_0, z^*) + \left(\frac{\ell'_2}{\sqrt{n}} + |\Delta| F'_1 \right) d(z_0, z^*) + \frac{2F'_2 |\Delta|}{\sqrt{n}} + \frac{\ell'_3}{n} \quad (\text{A.113})$$

$$\leq \frac{1}{4\ell'_1} \left(\frac{\ell'_2}{\sqrt{n}} + |\Delta| F'_1 \right)^2 + \frac{2F'_2 |\Delta|}{\sqrt{n}} + \frac{\ell'_3}{n} \quad (\text{A.114})$$

where (A.114) follows because the maximum of its left side is achieved at $d(z_0, z^*) = \frac{1}{2\ell'_1} \left(\frac{\ell'_2}{\sqrt{n}} + |\Delta| F'_1 \right)$.

Using (A.43), (A.46), (A.98), we upper-bound

$$\nu_n(z^*) \leq \nu_n^* + \frac{F'_2 |\Delta|}{\sqrt{n}} + \frac{\ell_3}{n V_{\min}} + \frac{\ell_3 F'_2}{n^{\frac{3}{2}}} \quad (\text{A.115})$$

Applying (A.114) and (A.115) to upper-bound $\max_{z \in \mathcal{D}} \nu_n(z)$, we have established (A.92) in which

$$b = \frac{F_1'^2 T_{\max}^{\frac{2}{3}}}{4\ell'_1} \quad (\text{A.116})$$

where we used (A.45) and (A.105) to upper-bound $\Delta^2 = \nu_n^{*2} V_n^*$, thereby completing the proof. \square

Appendix B

Lossy data compression: proofs

B.1 Properties of d-tilted information

Proof of Theorem 2.1. Although (2.10) holds under more general conditions, this proof of (2.10) requires that $P_{Z^*|S}$ is such that $P_{Z|S}P_S \ll P_{Z^*|S}P_S$. The proof follows the treatment in [24].

Equality in (2.9) is a standard result in convex optimization (Lagrange duality). By the assumption, the minimum in the right side of (2.9) is attained by $P_{Z^*|S}$, therefore $\mathbb{R}_S(d)$ is equal to the right side of (2.11).

To show (2.10), fix $0 \leq \alpha \leq 1$. Denote

$$P_S \rightarrow P_{\bar{Z}|S} \rightarrow P_{\bar{Z}} \tag{B.1}$$

$$P_{\hat{Z}|S} = \alpha P_{\bar{Z}|S} + (1 - \alpha) P_{Z^*|S} \tag{B.2}$$

$$P_S \rightarrow P_{\hat{Z}|S} \rightarrow P_{\hat{Z}} = \alpha P_{\bar{Z}} + (1 - \alpha) P_{Z^*} \tag{B.3}$$

and write

$$\begin{aligned} & \alpha (\mathbb{E} [\iota_{S;Z^*}(S; \bar{Z}) + \lambda^* d(S, \bar{Z})] - \mathbb{E} [\iota_{S;Z^*}(S; Z^*) + \lambda^* d(S, Z^*)]) + D(P_{S\hat{Z}} \| P_{SZ^*}) - D(P_{\hat{Z}} \| P_{Z^*}) \\ &= \alpha \mathbb{E} [\iota_{S;Z^*}(S; \bar{Z})] - \alpha D(P_{Z^*|S} \| P_{Z^*} | P_S) + D(P_{S\hat{Z}} \| P_{SZ^*}) - D(P_{\hat{Z}} \| P_{Z^*}) \\ &+ \lambda^* \alpha \mathbb{E} [d(S, \bar{Z})] - \lambda^* \alpha \mathbb{E} [d(S, Z^*)] \end{aligned} \tag{B.4}$$

$$= D(P_{\hat{Z}|S} \| P_{\hat{Z}} | P_S) - D(P_{Z^*|S} \| P_{Z^*} | P_S) + \lambda^* \alpha \mathbb{E} [d(S, \bar{Z})] - \lambda^* \alpha \mathbb{E} [d(S, Z^*)] \tag{B.5}$$

$$= \mathbb{E} [\iota_{S;\hat{Z}}(S; \hat{Z}) + \lambda^* d(S, \hat{Z})] - \mathbb{E} [\iota_{S;Z^*}(S; Z^*) + \lambda^* d(S, Z^*)] \tag{B.6}$$

$$\geq 0 \tag{B.7}$$

where (B.7) holds because Z^* achieves the minimum in the right side of (2.9). Since the left side of (B.4) is nonnegative, $D(P_{\hat{Z}}\|P_{Z^*}) < \infty$, and Lemma A.1 implies that $D(P_{S\hat{Z}}\|P_{SZ^*}) = o(\alpha)$ and $D(P_{\hat{Z}|S}\|P_{\hat{Z}}|P_S) = o(\alpha)$. Thus, supposing that

$$\mathbb{E} [\iota_{S;Z^*}(S; \bar{Z}) + \lambda^* d(S, \bar{Z})] < \mathbb{E} [\iota_{S;Z^*}(S; Z^*) + \lambda^* d(S, Z^*)] \quad (\text{B.8})$$

would lead to a contradiction, since then the left side of (B.4) would be negative for a sufficiently small α . We thus infer that (2.10) holds.

Let us show that if Z^* achieves $\mathbb{R}_S(d)$, then it must necessarily satisfy (2.8).

Consider the function

$$F(P_{Z|S}, P_{\bar{Z}}) = D(P_{Z|S}\|P_{\bar{Z}}|P_S) + \lambda^* \mathbb{E} [d(S, Z)] - \lambda^* d \quad (\text{B.9})$$

$$= I(S; Z) + D(Z\|\bar{Z}) + \lambda^* \mathbb{E} [d(S, Z)] - \lambda^* d \quad (\text{B.10})$$

$$\geq I(S, Z) + \lambda^* \mathbb{E} [d(S, Z)] - \lambda^* d \quad (\text{B.11})$$

Since equality in (B.11) holds if and only if $P_Z = P_{\bar{Z}}$, $\mathbb{R}_S(d)$ can be expressed as

$$\mathbb{R}_S(d) = \min_{P_{\bar{Z}}} \min_{P_{Z|S}} F(P_{Z|S}, P_{\bar{Z}}) \quad (\text{B.12})$$

$$\leq \min_{P_{\bar{Z}}} F(P_{Z^*|S}, P_{\bar{Z}}) \quad (\text{B.13})$$

$$= F(P_{Z^*|S}, P_{Z^*}) \quad (\text{B.14})$$

$$= \mathbb{R}_S(d) \quad (\text{B.15})$$

where (B.15) holds by the assumption. Therefore, equality holds in (B.13).

We now show (2.8), which would also automatically show (2.11). Fix $0 \leq \alpha \leq 1$.

For an arbitrary $P_{\bar{Z}}$, define the conditional distribution $P_{\bar{Z}^*|S}$ through (2.23), for $\lambda = \lambda^*$. Although for a fixed $P_{\bar{Z}}$ we can always define $P_{\bar{Z}^*|S}$ via (2.23), in general we cannot claim that $P_{\bar{Z}^*} = P_{\bar{Z}}$, where $P_S \rightarrow P_{\bar{Z}^*|S} \rightarrow P_{\bar{Z}^*}$, unless $P_{\bar{Z}}$ is such that for $P_{\bar{Z}}$ -a.s. z ,

$$\mathbb{E} [\exp (J_{\bar{Z}}(S, \lambda^*) - \lambda^* d(S, z))] = 1 \quad (\text{B.16})$$

where the function $J_{\bar{Z}}(s, \lambda^*)$ is defined in (2.24). Lemma A.2 implies that $P_{\bar{Z}^*|S}$ achieves equality in

$$D(P_{Z|S=s}\|P_{\bar{Z}}) + \lambda^* \mathbb{E} [d(s, Z)|S = s] \geq J_{\bar{Z}}(s, \lambda^*) \quad (\text{B.17})$$

Applying (B.17) to solve for the inner minimizer in (B.12), we have

$$\mathbb{R}_S(d) = \min_{P_{\bar{Z}}} F(P_{\bar{Z}^*|S}, P_{\bar{Z}}) \quad (\text{B.18})$$

$$= F(P_{Z^*|S}, P_{Z^*}) \quad (\text{B.19})$$

where (B.19) is the same as (B.14). Since the relation (B.17) holds for all pairs $(P_{\bar{Z}^*|S}, P_{\bar{Z}})$ in the right side of (B.18), and we know by the assumption that the minimum in (B.18) is actually achieved as indicated in (B.19), the pair $(P_{Z^*|S}, P_{Z^*})$ that achieves the rate-distortion function must satisfy (2.8), which is a particularization of (2.23).

Since $P_S \rightarrow P_{Z^*|S} \rightarrow P_{Z^*}$, equality in (B.16) particularized to P_{Z^*} holds for P_{Z^*} -a.s. z , which is equivalent to equality in (2.12). To show (2.12) for all z , write, using (2.8) and (2.11)

$$\mathbb{R}_S(d) = \mathbb{E}[J_{Z^*}(S, \lambda^*)] - \lambda^* d \quad (\text{B.20})$$

$$\leq \min_{P_{Z|S}} F(P_{Z|S}, P_{\bar{Z}}) \quad (\text{B.21})$$

$$= \mathbb{E}[J_{\bar{Z}}(S, \lambda^*)] - \lambda^* d \quad (\text{B.22})$$

For an arbitrary $\bar{z} \in \widehat{\mathcal{M}}$ and $0 \leq \alpha \leq 1$, let

$$P_{\bar{Z}} = (1 - \alpha)P_{Z^*} + \alpha\delta_{\bar{z}} \quad (\text{B.23})$$

for which (2.24) becomes

$$J_{\bar{Z}}(s, \lambda^*) = -\log[(1 - \alpha)\exp(-J_{Z^*}(s, \lambda^*)) + \alpha\exp(-\lambda^*d(s, \bar{z}))] \quad (\text{B.24})$$

Substituting (B.24) in (B.22), we obtain

$$0 \geq \mathbb{E}[J_{Z^*}(S, \lambda^*) - J_{\bar{Z}}(S, \lambda^*)] \quad (\text{B.25})$$

$$= \mathbb{E}[\log[1 - \alpha + \alpha\exp(J_{Z^*}(S, \lambda^*) - \lambda^*d(S, \bar{z}))]] \quad (\text{B.26})$$

Since $\frac{1}{\alpha} \log(1 + \alpha x) \leq x \log e$ for all $x \geq -1$, by the bounded convergence convergence theorem, the derivative of the right side of (B.26) with respect to α evaluated at $\alpha = 0$ is

$$\mathbb{E}[-1 + \exp(J_{Z^*}(S, \lambda^*) - \lambda^*d(S, \bar{z}))] \log e \leq 0 \quad (\text{B.27})$$

where the inequality holds because otherwise (B.25) would be violated for sufficiently small α . This concludes the proof of (2.12). \square

Proof of Theorem 2.2. Since by the assumption (2.8) particularized to $P_{\bar{S}}$ holds for P_{Z^*} -almost every z , we may write

$$\mathbb{E}[j_{\bar{S}}(S, d)] = \mathbb{E}[i_{\bar{S}; Z^*}(S; Z^*)] - \mathbb{R}'_{\bar{S}}(d) \mathbb{E}[d(S, Z^*) - d] \quad (\text{B.28})$$

$$= \mathbb{E}[i_{\bar{S}; Z^*}(S; Z^*)] \quad (\text{B.29})$$

Therefore, for $a \in \mathcal{M}$ (in nats)

$$\begin{aligned} & \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E}[j_{\bar{S}}(S, d)] \right|_{P_{\bar{S}}=P_S} \\ &= \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E}[\log P_{\bar{S}|Z^*}(S; Z^*)] \right|_{P_{\bar{S}}=P_S} - \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E}[\log P_{\bar{S}}(S)] \right|_{P_{\bar{S}}=P_S} \end{aligned} \quad (\text{B.30})$$

$$= \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E} \left[\frac{P_{\bar{S}|Z^*}(S; Z^*)}{P_{S|Z^*}(S; Z^*)} \right] \right|_{P_{\bar{S}}=P_S} - \mathbb{E} \left[\frac{1}{P_S(S)} \frac{\partial}{\partial P_{\bar{S}}(a)} P_{\bar{S}}(S) \right] \Big|_{P_{\bar{S}}=P_S} \quad (\text{B.31})$$

$$= \left. \frac{\partial}{\partial P_{\bar{S}}(a)} 1 \right|_{P_{\bar{S}}=P_S} - 1 \quad (\text{B.32})$$

$$= -1 \quad (\text{B.33})$$

This proves (2.15). To show (2.16), we invoke (2.15) to write

$$\dot{\mathbb{R}}_S(a) = \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E}[j_{\bar{S}}(\bar{S}, d)] \right|_{P_{\bar{S}}=P_S} \quad (\text{B.34})$$

$$= j_S(a, d) + \left. \frac{\partial}{\partial P_{\bar{S}}(a)} \mathbb{E}[j_{\bar{S}}(S, d)] \right|_{P_{\bar{S}}=P_S} \quad (\text{B.35})$$

$$= j_S(a, d) - \log e \quad (\text{B.36})$$

Finally, (2.17) is an immediate corollary to (2.16). \square

B.2 Hypothesis testing and almost lossless data compression

To show (2.94), without loss of generality, assume that the letters of the alphabet \mathcal{M} are labeled $1, 2, \dots$ in order of decreasing probabilities:

$$P_S(1) \geq P_S(2) \geq \dots \quad (\text{B.37})$$

Observe that

$$M^*(0, \epsilon) = \min \{m \geq 1 : \mathbb{P}[S \leq m] \geq 1 - \epsilon\}, \quad (\text{B.38})$$

and the optimal randomized test to decide between P_S and U is given by

$$P_{W|S}(1|a) = \begin{cases} 1, & a < M^*(0, \epsilon) \\ \alpha, & a = M^*(0, \epsilon) \\ 0, & a > M^*(0, \epsilon) \end{cases} \quad (\text{B.39})$$

for $a = 1, 2, \dots$. It follows that

$$\beta_{1-\epsilon}(P_S, U) = M^*(0, \epsilon) - 1 + \alpha \quad (\text{B.40})$$

where $\alpha \in (0, 1]$ is the solution to

$$\mathbb{P}[S \leq M^*(0, \epsilon) - 1] + \alpha P_S(M^*(0, \epsilon)) = 1 - \epsilon, \quad (\text{B.41})$$

hence (2.94).

B.3 Gaussian approximation analysis of almost lossless data compression

In this appendix we strengthen the remainder term in Theorem 2.22 for $d = 0$ (cf. (2.147)). Taking the logarithm of (2.94), we have

$$\log \beta_{1-\epsilon}(P_S, U) \leq \log M^*(0, \epsilon) \quad (\text{B.42})$$

$$\leq \log(\beta_{1-\epsilon}(P_S, U) + 1) \quad (\text{B.43})$$

$$= \log \beta_{1-\epsilon}(P_S, U) + \log \left(1 + \frac{1}{\beta_{1-\epsilon}(P_S, U)} \right) \quad (\text{B.44})$$

$$\leq \log \beta_{1-\epsilon}(P_S, U) + \frac{1}{\beta_{1-\epsilon}(P_S, U)} \log e \quad (\text{B.45})$$

where in (B.45) we used $\log(1+x) \leq x \log e$, $x > -1$.

Let $P_{S^k} = P_{\mathfrak{S}} \times \dots \times P_{\mathfrak{S}}$ be the source distribution, and let U^k to be the counting measure on \mathcal{S}^k . Examining the proof of Lemma 58 of [3] on the asymptotic behavior of $\beta_{1-\epsilon}(P, Q)$ it is not hard

to see that it extends naturally to σ -finite Q 's; thus if $\text{Var}[\iota_S(S)] > 0$,

$$\log \beta_{1-\epsilon}(P_{S^k}, U^k) = kH(S) + \sqrt{k \text{Var}[\iota_S(S)]} Q^{-1}(\epsilon) - \frac{1}{2} \log k + O(1) \quad (\text{B.46})$$

and if $\text{Var}[\iota_S(S)] = 0$,

$$\log \beta_{1-\epsilon}(P_{S^k}, U^k) = kH(S) - \log \frac{1}{1-\epsilon} \quad (\text{B.47})$$

Letting P_{S^k} and U^k play the roles of P_S and U in (B.42) and (B.45) and invoking (B.46) and (B.47), we obtain (2.147) and (2.148), respectively.

B.4 Generalization of Theorems 2.12 and 2.22

We show that even if the rate-distortion function is not achieved by any output distribution, the definition of \mathbf{d} -tilted information can be extended appropriately, so that Theorem 2.12 and the converse part of Theorem 2.22 still hold.

We use the following general representation of the rate-distortion function due to Csiszár [23].

Theorem B.1 (Alternative representation of $\mathbb{R}_S(d)$ [23]). *If the basic restriction (a) of Section 2.2 holds and in addition,*

(A) *The distortion measure is such that there exists a finite set $E \subset \widehat{\mathcal{M}}$ such that*

$$\mathbb{E} \left[\min_{z \in E} \mathbf{d}(S, z) \right] < \infty \quad (\text{B.48})$$

then for each $d > d_{\min}$, it holds that

$$\mathbb{R}_S(d) = \max_{J(s), \lambda} \{ \mathbb{E}[J(S)] - \lambda d \} \quad (\text{B.49})$$

where the maximization is over $J(s) \geq 0$ and $\lambda \geq 0$ satisfying the constraint

$$\mathbb{E}[\exp \{J(S) - \lambda \mathbf{d}(S, z)\}] \leq 1 \quad \forall z \in \widehat{\mathcal{M}} \quad (\text{B.50})$$

Let $(J^*(s), \lambda^*)$ achieve the maximum in (B.49) for some $d > d_{\min}$, and define the \mathbf{d} -tilted information in s by

$$j_S(s, d) \triangleq J^*(s) - \lambda^* d \quad (\text{B.51})$$

Note that (2.12), the only property of \mathbf{d} -tilted information we used in the proof of Theorem 2.12, still holds due to (B.50), thus Theorem 2.12 remains true.

The proof of the converse part of Theorem 2.22 generalizes immediately upon making the following two observations. First, (2.142) is still valid due to (B.49). Second, \mathbf{d} -tilted information in (B.51) still single-letterizes for memoryless sources:

Lemma B.2. *Under restrictions (i) and (ii) in Section 2.6.2, (2.163) holds.*

Proof. Let $(J^*(\mathbf{s}), \lambda^*)$ attain the maximum in (B.49) for the single-letter distribution P_S . It suffices to check that $(\sum_{i=1}^k J^*(s_i), k\lambda^*)$ attains the maximum in (B.49) for $P_{S^k} = P_S \times \dots \times P_S$.

As desired,

$$\mathbb{E} \left[\sum_{i=1}^k J^*(S_i) \right] - k\lambda^*d = k\mathbb{R}_S(d) = \mathbb{R}_{S^k}(d) \quad (\text{B.52})$$

and we just need to verify the constraints in (B.50) are satisfied:

$$\mathbb{E} \left[\exp \left\{ \sum_{i=1}^k J^*(S_i) - \lambda^* \sum_{i=1}^k \mathbf{d}(S_i, z) \right\} \right] = \prod_{i=1}^k \mathbb{E} [\exp \{ J^*(S_i) - \lambda^* \mathbf{d}(S_i, z) \}] \quad (\text{B.53})$$

$$\leq 1 \quad \forall z^k \in \hat{\mathcal{S}}^k \quad (\text{B.54})$$

□

B.5 Proof of Lemma 2.24

Before we prove Lemma 2.24, let us present some background results we will use. For $j = 1, 2, \dots$, denote, for an arbitrary $P_{\bar{Z}}$

$$\bar{\mathbf{d}}_{\bar{Z},j}(s, \lambda) = \frac{\mathbb{E} [\mathbf{d}^j(s, \bar{Z}) \exp(-\lambda \mathbf{d}(s, \bar{Z}))]}{\mathbb{E} [\exp(-\lambda \mathbf{d}(s, \bar{Z}))]} \quad (\text{B.55})$$

$$= \mathbb{E} [\mathbf{d}^j(s, \bar{Z}^*)] \quad (\text{B.56})$$

where $P_{\bar{Z}|S} = P_{\bar{Z}}$, and $P_{\bar{Z}^*|S}$ is defined in (2.23). Observe that

$$\bar{\mathbf{d}}_{\bar{Z},j}(s, 0) = \mathbb{E} [\mathbf{d}^j(s, \bar{Z})] \quad (\text{B.57})$$

Denoting by $(\cdot)'$ differentiation with respect to $\lambda > 0$, we state the following properties that are obtained by direct computation and whose proofs can be found in [37].

- A. $\left(\mathbb{E} \left[J_{\bar{Z}}(S, \lambda_{S, \bar{Z}}^*) \right] \right)' = d$ where $\lambda_{S, \bar{Z}}^* = -\mathbb{R}'_{S, \bar{Z}}(d)$.
- B. $\mathbb{E} [J_{\bar{Z}}''(S, \lambda)] < 0$ for all $\lambda > 0$ if $\mathbb{E} [\bar{d}_{Z, 2}(S, 0)] < \infty$.
- C. $J_{\bar{Z}}'(s, \lambda) = \bar{d}_{\bar{Z}, 1}(s, \lambda)$.
- D. $J_{\bar{Z}}''(s, \lambda) = \left[\bar{d}_{\bar{Z}, 1}^2(s, \lambda) - \bar{d}_{\bar{Z}, 2}(s, \lambda) \right] (\log e)^{-1} \leq 0$
if $\bar{d}_{\bar{Z}, 1}(s, 0) < \infty$.
- E. $\bar{d}'_{\bar{Z}, j}(s, \lambda) \leq 0$ if $\bar{d}_{\bar{Z}, j}(s, 0) < \infty$.
- F. $d_{\min|S, \bar{Z}} = \mathbb{E} [\alpha_{\bar{Z}}(S)]$, where $\alpha_{\bar{Z}}(s) = \text{ess inf } d(s, Z)$.

Remark B.1. By Properties A and B,

$$\mathbb{E} [J_{\bar{Z}}(S, \lambda_{S, \bar{Z}}^*)] - \lambda_{S, \bar{Z}}^* d = \sup_{\lambda > 0} \{ \mathbb{E} [J_{\bar{Z}}(S, \lambda)] - \lambda d \} \quad (\text{B.58})$$

Remark B.2. Properties C and D imply that

$$0 \leq J_{\bar{Z}}'(s, \lambda) \leq \bar{d}_{\bar{Z}, 1}(s, 0) \quad (\text{B.59})$$

Therefore, as long as $\mathbb{E} [\bar{d}_{\bar{Z}, 1}(S, 0)] < \infty$, the differentiation in Property A can be brought inside the expectation invoking the dominated convergence theorem. Keeping this in mind while taking the expectation of the equation in Property C with $\lambda = \lambda_{S, \bar{Z}}^*$ with respect to P_S , we confirm that

$$\mathbb{E} [\bar{d}_{Z, 1}(S, \lambda_{S, \bar{Z}}^*)] = d \quad (\text{B.60})$$

which is also a consequence of (B.56) and the assumption that the constraint in (2.28) is satisfied with equality when the minimum is achieved.

Remark B.3. By virtue of Properties D and E we have

$$-\bar{d}_{\bar{Z}, 2}(s, 0) \leq J_{\bar{Z}}''(s, \lambda) \log e \leq 0 \quad (\text{B.61})$$

Remark B.4. Using (B.60), derivatives of $\mathbb{R}_{S, \bar{Z}}(d)$ are conveniently expressed via $\mathbb{E} [\bar{d}_{Z, j}(s, \lambda_{S, \bar{Z}}^*)]$; in particular, at any

$$d_{\min|S, \bar{Z}} < d \leq d_{\max|S, \bar{Z}} = \mathbb{E} [\bar{d}_{\bar{Z}, 1}(S, 0)] \quad (\text{B.62})$$

we have

$$\mathbb{R}_{S,\bar{Z}}''(d) = -\frac{1}{\left(\mathbb{E}\left[\bar{d}_{\bar{Z},1}(S, \lambda_{S,\bar{Z}}^*)\right]\right)'} \quad (\text{B.63})$$

$$= \frac{\log e}{\mathbb{E}\left[\bar{d}_{\bar{Z},2}(S, \lambda_{S,\bar{Z}}^*)\right] - \mathbb{E}\left[\bar{d}_{\bar{Z},1}^2(S, \lambda_{S,\bar{Z}}^*)\right]} \quad (\text{B.64})$$

$$> 0 \quad (\text{B.65})$$

where (B.64) holds by Property D and the dominated convergence theorem due to (B.61) as long as $\mathbb{E}[\bar{d}_{\bar{Z},2}(S, 0)] < \infty$, and (B.65) is by Property B.

The proof of Lemma 2.24 consists of Gaussian approximation analysis of the bound (2.34) in Lemma 2.3. First, we weaken (2.34) by choosing $P_{\bar{Z}}$, δ and λ in the following manner. Fix $\tau > 0$, and let $\delta = \frac{\tau}{k}$, $P_{\bar{Z}} = P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, where Z^* achieves $\mathbb{R}_S(d)$, and choose $\lambda = k\lambda_{\bar{S}, Z^*}^*$, where $P_{\bar{S}}$ is the measure on \mathcal{S} generated by the empirical distribution of $s^k \in \mathcal{S}^k$:

$$P_{\bar{S}}(a) = \frac{1}{k} \sum_{i=1}^k 1\{s_i = a\} \quad (\text{B.66})$$

Since the distortion measure is separable, for any $\lambda > 0$ we have

$$J_{Z^{k*}}(s^k, \lambda k) = \sum_{i=1}^k J_{Z^*}(s_i, \lambda) \quad (\text{B.67})$$

so by Lemma 2.3, for all

$$d > d_{\min|\bar{S}^k, Z^{k*}} \quad (\text{B.68})$$

it holds that

$$\begin{aligned} P_{Z^{k*}}(B_d(s^k)) &\geq \exp\left(-\sum_{i=1}^k J_{Z^*}(s_i, \lambda(s^k)) + \lambda(s^k)kd - \lambda(s^k)\tau\right) \\ &\cdot \mathbb{P}\left[kd - \tau < \sum_{i=1}^k d(s_i, \bar{Z}_i^*) \leq kd|\bar{S}^k = s^k\right] \end{aligned} \quad (\text{B.69})$$

where we denoted

$$\lambda(s^k) = -\mathbb{R}'_{\bar{S}, Z^*}(d) \quad (\text{B.70})$$

($\lambda(s^k)$ depends on s^k through the distribution of \bar{S} in (B.66)), and $P_{\bar{Z}^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, where $P_{Z^*|\bar{S}}$ achieves $\mathbb{R}_{\bar{S}, Z^*}(d)$. The probability appearing in the right side of (B.69) can be lower bounded

by the following lemma.

Lemma B.3. *Assume that restrictions (i)-(iv) in Section 2.6.2 hold. Then, there exist $\delta_0, k_0 > 0$ such that for all $\delta \leq \delta_0, k \geq k_0$, there exist a set $F_k \subseteq \mathcal{S}^k$ and constants $\tau, C_1, K_1 > 0$ such that*

$$\mathbb{P} [S^k \notin F_k] \leq \frac{K_1}{\sqrt{k}} \quad (\text{B.71})$$

and for all $s^k \in F_k$,

$$\mathbb{P} \left[kd - \tau < \sum_{i=1}^k d(s_i, \bar{Z}_i^*) \leq kd \mid \bar{S}^k = s^k \right] \geq \frac{C_1}{\sqrt{k}} \quad (\text{B.72})$$

$$|\lambda(s^k) - \lambda^*| < \delta \quad (\text{B.73})$$

where $\lambda^* = -\mathbb{R}'_S(d)$, and $\lambda(s^k)$ is defined in (B.70).

Proof. The reasoning is similar the proof of [37, (4.6)]. Fix

$$0 < \Delta < \frac{1}{3} \min \{d - d_{\min|\mathcal{S}, \mathcal{Z}^*}, d_{\max|\mathcal{S}, \mathcal{Z}^*} - d\} \quad (\text{B.74})$$

(the right side of (B.74) is guaranteed to be positive by restriction (iii) in Section 2.6.2) and denote

$$\underline{\Delta} = -\mathbb{R}'_{\mathcal{S}, \mathcal{Z}^*} \left(d + \frac{3\Delta}{2} \right) \quad (\text{B.75})$$

$$\bar{\Delta} = -\mathbb{R}'_{\mathcal{S}, \mathcal{Z}^*} \left(d - \frac{3\Delta}{2} \right) \quad (\text{B.76})$$

$$\mu'' = \mathbb{E} [|J''_{\mathcal{Z}^*}(\mathcal{S}, \lambda^*)|] \quad (\text{B.77})$$

$$\delta = \frac{3\Delta}{2} \sup_{|\theta| < \frac{3\Delta}{2}} \mathbb{R}''_{\mathcal{S}, \mathcal{Z}^*}(d + \theta) \quad (\text{B.78})$$

$$\bar{V}(s^k) = \frac{1}{k} \sum_{i=1}^k \sup_{|\theta| < \delta} |J''_{\mathcal{Z}^*}(s_i, \lambda^* + \theta)| \log e \quad (\text{B.79})$$

$$\underline{V}(s^k) = \frac{1}{k} \sum_{i=1}^k \inf_{|\theta| < \delta} |J''_{\mathcal{Z}^*}(s_i, \lambda^* + \theta)| \log e \quad (\text{B.80})$$

Next we construct F_k as the set of all s^k that satisfy all of the following conditions:

$$\frac{1}{k} \sum_{i=1}^k \alpha_{Z^*}(s_i) < d_{\min|S, Z^*} + \Delta \quad (\text{B.81})$$

$$\frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, 0) > d_{\max|S, Z^*} - \Delta \quad (\text{B.82})$$

$$\frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \underline{\Delta}) > d + \Delta \quad (\text{B.83})$$

$$\frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \bar{\lambda}) < d - \Delta \quad (\text{B.84})$$

$$\frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,3}(s_i, 0) \leq \mathbb{E} [\bar{d}_{Z^*,3}(S, 0)] + \Delta \quad (\text{B.85})$$

$$\bar{V}(s^k) \geq \frac{\mu''}{2} \log e \quad (\text{B.86})$$

$$\underline{V}(s^k) \leq \frac{3\mu''}{2} \log e \quad (\text{B.87})$$

Let us first show that (B.73) holds with δ given by (B.78) for all s^k satisfying the conditions (B.81)–(B.84). From (B.83) and (B.84),

$$\frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \bar{\lambda}) < d < \frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \underline{\Delta}) \quad (\text{B.88})$$

On the other hand, from (B.60) we have

$$d = \frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \lambda(s^k)) \quad (\text{B.89})$$

Therefore, since the right side of (B.89) is decreasing (Property B),

$$\underline{\Delta} < \lambda(s^k) < \bar{\lambda} \quad (\text{B.90})$$

Finally, an application Taylor's theorem to (B.75) and (B.76) using $\lambda^* = \lambda_{S, Z^*}^*$ expands (B.90) as

$$-\frac{3\Delta}{2} \mathbb{R}_{S, Z^*}''(\bar{d}) + \lambda^* < \lambda(s^k) < \lambda^* + \frac{3\Delta}{2} \mathbb{R}_{S, Z^*}''(\underline{d}) \quad (\text{B.91})$$

for some $\bar{d} \in [d, d + \frac{3\Delta}{2}]$, $\underline{d} \in [d, d - \frac{3\Delta}{2}]$. Note that (B.74), (B.81) and (B.82) ensure that

$$d_{\min|S, Z^*} + 2\Delta < d < d_{\max|S, Z^*} - 2\Delta \quad (\text{B.92})$$

so the derivatives in (B.91) exist and are positive by Remark B.4. Therefore (B.73) holds with δ given by (B.78).

We are now ready to show that as long as Δ (and, therefore, δ) is small enough, there exists a $K_1 \geq 0$ such that (B.71) holds. Hölder's inequality and assumption (iv) in Section 2.6.2 imply that the third moments of the random variables involved in conditions (B.83)–(B.85) are finite. By the Berry-Esséen inequality, the probability of violating these conditions is $O\left(\frac{1}{\sqrt{k}}\right)$. To bound the probability of violating conditions (B.86) and (B.87), observe that since $J''_{Z^*}(S, \lambda)$ is dominated by integrable functions due to (B.61), we have by Fatou's lemma and continuity of $J''_{Z^*}(s, \cdot)$

$$\mu'' \leq \liminf_{\delta \downarrow 0} \mathbb{E} \left[\inf_{|\theta| \leq \delta} |J''_{Z^*}(S, \lambda^* + \theta)| \right] \quad (\text{B.93})$$

$$\leq \limsup_{\delta \downarrow 0} \mathbb{E} \left[\sup_{|\theta| \leq \delta} |J''_{Z^*}(S, \lambda^* + \theta)| \right] \quad (\text{B.94})$$

$$\leq \mu'' \quad (\text{B.95})$$

Therefore, if δ is small enough,

$$\frac{3\mu''}{4} \log e \leq \mathbb{E} [\underline{V}(S^k)] \leq \mathbb{E} [\overline{V}(S^k)] \leq \frac{5\mu''}{4} \log e \quad (\text{B.96})$$

The third absolute moments of $\overline{V}(S^k)$ and $\underline{V}(S^k)$ are finite by Hölder's inequality, (B.61) and assumption (iv) in Section 2.6.2. Thus, the probability of violating conditions (B.86) and (B.87) is also $O\left(\frac{1}{\sqrt{k}}\right)$. Now, (B.71) follows via the union bound.

To complete the proof of Lemma B.3, it remains to show (B.72). Toward this end, observe, recalling Properties D and E that the corresponding moments in the Berry-Esséen theorem are

given by

$$\mu(s^k) = \frac{1}{k} \sum_{i=1}^k \mathbb{E} [d(s_i, \bar{Z}^*) | \bar{S} = s_i] \quad (\text{B.97})$$

$$= \frac{1}{k} \sum_{i=1}^k \bar{d}_{Z^*,1}(s_i, \lambda(s^k)) \quad (\text{B.98})$$

$$= d \quad (\text{B.99})$$

$$V(s^k) = \frac{1}{k} \sum_{i=1}^k [\bar{d}_{Z^*,2}(s_i, \lambda(s^k)) - \bar{d}_{Z^*,1}^2(s_i, \lambda(s^k))] \quad (\text{B.100})$$

$$= -\frac{1}{k} \sum_{i=1}^k J_{Z^*}''(s_i, \lambda(s^k)) \log e \quad (\text{B.101})$$

$$\begin{aligned} T(s^k) &= \frac{1}{k} \sum_{i=1}^k \mathbb{E} \left[\left| d(s_i, \bar{Z}^*) - \mathbb{E} [d(s_i, \bar{Z}^*) | \bar{S} = s_i] \right|^3 | \bar{S} = s_i \right] \\ &\leq \frac{8}{k} \sum_{i=1}^k \mathbb{E} \left[\left| d(s_i, \bar{Z}^*) \right|^3 | \bar{S} = s_i \right] \end{aligned} \quad (\text{B.102})$$

$$= \frac{8}{k} \sum_{i=1}^k \bar{d}_{Z^*,3}(s_i, \lambda(s^k)) \quad (\text{B.103})$$

$$\leq \frac{8}{k} \sum_{i=1}^k \bar{d}_{Z^*,3}(s_i, 0) \quad (\text{B.104})$$

As long as $s^k \in F_k$,

$$\frac{\mu''}{2} \log e \leq V(s^k) \leq \frac{3\mu''}{2} \log e \quad (\text{B.105})$$

$$T(s^k) \leq 8\mathbb{E} [\bar{d}_{Z^*,3}(\bar{S}, 0)] + 8\Delta \quad (\text{B.106})$$

where (B.105) is due to (B.73), (B.86) and (B.87), and (B.106) is due to (B.85). Therefore, Lemma A.4.1 applies to $W_i = d(s_i, \bar{Z}_i^*) - d$, which concludes the proof of (B.72). \square

To upper-bound $\sum_{i=1}^k J_{Z^*}(s_i, \lambda(s^k))$ appearing in (B.69), we invoke the following result.

Lemma B.4. *Assume that restrictions (i)-(iv) in Section 2.6.2 hold. There exist constants $k_0, K_2 > 0$ such that for $k \geq k_0$,*

$$\mathbb{P} \left[\sum_{i=1}^k (J_{Z^*}(S_i, \lambda(S^k)) - \lambda(S^k)d) \leq \sum_{i=1}^k \mathcal{J}_{\mathcal{S}}(S_i, d) + C_2 \log k \right] > 1 - \frac{K_2}{\sqrt{k}} \quad (\text{B.107})$$

where $\lambda(s^k)$ is defined in (B.70), and

$$C_2 = \frac{\text{Var}[J'_{Z^*}(\mathbf{S}, \lambda^*)]}{\mathbb{E}[|J''_{Z^*}(\mathbf{S}, \lambda^*)|] \log e} \quad (\text{B.108})$$

Proof. Using (B.73), we have for all $x_k \in F_k$,

$$\begin{aligned} & \sum_{i=1}^k [J_{Z^*}(s_i, \lambda(s^k)) - J_{Z^*}(s_i, \lambda^*) - \lambda(s^k)d + \lambda^*d] \\ &= \sup_{|\theta| < \delta} \sum_{i=1}^k [J_{Z^*}(s_i, \lambda^* + \theta) - J_{Z^*}(s_i, \lambda^*) - \theta d] \end{aligned} \quad (\text{B.109})$$

$$= \sup_{|\theta| < \delta} \left\{ \theta \sum_{i=1}^k (J'_{Z^*}(s_i, \lambda^*) - d) + \frac{\theta^2}{2} \sum_{i=1}^k J''_{Z^*}(s_i, \lambda^* + \xi_k) \right\} \quad (\text{B.110})$$

$$\leq \sup_{|\theta| < \delta} \left\{ \theta \Sigma'(s^k) - \frac{\theta^2}{2} \Sigma''(s^k) \right\} \quad (\text{B.111})$$

$$\leq \frac{(\Sigma'(s^k))^2}{2\Sigma''(s^k)} \quad (\text{B.112})$$

where

- (B.109) is due to (B.58);
- (B.110) holds for some $|\xi_k| \leq \delta$ by Taylor's theorem;
- in (B.111) we denoted

$$\Sigma'(s^k) = \sum_{i=1}^k (J'_{Z^*}(s_i, \lambda^*) - d) \quad (\text{B.113})$$

$$\Sigma''(s^k) = - \sum_{i=1}^k \inf_{|\theta'| < \delta} |J''_{Z^*}(s_i, \lambda^* + \theta')| \quad (\text{B.114})$$

and used Property D;

- in (B.112) we maximized the quadratic equation in (B.111) with respect to θ .

Note that the reasoning leading to (B.112) is due to [38, proof of Theorem 3]. We now proceed to upper-bound the ratio in the right side of (B.112). Since $\mathbb{E}[\bar{d}_{Z^*,1}(\mathbf{S}, 0)] < \infty$ by assumption (iv) in Section 2.6.2, the differentiation in Property A can be brought inside the expectation by (B.59) and

the dominated convergence theorem, so

$$\mathbb{E} \left[\frac{1}{k} \Sigma'(S^k) \right] = \mathbb{E} [J'_{Z^*}(S, \lambda^*)] - d \quad (\text{B.115})$$

$$= 0 \quad (\text{B.116})$$

Denote

$$V' = \text{Var} [J'_{Z^*}(S, \lambda^*)] \quad (\text{B.117})$$

$$T' = \mathbb{E} \left[|J'_{Z^*}(S, \lambda^*) - \mathbb{E} [J'_{Z^*}(S, \lambda^*)]|^3 \right] \quad (\text{B.118})$$

If $V' = 0$, there is nothing to prove as that means $S'(S^k) = 0$ a.s. Otherwise, since (B.59) with Hölder's inequality and assumption (iv) in Section 2.6.2 guarantee that T' is finite, Lemma A.4.2 implies that there exists K'_2 such that for all k large enough

$$\mathbb{P} \left[(\Sigma'(S^k))^2 > V' k \log_e k \right] \leq \frac{K'_2}{\sqrt{k}} \quad (\text{B.119})$$

To treat $\Sigma''(S^k)$, observe that $\Sigma''(s^k) = k \underline{V}(s^k) (\log e)^{-1}$ (see (B.80)), so as before, the variance V'' and the third absolute moment T'' of $W_i = \inf_{|\theta| \leq \delta} |J''_{Z^*}(S_i, \lambda^* + \theta)|$ are finite, and $\mathbb{E} [W_i] \geq \frac{3\mu''}{4}$ by (B.96), where $\mu'' > 0$ is defined in (B.77). If $V'' = 0$, we have $W_i > \frac{\mu''}{2} \log e$ almost surely.

Otherwise, applying Lemma A.4.3 we conclude that there exists K''_2 such that

$$\mathbb{P} \left[\Sigma''(S^k) < k \frac{\mu''}{2} \right] \leq \mathbb{P} \left[\mathbb{E} [\Sigma''(S^k)] - \Sigma''(S^k) > k \frac{\mu''}{4} \right] \quad (\text{B.120})$$

$$\leq \frac{K''_2}{k} \quad (\text{B.121})$$

Finally, denoting

$$g(s^k) = \sum_{i=1}^k J_{Z^*}(s_i, \lambda(s^k)) - \sum_{i=1}^k J_{Z^*}(s_i, \lambda^*) - (\lambda(s^k) - \lambda^*) kd \quad (\text{B.122})$$

and letting G_k be the set of $s^k \in \mathcal{S}^k$ satisfying both

$$(\Sigma'(s^k))^2 \leq V' k \log_e k \quad (\text{B.123})$$

$$\Sigma''(s^k) \geq k \frac{\mu''}{2} \quad (\text{B.124})$$

we see from (B.71), (B.119), (B.121) applying elementary probability rules that

$$\begin{aligned} \mathbb{P}[g(S^k) > C_2 \log k] &= \mathbb{P}\left[g(S^k) > C_2 \log k, g(S^k) \leq \frac{(\Sigma'(S^k))^2}{2\Sigma''(S^k)}\right] \\ &+ \mathbb{P}\left[g(S^k) > C_2 \log k, g(S^k) > \frac{(\Sigma'(S^k))^2}{2\Sigma''(S^k)}\right] \end{aligned} \quad (\text{B.125})$$

$$\leq \mathbb{P}\left[\frac{(\Sigma'(S^k))^2}{2\Sigma''(S^k)} > C_2 \log k\right] + \frac{K_1}{\sqrt{k}} \quad (\text{B.126})$$

$$\begin{aligned} &= \mathbb{P}\left[\frac{(\Sigma'(S^k))^2}{2\Sigma''(S^k)} > C_2 \log k, S^k \in G_k\right] \\ &+ \mathbb{P}\left[\frac{(\Sigma'(S^k))^2}{2\Sigma''(S^k)} > C_2 \log k, S^k \notin G_k\right] + \frac{K_1}{\sqrt{k}} \end{aligned} \quad (\text{B.127})$$

$$< 0 + \frac{K'_2}{\sqrt{k}} + \frac{K''_2}{k} + \frac{K_1}{\sqrt{k}} \quad (\text{B.128})$$

We conclude that (B.107) holds for $k \geq k_0$ with $K_2 = K_1 + K'_2 + K''_2$. \square

To apply Lemmas B.3 and B.4 to (B.69), note that (B.68) (and hence (B.69)) holds for $s^k \in F_k$ due to (B.92). Weakening (B.69) using Lemmas B.3 and B.4 and the union bound we conclude that Lemma 2.24 holds with

$$C_0 = \frac{1}{2} + C_2 \quad (\text{B.129})$$

$$K = K_1 + K_2 \quad (\text{B.130})$$

$$c = (\lambda^* + \delta)\tau - \log C_1 \quad (\text{B.131})$$

B.6 Proof of Theorem 2.25

In this appendix, we show that (2.177) follows from (2.141). Fix a point (d_∞, R_∞) on the rate-distortion curve such that $d_\infty \in (\underline{d}, \bar{d})$. Let $d_k = D(k, R_\infty, \epsilon)$, and let α be the acute angle between the tangent to the $R(d)$ curve at $d = d_k$ and the d axis (see Fig. B.1). We are interested in the difference $d_k - d_\infty$. Since [31]

$$\lim_{k \rightarrow \infty} D(k, R, \epsilon) = D(R), \quad (\text{B.132})$$

there exists a $\delta > 0$ such that for large enough k ,

$$d_k \in \mathbb{B}_\delta(d_\infty) = [d_\infty - \delta, d_\infty + \delta] \subset (\underline{d}, \bar{d}) \quad (\text{B.133})$$

For such d_k ,

$$|d_k - d_\infty| \leq \left| \frac{R(d_k) - R_\infty}{\tan \alpha_k} \right| \quad (\text{B.134})$$

$$\leq \left| \frac{R(k, d_k, \epsilon) - R(d_k)}{\min_{d \in \mathbb{B}_\delta(d_\infty)} R'(d)} \right| \quad (\text{B.135})$$

$$= O\left(\frac{1}{\sqrt{k}}\right) \quad (\text{B.136})$$

where

- (B.134) is by convexity of $R(d)$;
- (B.135) follows by substituting $R(k, d_k, \epsilon) = R_\infty$ and $\tan \alpha_k = |R'(d_k)|$;
- (B.136) follows by Theorem 2.22. Note that we are allowed to plug d_k into (2.141) because the remainder in (2.141) can be uniformly bounded over all d from the compact set $\mathbb{B}_\delta(d_\infty)$ (just swap B_k in (2.165) for the maximum of B_k 's over $\mathbb{B}_\delta(d_\infty)$, and similarly swap c, j, B_k in (2.172) and (2.173) for the corresponding maxima); thus (2.141) holds not only for a fixed d but also for any sequence $d_k \in \mathbb{B}_\delta(d_\infty)$.

It remains to refine (B.136) to show (2.177). Write

$$\mathcal{V}(d_k) = \mathcal{V}(d_\infty) + O\left(\frac{1}{\sqrt{k}}\right) \quad (\text{B.137})$$

$$R(d_k) = R(d_\infty) + R'(d_\infty)(d_k - d_\infty) + O\left(\frac{1}{k}\right) \quad (\text{B.138})$$

$$= R(d_k) + \sqrt{\frac{\mathcal{V}(d_k)}{k}} Q^{-1}(\epsilon) + R'(d_\infty)(d_k - d_\infty) + \theta\left(\frac{\log k}{k}\right) \quad (\text{B.139})$$

$$= R(d_k) + \sqrt{\frac{\mathcal{V}(d_\infty)}{k}} Q^{-1}(\epsilon) + R'(d_\infty)(d_k - d_\infty) + \theta\left(\frac{\log k}{k}\right) \quad (\text{B.140})$$

where

- (B.137) and (B.138) follow by Taylor's theorem and (B.136) using finiteness of $\mathcal{V}'(d)$ and $R''(d)$ for all $d \in \mathbb{B}_\delta(d_\infty)$;
- (B.139) expands $R_\infty = R(k, d_k, \epsilon)$ using (2.141);
- (B.140) invokes (B.137).

Rearranging (B.140), we obtain the desired approximation (2.177) for the difference $d_k - d_\infty$.

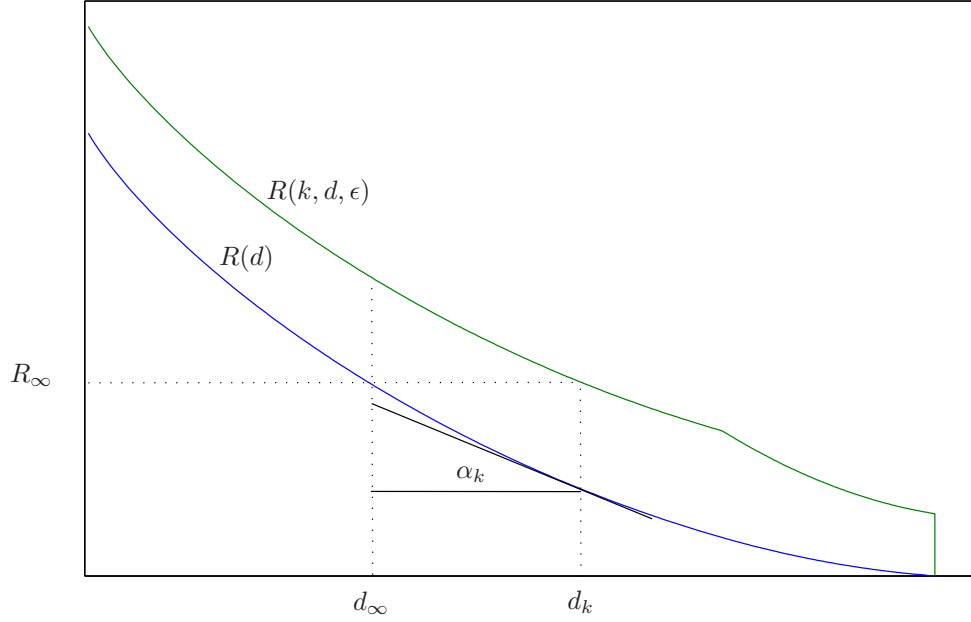


Figure B.1: Estimating $d_k - d_\infty$ from $R(k, d, \epsilon) - R(d)$.

B.7 Proof of Theorem 2.29

From the Stirling approximation, it follows that (e.g. [7])

$$\sqrt{\frac{k}{8k(k-j)}} \exp \left\{ kh \left(\frac{j}{k} \right) \right\} \leq \binom{k}{j} \quad (\text{B.141})$$

$$\leq \sqrt{\frac{k}{2\pi j(k-j)}} \exp \left\{ kh \left(\frac{j}{k} \right) \right\} \quad (\text{B.142})$$

In view of the inequality [3, (586)]

$$\binom{k}{j-i} \leq \binom{k}{j} \left(\frac{j}{k-j} \right)^i \quad (\text{B.143})$$

we can write [3, (599)-(601)]

$$\binom{k}{j} \leq \left\langle \frac{k}{j} \right\rangle \quad (\text{B.144})$$

$$\leq \binom{k}{j} \sum_{j=0}^{\infty} \left(\frac{j}{k-j} \right)^i \quad (\text{B.145})$$

$$= \binom{k}{j} \frac{k-j}{k-2j} \quad (\text{B.146})$$

where (B.146) holds as long as the series converges, i.e. as long as $2k < k$. Furthermore, combining (B.144) and (B.146) with Stirling's approximation (B.141) and (B.142), we conclude that for any $0 < \alpha < \frac{1}{2}$,

$$\log \left\langle \frac{k}{\lfloor k\alpha \rfloor} \right\rangle = kh(\alpha) - \frac{1}{2} \log k + O(1) \quad (\text{B.147})$$

Taking logarithms in (2.186) and letting $\log M = kR$ for any $R \geq R(k, d, \epsilon)$, we obtain

$$\log(1 - \epsilon) \leq k(R - \log 2) + \log \left\langle \frac{k}{\lfloor kd \rfloor} \right\rangle \quad (\text{B.148})$$

$$\leq k(R - \log 2 + h(d)) - \frac{1}{2} \log k + O(1) \quad (\text{B.149})$$

Since (B.149) holds for any $R \geq R(k, d, \epsilon)$, we conclude that

$$R(k, d, \epsilon) \geq R(d) + \frac{1}{2} \frac{\log k}{k} + O\left(\frac{1}{k}\right) \quad (\text{B.150})$$

Similarly, Corollary 2.28 implies that there exists an $(\exp(kR), d, \epsilon)$ code with

$$\log \epsilon \leq \exp(kR) \log \left(1 - \frac{\left\langle \frac{k}{\lfloor kd \rfloor} \right\rangle}{2^k} \right) \quad (\text{B.151})$$

$$\leq -\exp(kR) \frac{\left\langle \frac{k}{\lfloor kd \rfloor} \right\rangle}{2^k} \log e \quad (\text{B.152})$$

where we used $\log(1 + x) \leq x \log e$, $x > -1$. Taking the logarithm of the negative of both sides in (B.152), we have

$$\log \log \frac{1}{\epsilon} \geq k(R - \log 2) + \log \left\langle \frac{k}{\lfloor kd \rfloor} \right\rangle + \log \log e \quad (\text{B.153})$$

$$= k(R - \log 2 + h(d)) - \frac{1}{2} \log k + O(1), \quad (\text{B.154})$$

where (B.154) follows from (B.147). Therefore,

$$R(k, d, \epsilon) \leq R(d) + \frac{1}{2} \frac{\log k}{k} + O\left(\frac{1}{k}\right) \quad (\text{B.155})$$

The case $d = 0$ follows directly from (2.148). Alternatively, it can be easily checked by substituting $\left\langle \frac{k}{0} \right\rangle = 1$ in the analysis above.

B.8 Gaussian approximation of the bound in Theorem 2.33

By analyzing the asymptotic behavior of (2.209), we prove that

$$R(k, d, \epsilon) \leq h(p) - h(d) + \sqrt{\frac{\mathcal{V}(d)}{k}} Q^{-1}(\epsilon) + \frac{1}{2} \frac{\log k}{k} + \frac{\log \log k}{k} + O\left(\frac{1}{k}\right) \quad (\text{B.156})$$

where $\mathcal{V}(d)$ is as in (2.211), thereby showing that a constant composition code that attains the rate-dispersion function exists. Letting $M = \exp(kR)$ and using $(1-x)^M \leq e^{-Mx}$ in (2.209), we can guarantee existence of a (k, M, d, ϵ') code with

$$\epsilon' \leq \sum_{j=0}^k \binom{k}{j} p^j (1-p)^{k-j} e^{-\binom{k}{\lceil kq \rceil}^{-1} L_k(j, \lceil kq \rceil) \exp(kR)} \quad (\text{B.157})$$

In what follows we will show that one can choose an R satisfying the right side of (B.156) so that the right side of (B.157) is upper bounded by ϵ when k is large enough. Letting $j = np + k\Delta$, $t = \lceil kq \rceil$, $t_0 = \lceil \frac{\lceil kq \rceil + j - kd}{2} \rceil^+$ and using Stirling's formula (B.141), it is an algebraic exercise to show that there exist positive constants δ and C such that for all $\Delta \in [-\delta, \delta]$,

$$\binom{k}{t}^{-1} \binom{j}{t_0} \binom{k-j}{t-t_0} = \binom{k}{j}^{-1} \binom{t}{t_0} \binom{k-t}{j-t_0} \quad (\text{B.158})$$

$$\geq \frac{C}{\sqrt{k}} \exp\{kg(\Delta)\} \quad (\text{B.159})$$

where

$$g(\Delta) = h(p + \Delta) - qh\left(d - \frac{\Delta}{2q}\right) - (1-q)h\left(d + \frac{\Delta}{2(1-q)}\right)$$

It follows that

$$\binom{k}{\lceil kq \rceil}^{-1} L_k(np + k\Delta, \lceil kq \rceil) \geq \frac{C}{\sqrt{k}} \exp\{-kg(\Delta)\} \quad (\text{B.160})$$

whenever $L_k(j, \lceil kq \rceil)$ is nonzero, that is, whenever $\lceil kq \rceil - kd \leq j \leq \lceil kq \rceil + kd$, and $g(\Delta) = 0$ otherwise.

Applying a Taylor series expansion in the vicinity of $\Delta = 0$ to $g(\Delta)$, we get

$$g(\Delta) = h(p) - h(d) + h'(p)\Delta + O(\Delta^2) \quad (\text{B.161})$$

Since $g(\Delta)$ is continuously differentiable with $g'(0) = h'(p) > 0$, there exist constants $\underline{b}, \bar{b} > 0$ such

that $g(\Delta)$ is monotonically increasing on $(-\underline{b}, \bar{b})$ and (B.159) holds. Let

$$b_k = \sqrt{\frac{p(1-p)}{k}} Q^{-1}(\epsilon_k) \quad (\text{B.162})$$

$$\epsilon_k = \epsilon - \frac{2B_k}{\sqrt{k}} - \sqrt{\frac{\mathcal{V}(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2\mathcal{V}(d)}} - \frac{1}{\sqrt{k}} \quad (\text{B.163})$$

$$B_k = 6 \frac{1-2p+2p^2}{\sqrt{p(1-p)}} \quad (\text{B.164})$$

$$R = g(b_k) + \frac{1}{2} \frac{\log k}{k} + \frac{1}{k} \log \left(\frac{\log_e k}{2C} \right) \quad (\text{B.165})$$

Using (B.161) and applying a Taylor series expansion to $Q^{-1}(\cdot)$, it is easy to see that R in (B.165) can be rewritten as the right side of (B.156). Splitting the sum in (B.157) into three sums and upper bounding each of them separately, we have

$$\begin{aligned} & \sum_{j=0}^k \binom{k}{j} p^j (1-p)^{k-j} e^{-\binom{k}{kq}^{-1} L_k(j, \lceil kq \rceil) \exp(kR)} \\ &= \sum_{j=0}^{\lfloor np-k\underline{b} \rfloor} + \sum_{j=\lfloor np-k\underline{b} \rfloor+1}^{\lfloor kp+kb_k \rfloor} + \sum_{j=\lfloor kp+kb_k \rfloor+1}^k \end{aligned} \quad (\text{B.166})$$

$$\begin{aligned} &\leq \mathbb{P} \left[\sum_{i=1}^k S_i \leq np - k\underline{b} \right] + \sum_{j=\lfloor np-k\underline{b} \rfloor+1}^{\lfloor kp+kb_k \rfloor} \binom{k}{j} p^j (1-p)^{k-j} e^{-\frac{C}{\sqrt{k}} \exp\{kR - kg(\frac{j}{k} - p)\}} \\ &+ \mathbb{P} \left[\sum_{i=1}^k S_i \geq np + kb_k \right] \end{aligned} \quad (\text{B.167})$$

$$\leq \frac{B_k}{\sqrt{k}} + \sqrt{\frac{\mathcal{V}(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2\mathcal{V}(d)}} + \frac{1}{\sqrt{k}} + \epsilon_k + \frac{B_k}{\sqrt{k}} \quad (\text{B.168})$$

$$= \epsilon \quad (\text{B.169})$$

where $\{S_i\}$ are i.i.d. Bernoulli random variables with bias p . The first and third probabilities in the right side of (B.167) are bounded using the Berry-Esséen bound (2.155) and (A.28), while the second probability is bounded using the monotonicity of $g(\Delta)$ in $(-\underline{b}, b_k]$ for large enough k , in which case the minimum difference between R and $g(\Delta)$ in $(-\underline{b}, b_k)$ is $\frac{1}{2k} \log k + \frac{1}{k} \log \left(\frac{\log_e k}{2C} \right)$.

B.9 Proof of Theorem 2.38

In order to study the asymptotics of (2.223) and (2.225), we need to analyze the asymptotic behavior of $S_{\lfloor kd \rfloor}$ which can be carried out similarly to the binary case. Recalling the inequality (B.143), we

have

$$S_j = \sum_{i=0}^j \binom{k}{i} (m-1)^i \quad (\text{B.170})$$

$$\leq \binom{k}{j} \sum_{i=0}^j \left(\frac{j}{k-j} \right)^i (m-1)^{j-i} \quad (\text{B.171})$$

$$\leq \binom{k}{j} (m-1)^j \sum_{i=0}^{\infty} \left(\frac{j}{(k-j)(m-1)} \right)^i \quad (\text{B.172})$$

$$= \binom{k}{j} (m-1)^j \frac{k-j}{k-j \frac{m}{m-1}} \quad (\text{B.173})$$

where (B.173) holds as long as the series converges, i.e. as long as $\frac{j}{k} < \frac{m-1}{m}$. Using

$$S_j \geq \binom{k}{j} (m-1)^j \quad (\text{B.174})$$

and applying Stirling's approximation (B.141) and (B.142), we have for $0 < d < \frac{m-1}{m}$

$$\log S_{\lfloor kd \rfloor} = \log \binom{k}{\lfloor kd \rfloor} + kd \log(m-1) + O(1) \quad (\text{B.175})$$

$$= kh(d) + kd \log(m-1) - \frac{1}{2} \log k + O(1) \quad (\text{B.176})$$

Taking logarithms in (2.223) and letting $\log M = kR$ for any $R \geq R(k, d, \epsilon)$, we obtain

$$\log(1 - \epsilon) \leq k(R - \log m) + \log S_{\lfloor kd \rfloor} \quad (\text{B.177})$$

$$\leq k(R - \log m + h(d) + d \log(m-1)) - \frac{1}{2} \log k + O(1) \quad (\text{B.178})$$

Since (B.178) holds for any $R \geq R(k, d, \epsilon)$, we conclude that

$$R(k, d, \epsilon) \geq R(d) + \frac{1}{2k} \log k + O\left(\frac{1}{k}\right) \quad (\text{B.179})$$

Similarly, Theorem 2.37 implies that there exists an $(\exp(kR), d, \epsilon)$ code with

$$\log \epsilon \leq \exp(kR) \log \left(1 - \frac{S_{\lfloor kd \rfloor}}{m^k} \right) \quad (\text{B.180})$$

$$\leq -\exp(kR) \frac{S_{\lfloor kd \rfloor}}{m^k} \log e \quad (\text{B.181})$$

where we used $\log(1+x) \leq x \log e$, $x > -1$. Taking the logarithm of the negative of both sides of

(B.181), we have

$$\log \log \frac{1}{\epsilon} \geq k(R - \log m) + \log S_{\lfloor kd \rfloor} + \log \log e \quad (\text{B.182})$$

$$= k(R - \log m + h(d)) - \frac{1}{2} \log k + O(1), \quad (\text{B.183})$$

where (B.183) follows from (B.176). Therefore,

$$R(k, d, \epsilon) \leq R(d) + \frac{1}{2k} \log k + O\left(\frac{1}{k}\right) \quad (\text{B.184})$$

The case $d = 0$ follows directly from (2.148), or can be obtained by observing that $S_0 = 1$ in the analysis above.

B.10 Gaussian approximation of the bound in Theorem 2.41

Using Theorem 2.41, we show that

$$R(k, d, \epsilon) \leq R(d) + \sqrt{\frac{\mathcal{V}(d)}{k}} Q^{-1}(\epsilon) + \frac{(m-1)(M_S(\eta) - 1)}{2} \frac{\log k}{k} + \frac{\log \log k}{k} + O\left(\frac{1}{k}\right) \quad (\text{B.185})$$

where $M_S(\eta)$ is defined in (2.231), and $\mathcal{V}(d)$ is as in (2.252). Similar to the binary case, we express $L_k(\mathbf{j}, \mathbf{t}^*)$ in terms of the rate-distortion function. Observe that whenever $L_k(\mathbf{j}, \mathbf{t}^*)$ is nonzero,

$$\binom{k}{\mathbf{t}^*}^{-1} L_k(\mathbf{j}, \mathbf{t}^*) = \binom{k}{\mathbf{t}^*}^{-1} \prod_{a=1}^m \binom{j_a}{t_a^*} \quad (\text{B.186})$$

$$= \binom{k}{\mathbf{j}}^{-1} \prod_{a=1}^{M_S(\eta)} \binom{t_b^*}{j_b} \quad (\text{B.187})$$

where $\mathbf{j}_b = (t_{1,b}, \dots, t_{m,b})$. It can be shown [55] that for k large enough, there exist positive constants C_1, C_2 such that

$$\binom{k}{\mathbf{j}} \leq C_1 k^{-\frac{m-1}{2}} \exp k \left\{ H(\mathbf{S}) + \sum_{a=1}^m \Delta_a \log \frac{1}{P_S(a)} + O(|\Delta|^2) \right\} \quad (\text{B.188})$$

$$\binom{t_b^*}{\mathbf{j}_b} \geq C_2 k^{-\frac{m-1}{2}} \exp k \left\{ P_Z^*(b) H(\mathbf{S} | Z^* = b) + \sum_{a=1}^m \delta(a, b) \log \frac{1}{P_{\mathbf{S} | Z}^*(a|b)} + O(|\Delta|^2) \right\} \quad (\text{B.189})$$

hold for small enough $|\Delta|$, where $\Delta = (\Delta_1, \dots, \Delta_m)$. A simple calculation using $\sum_{a=1}^m \Delta_a = 0$ reveals that

$$\sum_{a=1}^m \sum_{b=1}^{M_S(\eta)} \delta(a, b) \log \frac{1}{P_{S|Z}^*(a|b)} = \sum_{a=1}^{M_S(\eta)} \Delta_a \log \frac{1}{\eta} + \sum_{a=M_S(\eta)+1}^m \Delta_a \log \frac{1}{P_S(a)} \quad (\text{B.192})$$

so invoking (B.188) and (B.189) one can write

$$\binom{k}{\mathbf{j}}^{-1} \prod_{a=1}^{M_S(\eta)} \binom{t_b^*}{\mathbf{j}_b} \geq C k^{-\frac{(m-1)(M_S(\eta)-1)}{2}} \exp \{-kg(\Delta)\} \quad (\text{B.193})$$

where C is a constant, and $g(\Delta)$ is a twice differentiable function that satisfies

$$g(\Delta) = R(d) + \sum_{a=1}^m \Delta_a v(a) + O(|\Delta|^2) \quad (\text{B.194})$$

$$v(a) = \min \left\{ \iota_S(a), \log \frac{1}{\eta} \right\} \quad (\text{B.195})$$

Similar to the BMS case, $g(\Delta)$ is monotonic in $\sum_{a=1}^m \Delta_a v(a) \in (-\underline{b}, \bar{b})$ for some constants $\underline{b}, \bar{b} > 0$ independent of k . Let

$$b_k = \sqrt{\frac{\mathcal{V}(d)}{k}} Q^{-1}(\epsilon) \quad (\text{B.196})$$

$$\epsilon_k = \epsilon - \frac{2B_k}{\sqrt{k}} - \frac{1}{\sqrt{k}} - \sqrt{\frac{\mathcal{V}(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2\mathcal{V}(d)}} \quad (\text{B.197})$$

$$R = \max_{\substack{\Delta: \\ \sum_{a=1}^m \Delta_a v(a) \in (-\underline{b}, b_k]}} g(\Delta) + \frac{(m-1)(M_S(\eta)-1)}{2} \frac{\log k}{k} + \frac{1}{k} \log \left(\frac{\log_e k}{2C} \right) \quad (\text{B.198})$$

where B_k is the finite constant defined in (2.159). Using (B.194) and applying a Taylor series expansion to $Q^{-1}(\cdot)$, it is easy to see that R in (B.198) can be rewritten as the right side of (B.185).

Further, we use $kR = \log M$ and $(1-x)^M \leq e^{-Mx}$ to weaken the right side of (2.242) to obtain

$$\begin{aligned} & \sum_{\Delta} \binom{k}{k(\mathbf{p} + \Delta)} p^{k(\mathbf{p} + \Delta)} e^{-\binom{k}{\mathbf{t}^*}^{-1} L_k(k(\mathbf{p} + \Delta), \mathbf{t}^*) \exp(kR)} \\ &= \sum_{\sum_{a=1}^m \Delta_a v(a) \leq -\underline{b}} + \sum_{\sum_{a=1}^m \Delta_a v(a) \in (-\underline{b}, b_k)} + \sum_{\sum_{a=1}^m \Delta_a v(a) \geq b_k} \end{aligned} \quad (\text{B.199})$$

$$\begin{aligned} & \leq \mathbb{P} \left[\sum_{j=1}^k v(S_j) \leq \mathbb{E}[v(\mathbf{S})] - \underline{b} \right] \\ & + \sup_{\sum_{a=1}^m \Delta_a v(a) \in (-\underline{b}, b_k)} e^{-Ck - \frac{(m-1)(M_S(\eta)-1)}{2} \exp k\{R-g(\Delta)\}} \\ & + \mathbb{P} \left[\sum_{j=1}^k v(S_j) \geq \mathbb{E}[v(\mathbf{S})] + \underline{b}_k \right] \end{aligned} \quad (\text{B.200})$$

$$\leq \frac{B_k}{\sqrt{k}} + \sqrt{\frac{\mathcal{V}(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2\mathcal{V}(d)}} + \frac{1}{\sqrt{k}} + \epsilon_k + \frac{B_k}{\sqrt{k}} \quad (\text{B.201})$$

where $P_{S_j}(a) = P_S(a)$. The first and third probabilities in (B.199) are bounded using the Berry-Esséen bound (2.155) and (A.28). The middle probability is bounded by observing that the difference between R and $g(\Delta)$ in $\sum_{a=1}^m \Delta_a v(a) \in (-\underline{b}, b_k)$ is at least $\frac{(m-1)(M_S(\eta)-1)}{2} \frac{\log k}{k} + \frac{1}{k} \log \left(\frac{\log_e k}{2C} \right)$.

B.11 Gaussian approximation of the bound in Theorem 2.45

Using Theorem 2.45, we show that $R(k, d, \epsilon)$ does not exceed the right-hand side of (2.283) with the remainder satisfying (2.285). Since the excess-distortion probability in (2.270) depends on σ^2 only through the ratio $\frac{d}{\sigma^2}$, for simplicity we let $\sigma^2 = 1$. Using inequality $(1-x)^M \leq e^{-Mx}$, the right side of (2.270) can be upper bounded by

$$\int_0^\infty e^{-\rho(k, x) \exp(kR)} f_{\chi_k^2}(kx) k \, dx, \quad (\text{B.202})$$

From Stirling's approximation for the Gamma function

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \left(1 + O\left(\frac{1}{x}\right)\right) \quad (\text{B.203})$$

it follows that

$$\frac{\Gamma\left(\frac{k}{2} + 1\right)}{\sqrt{\pi k} \Gamma\left(\frac{k-1}{2} + 1\right)} = \frac{1}{\sqrt{2\pi k}} \left(1 + O\left(\frac{1}{k}\right)\right), \quad (\text{B.204})$$

which is clearly lower bounded by $\frac{1}{2\sqrt{\pi k}}$ when k is large enough. This implies that for all $a^2 \leq x \leq b^2$ and all k large enough

$$\rho(k, z) \geq \frac{1}{2\sqrt{\pi k}} \exp \left\{ (k-1) \log (1 - g(x))^{\frac{1}{2}} \right\} \quad (\text{B.205})$$

where

$$g(x) = \frac{(1+x-2d)^2}{4(1-d)z} \quad (\text{B.206})$$

It is easy to check that $g(x)$ attains its global minimum at $x = [1-2d]^+$ and is monotonically increasing for $x > [1-2d]^+$. Let

$$b_k = \sqrt{\frac{2}{k}} Q^{-1}(\epsilon_k) \quad (\text{B.207})$$

$$\epsilon_k = \epsilon - \frac{c_0 4\sqrt{2} + 1}{\sqrt{k}} - \frac{1}{4d\sqrt{\pi k}} e^{-2d^2 k} \quad (\text{B.208})$$

$$R = -\frac{1}{2} \log(1 - g(1 + b_k)) + \frac{1}{2} \frac{\log k}{k} + \frac{1}{k} \log(\sqrt{\pi} \log_e k) \quad (\text{B.209})$$

where c_0 is that in (2.159). Using a Taylor series expansion, it is not hard to check that R in (B.209) can be written as the right side of (2.283). So, the theorem will be proven if we show that with R in (B.209), (B.202) is upper bounded by ϵ for k sufficiently large.

Toward this end, we split the integral in (B.202) into three integrals and upper bound each separately:

$$\int_0^\infty = \int_0^{[1-2d]^+} + \int_{[1-2d]^+}^{1+b_k} + \int_{1+b_k}^\infty \quad (\text{B.210})$$

The first and the third integrals can be upper bounded using the Berry-Essén inequality (2.155) and (A.28):

$$\int_0^{[1-2d]^+} \leq \mathbb{P} \left[\sum_{i=1}^k S_i^2 < k(1-2d) \right] \quad (\text{B.211})$$

$$\leq \frac{B_k}{\sqrt{k}} + \frac{1}{4d\sqrt{\pi k}} e^{-2d^2 k} \quad (\text{B.212})$$

$$\int_{1+b_k}^\infty \leq \mathbb{P} \left[\sum_{i=1}^k S_i^2 > k(1+b_k) \right] \quad (\text{B.213})$$

$$\leq \epsilon_k + \frac{B_k}{\sqrt{k}} \quad (\text{B.214})$$

Finally, the second integral is upper bounded by $\frac{1}{\sqrt{k}}$ because by the monotonicity of $g(z)$,

$$e^{-\rho(k,x)\exp(kR)} \leq e^{-\frac{1}{2\sqrt{\pi k}}\exp\{\frac{1}{2}\log k + \log(\sqrt{\pi}\log_e k)\}} \quad (\text{B.215})$$

$$= \frac{1}{\sqrt{k}} \quad (\text{B.216})$$

for all $[1 - 2d]^+ \leq x \leq 1 + b_k$.

Appendix C

Lossy joint source-channel coding: proofs

C.1 Proof of the converse part of Theorem 3.10

Note that for the converse, restriction (iv) in Section 2.6.2 can be replaced by the following weaker one:

(iv') The random variable $J_S(S, d)$ has finite absolute third moment.

To verify that (iv) implies (iv'), observe that by the concavity of the logarithm,

$$0 \leq J_S(s, d) + \lambda^* d \leq \lambda^* \mathbb{E}[d(s, Z^*)] \quad (\text{C.1})$$

so

$$\mathbb{E} \left[|J_S(S, d) + \lambda^* d|^3 \right] \leq \lambda^{*3} \mathbb{E} [d^3(S, Z^*)] \quad (\text{C.2})$$

We now proceed to prove the converse by showing first that we can eliminate all rates exceeding

$$\frac{k}{n} \geq \frac{C}{R(d) - 3\tau} \quad (\text{C.3})$$

for any $0 < \tau < \frac{R(d)}{3}$. More precisely, we show that the excess-distortion probability of any code having such rate converges to 1 as $n \rightarrow \infty$, and therefore for any $\epsilon < 1$, there is an n_0 such that for all $n \geq n_0$, no (k, n, d, ϵ) code can exist for k, n satisfying (C.3).

We weaken (3.11) by fixing $\gamma = k\tau$ and choosing a particular channel output distribution, namely, $P_{\bar{Y}^n} = P_{Y^{n*}} = P_{Y^*} \times \dots \times P_{Y^*}$. Due to restrictions (i) and (ii) in Section 2.6.2, $P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, and the d-tilted information single-letterizes, that is, for a.e. s^k ,

$$J_{S^k}(s^k, d) = \sum_{i=1}^k J_S(s_i, d) \quad (\text{C.4})$$

Theorem 3.1 implies that error probability ϵ' of every (k, n, d, ϵ') code must be lower bounded by

$$\begin{aligned} & \mathbb{E} \left[\min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^k J_S(S_i, d) - \sum_{j=1}^n i_{X;Y}^*(x_i; Y_i) \geq k\tau \mid S^k \right] \right] - \exp(-k\tau) \\ & \geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{j=1}^n i_{X;Y}^*(x_i; Y_i) \leq nC + k\tau \right] \mathbb{P} \left[\sum_{i=1}^k J_S(S_i, d) \geq nC + 2k\tau \right] - \exp(-k\tau) \end{aligned} \quad (\text{C.5})$$

$$\geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{j=1}^n i_{X;Y}^*(x_i; Y_i) \leq nC + n\tau' \right] \mathbb{P} \left[\sum_{i=1}^k J_S(S_i, d) \geq kR(d) - k\tau \right] - \exp(-k\tau) \quad (\text{C.6})$$

where in (C.6), we used (C.3) and $\tau' = \frac{C\tau}{R(d)-3\tau} > 0$. Recalling (2.11) and the well-known

$$D(P_{Y|X=x} \| P_{Y^*}) \leq C \quad (\text{C.7})$$

with equality for P_{X^*} -a.e. x , we conclude using the law of large numbers that (C.6) tends to 1 as $k, n \rightarrow \infty$.

We proceed to show that for all large enough k, n , if there is a sequence of (k, n, d, ϵ') codes such that

$$-3k\tau \leq nC - kR(d) \quad (\text{C.8})$$

$$\leq \sqrt{nV + kV(d)}Q^{-1}(\epsilon) + \theta(n) \quad (\text{C.9})$$

then $\epsilon' \geq \epsilon$.

Note that in general the bound in Theorem 3.1 with the choice of $P_{\bar{Y}^n}$ as above does not lead to the correct channel dispersion term. We first consider the general case, in which we apply Corollary 3.3, and then we show the symmetric case, in which we apply Theorem 3.2.

Given a finite set \mathcal{A} , we say that $x^n \in \mathcal{A}^n$ has type P_X if the number of times each letter $a \in \mathcal{A}$ is encountered in x^n is $nP_X(a)$. In Corollary 3.3, we weaken the supremum over W by letting W map X^n to its type, $W = \text{type}(X^n)$. Note that the total number of types satisfies (e.g. [46])

$T \leq (n+1)^{|\mathcal{A}|-1}$. We weaken the supremum over \bar{Y}^n in (3.25) by fixing $P_{\bar{Y}^n|W=P_X} = P_Y \times \dots \times P_Y$, where $P_X \rightarrow P_{Y|X} \rightarrow P_Y$, i.e. P_Y is the output distribution induced by the type P_X . In this way, Corollary 3.3 implies that the error probability of any (k, n, d, ϵ') code must be lower bounded by

$$\epsilon' \geq \mathbb{E} \left[\min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_S(S_i, d) - \sum_{i=1}^n \iota_{X;Y}(x_i; Y_i) \geq \gamma \mid S^k \right] \right] - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \quad (\text{C.10})$$

Choose

$$\gamma = \left(|\mathcal{A}| - \frac{1}{2} \right) \log(n+1) \quad (\text{C.11})$$

We need to solve the minimization in (C.10) only in the following typical set of source sequences:

$$\mathcal{T}_{k,n} = \left\{ s^k \in \mathcal{S}^k : \left| \sum_{i=1}^k \mathcal{J}_S(s_i, d) - nC \right| \leq n\bar{\Delta} - \gamma \right\} \quad (\text{C.12})$$

Observe that

$$\mathbb{P}[S^k \notin \mathcal{T}_{k,n}] = \mathbb{P} \left[\left| \sum_{i=1}^k \mathcal{J}_S(S_i, d) - nC \right| > n\bar{\Delta} - \gamma \right] \quad (\text{C.13})$$

$$\leq \mathbb{P} \left[\left| \sum_{i=1}^k \mathcal{J}_S(S_i, d) - kR(d) \right| + |nC - kR(d)| + \gamma > n\bar{\Delta} \right] \quad (\text{C.14})$$

$$\leq \mathbb{P} \left[\left| \sum_{i=1}^k \mathcal{J}_S(S_i, d) - kR(d) \right| > k \frac{\bar{\Delta} R(d)}{2C} \right] \quad (\text{C.15})$$

$$\leq \frac{4C^2}{R^2(d)\bar{\Delta}^2} \frac{\mathcal{V}(d)}{k} \quad (\text{C.16})$$

where

- (C.15) follows by lower bounding

$$n\bar{\Delta} - \gamma - |nC - kR(d)| \geq n\bar{\Delta} - \gamma - 3k\tau \quad (\text{C.17})$$

$$\geq n \frac{3\bar{\Delta}}{4} - 3k\tau \quad (\text{C.18})$$

$$\geq k \frac{3\bar{\Delta}}{4C} (R(d) - 3\tau) - 3k\tau \quad (\text{C.19})$$

$$\geq k \frac{\bar{\Delta} R(d)}{2C} \quad (\text{C.20})$$

where

- (C.17) holds for large enough n due to (C.8) and (C.9);

- (C.18) holds for large enough n by the choice of γ in (C.11);
- (C.19) lower bounds n using (C.8);
- (C.20) holds for a small enough $\tau > 0$.

- (C.16) is by Chebyshev's inequality.

We perform the minimization on the right side of (E.30) separately for $\text{type}(x^n) \in \mathcal{P}_\delta^*$ and $\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*$, where

$$\mathcal{P}_\delta^* = \{P_X \in \mathcal{P}: |P_X - P_{X^*}| \leq \delta\} \quad (\text{C.21})$$

We now show that $\epsilon' \geq \epsilon$ holds for arbitrary $\epsilon < 1$ and n large enough as long as $\bar{\Delta} \leq \frac{a}{2}$ if the minimization is restricted to types in $\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*$, where $\delta > 0$ is arbitrary, and

$$a = C - \max_{P_X \in \mathcal{P} \setminus \mathcal{P}_\delta^*} I(P_X) > 0 \quad (\text{C.22})$$

Define the following functions $\mathcal{P} \mapsto \mathbb{R}_+$:

$$\mu(P_X) = \mathbb{E}[\iota_{X;Y}(X; Y)] \quad (\text{C.23})$$

$$V(P_X) = \mathbb{E}[\text{Var}[\iota_{X;Y}(X; Y) \mid X]] \quad (\text{C.24})$$

$$T(P_X) = \mathbb{E}[\left|\iota_{X;Y}(X; Y) - \mathbb{E}[\iota_{X;Y}(X; Y) \mid X]\right|^3] \quad (\text{C.25})$$

where $P_X \mapsto P_{Y|X} \mapsto P_Y$.

By Chebyshev's inequality, for all x^n whose type belongs to $\mathcal{P} \setminus \mathcal{P}_\delta^*$ and all $s^k \in \mathcal{T}_{k,n}$

$$\mathbb{P}\left[\sum_{i=1}^k J_S(s_i, d) - \sum_{i=1}^n \iota_{X;Y}(x_i; Y_i) < \gamma \mid S^k = s^k\right] \leq \mathbb{P}\left[\sum_{i=1}^n \iota_{X;Y}(x_i; Y_i) > n(C - \bar{\Delta})\right] \quad (\text{C.26})$$

$$= \mathbb{P}\left[\sum_{i=1}^n \iota_{X;Y}(x_i; Y_i) - n\mu(P_X) > n(C - I(P_X)) - n\bar{\Delta}\right] \quad (\text{C.27})$$

$$\leq \mathbb{P}\left[\sum_{i=1}^n \iota_{X;Y}(x_i; Y_i) - n\mu(P_X) > \frac{na}{2}\right] \quad (\text{C.28})$$

$$\leq \frac{4\bar{V}}{na^2} \quad (\text{C.29})$$

where in (C.28) we used

$$\Delta \leq \frac{1}{2}a < a \leq C - I(P_X) \quad (\text{C.30})$$

and (C.29) is by Lemma A.4.3, where

$$\overline{V} = \max_{P_X \in \mathcal{P}} V(P_X) \quad (\text{C.31})$$

Note that $\overline{V} < \infty$ because $V(P_X)$ is continuous on the compact set \mathcal{P} and therefore achieves its maximum, so

$$\begin{aligned} & \mathbb{E} \left[\min_{\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*} \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_S(s_i, d) - \sum_{i=1}^n \imath_{X;Y}(x_i; Y_i) \geq \gamma | S^k \right] \right] \\ & \geq \mathbb{E} \left[\min_{\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*} \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_S(s_i, d) - \sum_{i=1}^n \imath_{X;Y}(x_i; Y_i) \geq \gamma | S^k \right] 1_{\{S^k \in \mathcal{T}_{k,n}\}} \right] \end{aligned} \quad (\text{C.32})$$

$$> \left(1 - \frac{4\overline{V}}{na^2} \right) P_{S^k}(\mathcal{T}_{k,n}) \quad (\text{C.33})$$

$$\geq 1 - \frac{4\overline{V}}{na^2} - \frac{16C^2}{R^2(d)a^2} \frac{\mathcal{V}(d)}{k} \quad (\text{C.34})$$

We conclude that the excess-distortion probability approaches 1 arbitrarily closely if the minimization is restricted to types in $\mathcal{P} \setminus \mathcal{P}_\delta^*$.

To perform the minimization on the right side of (E.30) over \mathcal{P}_δ^* , we will invoke Theorem A.6 with $\mathcal{D} = \mathcal{P}_\delta^*$, the distance being the usual Euclidean distance between $|\mathcal{A}|$ -vectors, and

$$W_i = \imath_{X;Y}(x_i; Y_i) \quad (\text{C.35})$$

$$z = P_X = \text{type}(x^n) \quad (\text{C.36})$$

Denote by $P_{\hat{X}}$ the minimum Euclidean distance approximation of P_X in the set of n -types, that is,

$$P_{\hat{X}} = \arg \min_{\substack{P \in \mathcal{P}: \\ P \text{ is an } n\text{-type}}} |P_X - P| \quad (\text{C.37})$$

and denote $P_{\hat{X}} \mapsto P_{Y|X} \mapsto P_{\hat{Y}}$. The accuracy of approximation in (C.37) is controlled by the following inequality.

$$|P_X - P_{\hat{X}}| \leq \frac{\sqrt{|\mathcal{A}|(|\mathcal{A}| - 1)}}{n} \quad (\text{C.38})$$

With the choice in (C.35) and (C.36) the functions (A.40)–(A.42) are particularized to the following mappings $\mathcal{P} \mapsto \mathbb{R}_+$:

$$\mu_n(P_X) = \mu(P_{\hat{X}}) \quad (\text{C.39})$$

$$V_n(P_X) = V(P_{\hat{X}}) \quad (\text{C.40})$$

$$T_n(P_X) = T(P_{\hat{X}}) \quad (\text{C.41})$$

where $P_{\hat{X}} \rightarrow P_{Y|X} \rightarrow P_{\hat{Y}}$ and the functions $\mu(\cdot)$, $V(\cdot)$, $T(\cdot)$ are defined in (C.23)–(C.25).

Assuming without loss of generality that all outputs in \mathcal{B} are accessible (which implies that $P_{Y^*}(y) > 0$ for all $y \in \mathcal{B}$), we choose $\delta > 0$ so that

$$\min_{P_X \in \mathcal{P}_\delta^*} \min_{y \in \mathcal{B}} P_Y(y) = p_{\min} > 0 \quad (\text{C.42})$$

$$2 \min_{P_X \in \mathcal{P}_\delta^*} V(P_X) \geq V \quad (\text{C.43})$$

Let us check that the assumptions of Theorem A.6.3 are satisfied with μ_n^* , V_n^* being

$$\mu_n^* = C \quad (\text{C.44})$$

$$V_n^* = V \quad (\text{C.45})$$

It is easy to verify directly that the functions $\mu(\cdot)$, $V(\cdot)$, $T(\cdot)$ are continuous (and therefore bounded) on \mathcal{P} and infinitely differentiable on \mathcal{P}_δ^* . Therefore, assumptions (A.46) and (A.47) of Theorem A.6 are met.

To verify that (A.51) holds, write

$$C - \mu(P_{\hat{X}}) = C - \mu(P_X) + \mu(P_X) - \mu(P_{\hat{X}}) \quad (\text{C.46})$$

$$\geq C - \mu(P_X) - \frac{\ell_3}{n} \quad (\text{C.47})$$

$$\geq \ell_1 |v|^2 - \frac{\ell_3}{n} \quad (\text{C.48})$$

where ℓ_1 and ℓ_3 are positive constants,

$$v = P_X - P_{X^*}, \quad (\text{C.49})$$

and:

- (C.47) uses (C.38) and continuous differentiability of $\mu(\cdot)$ on the compact \mathcal{P}_δ^* ;
- (C.48) uses

$$\mu(P_X) \leq C - \ell'_1 |v|^2 \quad (\text{C.50})$$

Although (C.50) was shown in [3, (497)–(505)] by an explicit computation of the Hessian matrix of $\mu(P_X)$, here we provide a simpler proof using Pinsker's inequality. Viewing P_X as a vector and $P_{Y|X}$ as a matrix, write

$$P_X = P_{X^*} + v_0 + v_\perp \quad (\text{C.51})$$

where v_0 and v_\perp are projections of v onto $\text{Ker} P_{Y|X}$ and $(\text{Ker} P_{Y|X})^\perp$ respectively, where

$$\text{Ker} P_{Y|X} = \left\{ v \in \mathbb{R}^{|\mathcal{A}|} : v^T P_{Y|X} = 0 \right\} \quad (\text{C.52})$$

We consider two cases $v_\perp = 0$ and $v_\perp \neq 0$ separately. Condition $v_\perp = 0$ implies $P_X \rightarrow P_{Y|X} \rightarrow P_{Y^*}$, which combined with $P_X \neq P_{X^*}$ and (C.7) means that the complement of $F = \text{supp}(P_{X^*})$ is nonempty and

$$a \triangleq C - \max_{x \notin F} D(P_{Y|X=x} \| P_{Y^*}) \quad (\text{C.53})$$

is positive. Therefore

$$\mu(P_X) = D(P_{Y|X} \| P_{Y^*} | P_X) \quad (\text{C.54})$$

$$\leq C P_X(F) + P_X(F^c) (C - a) \quad (\text{C.55})$$

$$\leq C - (\lambda_{\min}^+(P_F^2))^{1/2} a |v| \quad (\text{C.56})$$

$$\leq C - \frac{1}{4} (\lambda_{\min}^+(P_F^2))^{1/2} a |v|^2 \quad (\text{C.57})$$

where (C.55) uses (C.7), P_F is the orthogonal projection matrix onto F^c and $\lambda_{\min}^+(\cdot)$ is the minimum nonzero eigenvalue of the indicated positive semidefinite matrix, and (C.57) holds because the Euclidean distance between two distributions is bounded by 2.

If $v_\perp \neq 0$, write

$$\mu(P_X) = D(P_{Y|X} \| P_{Y^*} | P_X) - D(P_Y \| P_{Y^*}) \quad (\text{C.58})$$

$$\leq D(P_{Y|X} \| P_{Y^*} | P_X) - \frac{1}{2} |P_Y - P_{Y^*}|^2 \log e \quad (\text{C.59})$$

$$\leq C - \frac{1}{2} |P_Y - P_{Y^*}|^2 \log e \quad (\text{C.60})$$

where (C.59) is by Pinsker's inequality (given in (A.6)), and (C.60) is by (C.7). To conclude the proof of (C.50), we lower bound the second term in (C.60) as follows.

$$|P_Y - P_{Y^*}|^2 = \left| (P_X - P_{X^*})^T P_{Y|X} \right|^2 \quad (\text{C.61})$$

$$= |v_\perp^T P_{Y|X}|^2 \quad (\text{C.62})$$

$$\geq \lambda_{\min}(P_{Y|X}) |v_\perp|^2 \quad (\text{C.63})$$

$$\geq \lambda_{\min}^+(P_{Y|X} P_{Y|X}^T) \lambda_{\min}^+(P_\perp^2) |v|^2 \quad (\text{C.64})$$

where P_\perp is the orthogonal projection matrix onto $(\text{Ker } P_{Y|X})^\perp$.

Finally, conditions (A.44) and (A.52) (with $F_2 = 0$) hold due to continuous differentiability of $\mu(\cdot)$ and $V(\cdot)$ on the compact \mathcal{P}_δ^* .

Theorem A.6 is thereby applicable.

At this point we consider two cases separately, $V > 0$ and $V = 0$.

C.1.1 $V > 0$.

Let

$$\epsilon_{k,n} = \epsilon + \frac{B}{\sqrt{k}} + \frac{K}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{4C^2}{R^2(d)\Delta^2} \frac{\mathcal{V}(d)}{k} \quad (\text{C.65})$$

where $B > 0$ will be chosen in the sequel, K is the same as in (A.50), and k, n are chosen so that both (C.8) and the following version of (C.9) hold:

$$nC - kR(d) \leq \sqrt{nV + k\mathcal{V}(d)} Q^{-1}(\epsilon_{k,n}) - \gamma \quad (\text{C.66})$$

In order to apply Theorem A.6.3, let $W^* \sim \mathcal{N}(C, \frac{V}{n})$, independent of S^k . Weakening (C.10)

using (C.11) and Theorem A.6.3, we can lower bound ϵ' by

$$\begin{aligned} & \mathbb{E} \left[\min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\iota_{\mathbf{X}; \mathbf{Y}}(x_i; Y_i) - \sum_{i=1}^k j_S(S_i, d) \leq -\gamma \mid S^k \right] \cdot \mathbb{1} \{S^k \in \mathcal{T}_{k,n}\} \right] - \frac{1}{\sqrt{n+1}} \\ & \geq \mathbb{E} \left[\mathbb{P} \left[nW^* - \sum_{i=1}^k j_S(S_i, d) \leq -\gamma \mid S^k \right] \cdot \mathbb{1} \{S^k \in \mathcal{T}_{k,n}\} \right] - \frac{K}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \end{aligned} \quad (\text{C.67})$$

$$= \mathbb{P} \left[nW^* - \sum_{i=1}^k j_S(S_i, d) \leq -\gamma, S^k \in \mathcal{T}_{k,n} \right] - \frac{K}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \quad (\text{C.68})$$

$$\geq \mathbb{P} \left[nW^* - \sum_{i=1}^k j_S(S_i, d) \leq -\gamma \right] - \mathbb{P} [S^k \notin \mathcal{T}_{k,n}] - \frac{K}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \quad (\text{C.69})$$

$$\geq \mathbb{P} \left[nW^* - \sum_{i=1}^k j_S(S_i, d) \leq -\gamma \right] - \frac{4C^2}{R^2(d)\Delta^2} \frac{\mathcal{V}(d)}{k} - \frac{K}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \quad (\text{C.70})$$

$$\geq \epsilon \quad (\text{C.71})$$

where (C.67) is by Theorem A.6.4, (C.69) is by the union bound, and (C.70) makes use of (C.16). To justify (C.71), observe that if $\mathcal{V}(d) = 0$, $j_S(S_i, d) = R(d)$ a.s., and (C.71) is immediate with $B = 0$ in view of (C.65). Otherwise, (C.71) follows by the Berry-Esséen bound (Theorem 2.23) letting B to be the Berry-Esséen ratio (2.159) of the $k+1$ independent random variables $j_S(S_1, d), \dots, j_S(S_k, d), nW^*$.

C.1.2 $V = 0$.

If $V = \mathcal{V}(d) = 0$, fix $0 < \eta < 1 - \epsilon$ and let

$$\gamma = (|\mathcal{A}| - 1) \log(n+1) + \log \frac{1}{\eta} \quad (\text{C.72})$$

and choose k, n that satisfy

$$nC - kR(d) \geq \left(\frac{K}{1 - \epsilon - \eta} \right)^{\frac{2}{3}} n^{\frac{1}{3}} + \gamma \quad (\text{C.73})$$

where $K > 0$ is that in (A.53). Plugging $j_S(S_i, d) = R(d)$ a.s. in (C.10), we have

$$\epsilon' \geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^n \iota_{\mathbf{X}; \mathbf{Y}}(x_i; Y_i) \leq kR(d) - \gamma \right] - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \quad (\text{C.74})$$

$$\geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^n \iota_{\mathbf{X}; \mathbf{Y}}(x_i; Y_i) \leq nC + \left(\frac{K}{1 - \epsilon - \eta} \right)^{\frac{2}{3}} n^{\frac{1}{3}} \right] - \eta \quad (\text{C.75})$$

$$\geq \epsilon \quad (\text{C.76})$$

where (C.76) invokes Theorem A.6.4 with $\beta = \frac{1}{6}$, $A = \left(\frac{K}{1-\epsilon-\eta}\right)^{\frac{2}{3}}$.

If $\mathcal{V}(d) > 0$, we choose γ as in (C.11), and we let

$$\epsilon_{k,n} = \epsilon + \frac{B}{\sqrt{k}} + (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) + \frac{1}{n^{\frac{1}{4}-\frac{3}{2}\beta}} \quad (\text{C.77})$$

where $B > 0$ is the same as in (C.65), and k, n are chosen so that the following version of (C.9) holds:

$$nC - kR(d) \leq \sqrt{k\mathcal{V}(d)}Q^{-1}(\epsilon_{k,n}) - \gamma - An^{\frac{1}{2}-\beta} \quad (\text{C.78})$$

where $A = K^{\frac{2}{3}}$, where K is that in (A.53). Weakening (C.10) using Theorem A.6.4, we have

$$\begin{aligned} \epsilon' &\geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\iota_{\mathbf{X}; \mathbf{Y}}(x_i, Y_i) \geq nC + An^{\frac{1}{2}-\beta} \right] \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_{\mathbf{S}}(S_i, d) \geq nC + An^{\frac{1}{2}-\beta} + \gamma \right] \\ &\quad - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \end{aligned} \quad (\text{C.79})$$

$$\geq \left(1 - \frac{1}{n^{\frac{1}{4}-\frac{3}{2}\beta}}\right) \mathbb{P} \left[\sum_{i=1}^k \mathcal{J}_{\mathbf{S}}(S_i, d) \geq kR(d) + \sqrt{k\mathcal{V}(d)}Q^{-1}(\epsilon_{k,n}) \right] - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \quad (\text{C.80})$$

$$\geq \left(1 - \frac{1}{n^{\frac{1}{4}-\frac{3}{2}\beta}}\right) \left(\epsilon_{k,n} - \frac{B}{\sqrt{k}} \right) - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \quad (\text{C.81})$$

$$\geq \epsilon_{k,n} - \frac{B}{\sqrt{k}} - \frac{1}{n^{\frac{1}{4}-\frac{3}{2}\beta}} - (n+1)^{|\mathcal{A}|-1} \exp(-\gamma) \quad (\text{C.82})$$

$$= \epsilon \quad (\text{C.83})$$

where (C.80) uses (A.53) and (C.78), and (C.81) is by the Berry-Esséen bound.

C.1.3 Symmetric channel

We show that if the channel is such that the distribution of $\iota_{\mathbf{X}; \mathbf{Y}}^*(\mathbf{x}; \mathbf{Y})$ (according to $P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$) does not depend on the choice $\mathbf{x} \in \mathcal{A}$, Theorem 3.2 leads to a tighter third-order term than (3.93).

If either $V > 0$ or $\mathcal{V}(d) > 0$, let

$$\gamma = \frac{1}{2} \log n \quad (\text{C.84})$$

$$\epsilon_{k,n} = \epsilon + \frac{B}{\sqrt{n+k}} + \frac{1}{\sqrt{n}} \quad (\text{C.85})$$

where $B > 0$ can be chosen as in (C.65), and let k, n be such that the following version of (C.9)

(with the remainder $\theta(n)$ satisfying (3.93) with $\underline{c} = \frac{1}{2}$) holds:

$$nC - kR(d) \leq \sqrt{nV + k\mathcal{V}(d)}Q^{-1}(\epsilon_{k,n}) - \gamma \quad (\text{C.86})$$

Theorem 3.2 and Theorem 2.23 imply that the error probability of every (k, n, d, ϵ') code must satisfy, for an arbitrary sequence $x^n \in \mathcal{A}^n$,

$$\epsilon' \geq \mathbb{P} \left[\sum_{i=1}^k \mathcal{I}_{\mathcal{S}}(S_i, d) - \sum_{j=1}^n \mathcal{I}_{\mathbf{X}; \mathbf{Y}}^*(x_j; Y_j) \geq \gamma \right] - \exp(-\gamma) \quad (\text{C.87})$$

$$\geq \epsilon \quad (\text{C.88})$$

If both $V = 0$ and $\mathcal{V}(d) = 0$, choose k, n to satisfy

$$kR(d) - nC \geq \gamma \quad (\text{C.89})$$

$$= \log \frac{1}{1 - \epsilon} \quad (\text{C.90})$$

Substituting (C.90) and $\mathcal{I}_{\mathcal{S}}(S_i, d) = R(d)$, $\mathcal{I}_{\mathbf{X}; \mathbf{Y}}^*(x_i; Y_i) = C$ a.s. in (C.87), we conclude that the right side of (C.87) equals ϵ , so $\epsilon' \geq \epsilon$ whenever a (k, n, d, ϵ') code exists.

C.1.4 Gaussian channel

In view of Remark 3.14, it suffices to consider the equal power constraint (3.107). The spherically-symmetric $P_{Y^n} = P_{Y^{n*}} = P_{Y^*} \times \dots \times P_{Y^*}$, where $Y^* \sim \mathcal{N}(0, \sigma_{\mathbf{N}}^2(1 + P))$, satisfies the symmetry assumption of Theorem 3.2. In fact, for all $x^n \in \mathcal{F}(\alpha)$, $\mathcal{I}_{X^n; Y^n}^*(x^n; Y^n)$ has the same distribution under $P_{Y^n|X^n=x^n}$ as (cf. (3.142))

$$G_n = \frac{n}{2} \log(1 + P) - \frac{\log e}{2} \left(\frac{P}{1 + P} \sum_{i=1}^n \left(W_i - \frac{1}{\sqrt{P}} \right)^2 - n \right) \quad (\text{C.91})$$

where $W_i \sim \mathcal{N}\left(\frac{1}{\sqrt{P}}, 1\right)$, independent of each other. Since G_n is a sum of i.i.d. random variables, the mean of $\frac{G_n}{n}$ is equal to $C = \frac{1}{2} \log(1 + P)$ and its variance is equal to (3.92), the result follows analogously to (C.84)–(C.88).

C.2 Proof of the achievability part of Theorem 3.10

C.2.1 Almost lossless coding ($d = 0$) over a DMC.

The proof consists of an asymptotic analysis of the bound in Theorem 3.9 by means of Theorem 2.23. Weakening (3.87) by fixing $P_{X^n} = P_{X^n}^* = P_{X^*} \times \dots \times P_{X^*}$, we conclude that there exists a $(k, n, 0, \epsilon')$ code with

$$\epsilon' \leq \mathbb{E} \left[\exp \left(- \left| \sum_{i=1}^n \imath_{X^*, Y^*}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \imath_S(S_i) \right|^+ \right) \right] \quad (\text{C.92})$$

where (S^k, X^{n*}, Y^{n*}) are distributed according to $P_{S^k} P_{X^{n*}} P_{Y^{n*} | X^{n*}}$. The case of equiprobable S has been tackled in [3]. Here we assume that $\imath_S(S)$ is not a constant, that is, $\text{Var}[\imath_S(S)] > 0$.

Let k and n be such that

$$nC - kH(S) \geq \sqrt{nV + k\mathcal{V}} Q^{-1}(\epsilon_{k,n}) \quad (\text{C.93})$$

$$\epsilon_{k,n} = \epsilon - \frac{B_{k,n}}{\sqrt{n+k}} - \frac{2 \log 2}{\sqrt{2\pi(nV + k\mathcal{V})}} + \frac{4B_{k,n}}{n+k} \quad (\text{C.94})$$

where $\mathcal{V} = \text{Var}[\imath_S(S)]$, and $B_{k,n}$ is the Berry-Essén ratio (2.159) for the sum of $n+k$ independent random variables appearing in the right side of (C.92). Note that $B_{k,n}$ is bounded by a constant due to:

- $\text{Var}[\imath_S(S)] > 0$;
- the third absolute moment of $\imath_S(S)$ is finite;
- the third absolute moment of $\imath_{X^*, Y^*}^*(X^*; Y^*)$ is finite since the channel has finite input and output alphabets.

Therefore, (C.93) can be written as (3.88) with the remainder therein satisfying (3.97). So, it suffices to prove that if k, n satisfy (C.93), then the right side of (C.92) is upper bounded by ϵ . Let

$$S_{k,n} = nC - kH(S) - \sqrt{nV + k\mathcal{V}} Q^{-1}(\epsilon_{k,n}) \quad (\text{C.95})$$

Note that $S_{k,n} \geq 0$. We now further upper bound (C.92) as

$$\begin{aligned} \epsilon' &\leq \mathbb{E} \left[\exp \left(- \sum_{i=1}^n \iota_{X;Y}^*(X_i^*; Y_i^*) + \sum_{i=1}^k \iota_S(S_i) \right) 1 \left\{ \sum_{i=1}^n \iota_{X;Y}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \iota_S(S_i) > S_{k,n} \right\} \right] \\ &\quad + \mathbb{P} \left[\sum_{i=1}^n \iota_{X;Y}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \iota_S(S_i) \leq S_{k,n} \right] \end{aligned} \quad (\text{C.96})$$

$$\leq 2 \left(\frac{\log 2}{\sqrt{2\pi(nV + k\mathcal{V})}} + \frac{2B_{k,n}}{n+k} \right) \frac{1}{\exp(S_{k,n})} + \epsilon_{k,n} - \frac{B_{k,n}}{\sqrt{n+k}} \quad (\text{C.97})$$

$$\leq \epsilon \quad (\text{C.98})$$

where we invoked Lemma A.4.4 and the Berry-Esséen bound (Theorem 2.23) to upper-bound the first and the second term in the right side of (C.96), respectively.

C.2.2 Lossy coding over a DMC.

The proof consists of the asymptotic analysis of the bound in Theorem 3.8 using Theorem 2.23 and Lemma 2.24, which deals with asymptotic behavior of distortion d -balls. Note that Lemma 2.24 is the only step that requires finiteness of the ninth absolute moment of $d(S, Z^*)$ as required by restriction (iv) in Section 2.6.2. We weaken (3.82) by fixing

$$P_{X^n} = P_{X^{n*}} = P_{X^*} \times \dots \times P_{X^*} \quad (\text{C.99})$$

$$P_{Z^k} = P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*} \quad (\text{C.100})$$

$$\gamma = \frac{1}{2} \log_e k + 1 \quad (\text{C.101})$$

where $\Delta > 0$, so there exists a (k, n, d, ϵ') code with error probability ϵ' upper bounded by

$$\mathbb{E} \left[\exp \left(- \left| \sum_{i=1}^n \iota_{X;Y}^*(X_i^*; Y_i^*) - \log \frac{\gamma}{P_{Z^{k*}}(B_d(S^k))} \right|^+ \right) \right] + e^{1-\gamma} \quad (\text{C.102})$$

where $(S^k, X^{n*}, Y^{n*}, Z^{k*})$ are distributed according to $P_{S^k} P_{X^{n*}} P_{Y^{n*}|X^{n*}} P_{Z^{k*}}$. We need to show that for k, n satisfying (3.88), (C.102) is upper bounded by ϵ .

We apply Lemma 2.24 to upper bound (C.102) as follows:

$$\epsilon' \leq \mathbb{E} \left[\exp \left(- |U_{k,n}|^+ \right) \right] + \frac{K+1}{\sqrt{k}} \quad (\text{C.103})$$

with

$$U_{k,n} = \sum_{i=1}^n i_{\mathbf{X};\mathbf{Y}}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \mathcal{I}_{\mathcal{S}}(S_i, d) - C_0 \log k - \log \gamma - c \quad (\text{C.104})$$

where constants c , C_0 and K are defined in Lemma 2.24.

We first consider the (nontrivial) case $\mathcal{V}(d) + V > 0$. Let k and n be such that

$$nC - kR(d) \geq \sqrt{nV + k\mathcal{V}(d)} Q^{-1}(\epsilon_{k,n}) + C_0 \log k + \log \gamma + c \quad (\text{C.105})$$

$$\epsilon_{k,n} = \epsilon - \frac{B_{k,n}}{\sqrt{n+k}} - \frac{2 \log 2}{\sqrt{2\pi(nV + k\mathcal{V}(d))}} - \frac{4B_{k,n}}{n+k} - \frac{K+1}{\sqrt{k}} \quad (\text{C.106})$$

where $B_{k,n}$ is the Berry-Esséen ratio (2.159) for the sum of $n+k$ independent random variables appearing in (C.103). Note that $B_{k,n}$ is bounded by a constant because:

- either $\mathcal{V}(d) > 0$ or $V > 0$ by the assumption;
- the third absolute moment of $\mathcal{I}_{\mathcal{S}}(S, d)$ is finite by restriction (iv) in Section 2.6.2 as spelled out in (C.2);
- the third absolute moment of $i_{\mathbf{X};\mathbf{Y}}^*(X^*; Y^*)$ is finite since the channel has finite input and output alphabets.

Applying a Taylor series expansion to (C.105) with the choice of γ in (C.101), we conclude that (C.105) can be written as (3.88) with the remainder term satisfying (3.94).

It remains to further upper bound (C.103) using (C.105). Let

$$S_{k,n} = nC - kR(d) - \sqrt{nV + k\mathcal{V}(d)} Q^{-1}(\epsilon_{k,n}) \quad (\text{C.107})$$

Noting that $S_{k,n} \geq C_0 \log k + \log \gamma + c$, we upper-bound the expectation in the right side of (C.103) as

$$\begin{aligned} \mathbb{E} \left[\exp \left(-|U_{k,n}|^+ \right) \right] &\leq \mathbb{E} \left[\exp \left(-U_{k,n} \right) \mathbf{1} \left\{ \sum_{i=1}^n i_{\mathbf{X};\mathbf{Y}}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \mathcal{I}_{\mathcal{S}}(S_i, d) > S_{k,n} \right\} \right] \\ &\quad + \mathbb{P} \left[i_{\mathbf{X};\mathbf{Y}}^*(X_i^*; Y_i^*) - \sum_{i=1}^k \mathcal{I}_{\mathcal{S}}(S_i, d) \leq S_{k,n} \right] \end{aligned} \quad (\text{C.108})$$

$$\leq 2 \left(\frac{\log 2}{\sqrt{2\pi(nV + k\mathcal{V}(d))}} + \frac{2B_{k,n}}{n+k} \right) + \epsilon_{k,n} + \frac{B_{k,n}}{\sqrt{n+k}} \quad (\text{C.109})$$

where we used (C.105) and (C.107) to upper bound the exponent in the right side of (C.108).

Putting (C.103) and (C.109) together, we conclude that $\epsilon' \leq \epsilon$.

Finally, consider the case $V = \mathcal{V}(d) = 0$, which implies $\mathcal{I}_{\mathcal{S}}(S, d) = R(d)$ and $\mathcal{I}_{\mathbf{X}; \mathbf{Y}}^*(X_i^*; Y_i^*) = C$ almost surely, and let k and n be such that

$$nC - kR(d) \geq C_0 \log k + \log \gamma + c + \log \frac{1}{\epsilon - \frac{K+1}{\sqrt{k}}} \quad (\text{C.110})$$

where constants c and C_0 are defined in Lemma 2.24. Then

$$\mathbb{E} \left[\exp \left(-|U_{k,n}|^+ \right) \right] \leq \epsilon - \frac{K+1}{\sqrt{k}} \quad (\text{C.111})$$

which, together with (C.103), implies that $\epsilon' \leq \epsilon$, as desired.

C.2.3 Lossy or almost lossless coding over a Gaussian channel

In view of Remark 3.14, it suffices to consider the equal power constraint (3.107). As shown in the proof of Theorem 3.18, for any distribution of X^n on the power sphere,

$$\mathcal{I}_{X^n; Y^n}(X^n; Y^n) \geq G_n - F \quad (\text{C.112})$$

where G_n is defined in (C.91) (cf. (3.142)) and F is a (computable) constant.

Now, the proof for almost lossless coding in Appendix C.2.1 can be modified to work for the Gaussian channel by adding $\log F$ to the right side of (C.93) and replacing $\sum_{i=1}^n \mathcal{I}_{\mathbf{X}; \mathbf{Y}}^*(X_i^*; Y_i^*)$ in (C.92) and (C.96) with $G_n - \log F$, and in (C.95) with G_n .

Similarly, the proof for lossy coding in Appendix C.2.2 is adapted for the Gaussian channel by adding $\log F$ to the right side of (C.105) and replacing $\sum_{i=1}^n \mathcal{I}_{\mathbf{X}; \mathbf{Y}}^*(X_i^*; Y_i^*)$ in (C.102), (C.104) and (C.108) with $G_n - \log F$.

C.3 Proof of Theorem 3.22

Applying the Berry-Esséen bound to (3.164), we obtain

$$D_1(n, \epsilon, \alpha) \geq \min_{\substack{P_{\mathbf{Z}|\mathbf{S}}: \\ I(\mathbf{S}; \mathbf{Z}) \leq C(\alpha)}} \left\{ \mathbb{E}[\mathbf{d}(\mathbf{S}, \mathbf{Z})] + \sqrt{\frac{\text{Var}[\mathbf{d}(\mathbf{S}, \mathbf{Z})]}{n}} Q^{-1} \left(\epsilon + \frac{B}{\sqrt{n}} \right) \right\} \quad (\text{C.113})$$

$$= D(C(\alpha)) + \sqrt{\frac{W_1(\alpha)}{n}} Q^{-1} \left(\epsilon + \frac{B}{\sqrt{n}} \right) \quad (\text{C.114})$$

where B is the Berry-Esséen ratio, and (C.114) follows by the application of Lemma A.8 with

$$\mathcal{D} = \{P_{\mathbf{SZ}} = P_{\mathbf{Z}|\mathbf{S}}P_{\mathbf{S}} : I(\mathbf{S}; \mathbf{Z}) \leq R(\bar{d})\} \quad (\text{C.115})$$

$$f(P_{\mathbf{SZ}}) = -\mathbb{E}[d(\mathbf{S}, \mathbf{Z})] \quad (\text{C.116})$$

$$g(P_{\mathbf{SZ}}) = -\sqrt{\text{Var}[d(\mathbf{S}, \mathbf{Z})]}Q^{-1}\left(\epsilon + \frac{B}{\sqrt{n}}\right) \quad (\text{C.117})$$

$$\varphi = 1 \quad (\text{C.118})$$

$$\psi = \frac{1}{\sqrt{n}} \quad (\text{C.119})$$

Note that the mean and standard deviation of $d(\mathbf{S}, \mathbf{Z})$ are linear and continuously differentiable in $P_{\mathbf{SZ}}$, respectively, so conditions (A.68) and (A.70) hold with the metric being the usual Euclidean distance between vectors in $\mathbb{R}^{|\mathcal{S}| \times |\hat{\mathcal{S}}|}$. So, (C.114) follows immediately upon observing that by the definition of the rate-distortion function, $\mathbb{E}[d(\mathbf{S}, \mathbf{Z})] \geq \mathbb{E}[d(\mathbf{S}, \mathbf{Z}^*)] = D(C(\alpha))$ for all $P_{\mathbf{Z}|\mathbf{S}}$ such that $I(\mathbf{S}; \mathbf{Z}) \leq C(\alpha)$.

Appendix D

Noisy lossy source coding: proofs

D.1 Proof of the converse part of Theorem 4.6

The proof consists of an asymptotic analysis of the bound in Theorem 4.1 with a careful choice of tunable parameters.

The following auxiliary result will be instrumental.

Lemma D.1. *Let X_1, \dots, X_k be independent on \mathcal{A} and distributed according to P_X . For all k , it holds that*

$$\mathbb{P} \left[|\text{type}(X^k) - P_X|^2 > \frac{\log k}{k} \right] \leq \frac{|\mathcal{A}|}{\sqrt{k}} \quad (\text{D.1})$$

Proof. By Hoeffding's inequality, similar to Yu and Speed [132, (2.10)]. □

Let $P_{Z|X}: \mathcal{A} \mapsto \hat{\mathcal{S}}$ be a stochastic matrix whose entries are multiples of $\frac{1}{k}$. We say that the conditional type of z^k given x^k is equal to $P_{Z|X}$, $\text{type}(z^k|x^k) = P_{Z|X}$, if the number of a 's in x^k that are mapped to b in z^k is equal to the number of a 's in x^k times $P_{Z|X}(b|a)$, for all $(a, b) \in \mathcal{A} \times \hat{\mathcal{S}}$.

Let

$$\log M = kR(d) + \sqrt{k\tilde{\mathcal{V}}(d)Q^{-1}(\epsilon_k)} - \frac{1}{2} \log k - \log |\mathcal{P}_{[k]}| \quad (\text{D.2})$$

where $\epsilon_k = \epsilon + o(1)$ will be specified in the sequel, and $\mathcal{P}_{[k]}$ denotes the set of all conditional k -types $\hat{\mathcal{S}} \rightarrow \mathcal{A}$.

We weaken the bound in (4.20) by choosing

$$P_{\bar{X}^k|\bar{Z}^k=z^k}(x^k) = \frac{1}{|\mathcal{P}_{[k]}|} \sum_{P_{X|Z} \in \mathcal{P}_{[k]}} \prod_{i=1}^k P_{X|Z=z_i}(x_i) \quad (\text{D.3})$$

$$\lambda = k\lambda(x^k) = k\mathbb{R}'_{\text{type}(x^k)}(d) \quad (\text{D.4})$$

$$\gamma = \frac{1}{2} \log k \quad (\text{D.5})$$

By virtue of Theorem 4.1, the excess distortion probability of all (M, d, ϵ) codes where M is that in (D.2) must satisfy

$$\epsilon \geq \mathbb{E} \left[\min_{z^k \in \mathcal{S}^k} \mathbb{P} [\iota_{\bar{X}^k|\bar{Z}^k\|X^k}(X^k; z^k) + k\lambda(X^k)(d(S^k, z^k) - d) \geq \log M + \gamma|X^k] \right] - \exp(-\gamma) \quad (\text{D.6})$$

We identify the typical set of channel outputs:

$$\mathcal{T}_k = \left\{ x^k \in \mathcal{A}^k : |\text{type}(x^k) - P_X|^2 \leq \frac{\log k}{k} \right\} \quad (\text{D.7})$$

where $|\cdot|$ is the Euclidean norm.

We proceed to evaluate the minimum in in (D.6) for $x^k \in \mathcal{T}_k$.

For a given pair (x^k, z^k) , abbreviate

$$\text{type}(x^k) = P_{\bar{X}} \quad (\text{D.8})$$

$$\text{type}(z^k|x^k) = P_{\bar{Z}|\bar{X}} \quad (\text{D.9})$$

$$\lambda(x^k) = \lambda_{\bar{X}} \quad (\text{D.10})$$

We define $P_{\bar{X}|\bar{Z}}$ through $P_{\bar{X}}P_{\bar{Z}|\bar{X}}$ and lower-bound the sum in (D.3) by the term containing $P_{\bar{X}|\bar{Z}}$, concluding that

$$\begin{aligned} \iota_{\bar{X}^k|\bar{Z}^k\|X^k}(x^k; z^k) + \lambda(x^k)(d(S^k, z^k) - d) &\geq kI(\bar{X}; \bar{Z}) + kD(\bar{X}\|X) \\ &\quad + \lambda_{\bar{X}} \left(\sum_{i=1}^k \bar{d}_Z(S_i|x_i) - kd \right) - \log |\mathcal{P}_{[k]}| \end{aligned} \quad (\text{D.11})$$

$$= \sum_{i=1}^k W_i + kD(\bar{X}\|X) - \log |\mathcal{P}_{[k]}| \quad (\text{D.12})$$

where

$$W_i = I(\bar{\mathbf{X}}; \bar{\mathbf{Z}}) + \lambda_{\bar{\mathbf{X}}} (\bar{\mathbf{d}}_{\bar{\mathbf{Z}}}(S_i|x_i) - d) \quad (\text{D.13})$$

where $P_{\mathbf{S}\bar{\mathbf{X}}\bar{\mathbf{Z}}}(s, a, b) = P_{\bar{\mathbf{X}}}(a)P_{\mathbf{S}|\bar{\mathbf{X}}}(s|a)P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}(b|a)$.

Conditioned on $X^k = x^k$, the random variables W_i are independent with (in the notation of Theorem A.6 where $z = P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}$)

$$\mu_k(P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}) = I(\bar{\mathbf{X}}; \bar{\mathbf{Z}}) + \lambda_{\bar{\mathbf{X}}} (\mathbb{E} [\bar{\mathbf{d}}_{\bar{\mathbf{Z}}}(\mathbf{S}|\bar{\mathbf{X}})] - d) \quad (\text{D.14})$$

$$V_k(P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}) = \lambda_{\bar{\mathbf{X}}}^2 \mathbb{E} [\text{Var} [\bar{\mathbf{d}}_{\bar{\mathbf{Z}}}(\mathbf{S}|\bar{\mathbf{X}})|\bar{\mathbf{X}}]] \quad (\text{D.15})$$

$$T_k(P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}) = \lambda_{\bar{\mathbf{X}}}^3 \mathbb{E} [\left| \bar{\mathbf{d}}_{\bar{\mathbf{Z}}}(\mathbf{S}|\bar{\mathbf{X}}) - \mathbb{E} [\bar{\mathbf{d}}_{\bar{\mathbf{Z}}}(\mathbf{S}|\bar{\mathbf{X}})|\bar{\mathbf{X}}] \right|^3] \quad (\text{D.16})$$

Denote the backward conditional distribution that achieves $\mathbb{R}_{\bar{\mathbf{X}}}(d)$ by $P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}^*}$. Write

$$\mu_k(P_{\bar{\mathbf{Z}}|\bar{\mathbf{X}}}) = I(\bar{\mathbf{X}}; \bar{\mathbf{Z}}) + \lambda_{\bar{\mathbf{X}}} (\mathbb{E} [\bar{\mathbf{d}}(\bar{\mathbf{X}}, \bar{\mathbf{Z}})] - d) \quad (\text{D.17})$$

$$= \mathbb{E} [\iota_{\bar{\mathbf{X}}, \bar{\mathbf{Z}}^*}(\bar{\mathbf{X}}; \bar{\mathbf{Z}}) + \lambda_{\bar{\mathbf{X}}} \bar{\mathbf{d}}(\bar{\mathbf{X}}, \bar{\mathbf{Z}})] - \lambda_{\bar{\mathbf{X}}} d + D(P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}} \| P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}^*} | P_{\bar{\mathbf{Z}}}) \quad (\text{D.18})$$

$$\geq \mathbb{R}_{\bar{\mathbf{X}}}(d) + D(P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}} \| P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}^*} | P_{\bar{\mathbf{Z}}}) \quad (\text{D.19})$$

$$\geq \mathbb{R}_{\bar{\mathbf{X}}}(d) + \frac{1}{2} \left| P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}} P_{\bar{\mathbf{Z}}} - P_{\bar{\mathbf{X}}|\bar{\mathbf{Z}}^*} P_{\bar{\mathbf{Z}}} \right|^2 \log e \quad (\text{D.20})$$

where (D.19) is by Theorem 2.1, and (D.20) is by Pinsker's inequality (given in (A.6)). Similar to the proof of (C.50), we conclude that the conditions of Theorem A.6.4 are satisfied.

Denote

$$a_k = \log M + \frac{1}{2} \log k + \log |\mathcal{P}_{[k]}| - kD(\bar{\mathbf{X}}\|\mathbf{X}) - k\mathbb{R}_{\bar{\mathbf{X}}}(d) \quad (\text{D.21})$$

$$b_k = \log M + \frac{1}{2} \log k + \log |\mathcal{P}_{[k]}| - kD(\bar{\mathbf{X}}\|\mathbf{X}) - k\mathbb{R}_{\mathbf{X}}(d) - c \log k \quad (\text{D.22})$$

$$W_i^* = j_{\mathbf{X}}(X_i, d) - \mathbb{R}_{\mathbf{X}}(d) + \lambda_{\mathbf{X}} \bar{\mathbf{d}}_{\mathbf{Z}^*}(S_i|X_i) - \lambda_{\mathbf{X}} \mathbb{E} [\bar{\mathbf{d}}_{\mathbf{Z}^*}(S_i|X_i) | X_i] \quad (\text{D.23})$$

where M is that in (D.2), and constant $c > 0$ will be identified later. Weakening (D.6) further, we

have

$$\epsilon \geq \mathbb{E} \left[\min_{P_{Z|\bar{X}}} \mathbb{P} \left[\sum_{i=1}^k W_i \geq k\mathbb{R}_{\bar{X}}(d) + a_k |\text{type}(X^k) = P_{\bar{X}}| \mathbf{1} \{X^k \in \mathcal{T}_k\} \right] - \frac{1}{\sqrt{k}} \right] \quad (\text{D.24})$$

$$\geq \mathbb{E} \left[\mathbb{P} \left[\lambda_{\bar{X}} \left(\sum_{i=1}^k \bar{d}_{\bar{Z}^*}(S_i|X_i) - kd \right) \geq a_k |\text{type}(X^k) = P_{\bar{X}}| \mathbf{1} \{X^k \in \mathcal{T}_k\} \right] - \frac{K+1}{\sqrt{k}} \right] \quad (\text{D.25})$$

$$\geq \mathbb{E} \left[\mathbb{P} \left[\lambda_{\bar{X}} \left(\sum_{i=1}^k \bar{d}_{\bar{Z}^*}(S_i|X_i) - k\mathbb{E}[\bar{d}_{\bar{Z}^*}(\mathbf{S}|\bar{X})] \right) \geq a_k |\text{type}(X^k) = P_{\bar{X}}| \mathbf{1} \{X^k \in \mathcal{T}_k\} \right] - \frac{K_1 \log k + K + 1}{\sqrt{k}} \right] \quad (\text{D.26})$$

$$\geq \mathbb{E} \left[\mathbb{P} \left[\sum_{i=1}^k W_i^* \geq b_k |\text{type}(X^k) = P_{\bar{X}}| \mathbf{1} \{X^k \in \mathcal{T}_k\} \right] - \frac{K_1 \log k + K + 2B + 1}{\sqrt{k}} \right] \quad (\text{D.27})$$

$$\geq \mathbb{P} \left[\sum_{i=1}^k W_i^* \geq b_k \right] - \mathbb{P}[X^k \notin \mathcal{T}_k] - \frac{K_1 \log k + K + 2B + 1}{\sqrt{k}} \quad (\text{D.28})$$

$$\geq \mathbb{P} \left[\sum_{i=1}^k W_i^* \geq b_k \right] - \frac{K_1 \log k + K + 2B + |\mathcal{A}| + 1}{\sqrt{k}} \quad (\text{D.29})$$

$$\geq \epsilon_k - \frac{K_1 \log k + K + 2B + B^* + |\mathcal{A}| + 1}{\sqrt{k}} \quad (\text{D.30})$$

where

- (D.25) is by Theorem A.6.4, and $K > 0$ is defined therein.
- To show (D.26), which holds for some $K_1 > 0$, observe that since

$$\mathbb{E}[\bar{d}_{\bar{Z}^*}(\mathbf{S}|\bar{X})] = \mathbb{E}[\bar{d}(\bar{X}, \bar{Z}^*)] \quad (\text{D.31})$$

$$= d \quad (\text{D.32})$$

conditioned on x^k , both random variables $\lambda_{\bar{X}} \left(\sum_{i=1}^k \bar{d}_{\bar{Z}^*}(S_i|x_i) - kd \right)$

and $\lambda_{\bar{X}} \left(\sum_{i=1}^k \bar{d}_{\bar{Z}^*}(S_i|x_i) - k\mathbb{E}[\bar{d}_{\bar{Z}^*}(\mathbf{S}|\bar{X})] \right)$ are zero mean. By the Berry-Essén theorem and

the assumption that all alphabets are finite, there exists $B > 0$ such that

$$\begin{aligned} & \mathbb{P} \left[\lambda_{\mathbf{X}} \left(\sum_{i=1}^k \bar{d}_{\bar{\mathbf{Z}}^*}(S_i|X_i) - kd \right) \geq a_k | \text{type}(X^k) = P_{\bar{\mathbf{X}}} \right] \\ & \geq Q \left(\frac{a_k}{\lambda_{\bar{\mathbf{X}}} \sqrt{k \text{Var}[\bar{d}_{\bar{\mathbf{Z}}^*}(\mathbf{S}|\bar{\mathbf{X}})|\bar{\mathbf{X}}]}} \right) - \frac{B}{\sqrt{k}} \end{aligned} \quad (\text{D.33})$$

$$\geq Q \left(\frac{a_k}{\lambda_{\mathbf{X}} \sqrt{k \text{Var}[\bar{d}_{\mathbf{Z}^*}(\mathbf{S}|\mathbf{X})|\mathbf{X}]}} \left(1 + a \sqrt{\frac{\log k}{k}} \right) \right) - \frac{B}{\sqrt{k}} \quad (\text{D.34})$$

$$\geq Q \left(\frac{a_k}{\lambda_{\mathbf{X}} \sqrt{k \text{Var}[\bar{d}_{\mathbf{Z}^*}(\mathbf{S}|\mathbf{X})|\mathbf{X}]}} \right) - \frac{B}{\sqrt{k}} - K_1 \frac{\log k}{\sqrt{k}} \quad (\text{D.35})$$

$$\begin{aligned} & \geq \mathbb{P} \left[\lambda_{\mathbf{X}} \left(\sum_{i=1}^k \bar{d}_{\mathbf{Z}^*}(S_i|X_i) - k \mathbb{E}[\bar{d}_{\mathbf{Z}^*}(\mathbf{S}|\bar{\mathbf{X}})] \right) \geq a_k | \text{type}(X^k) = P_{\bar{\mathbf{X}}} \right] \\ & - \frac{2B}{\sqrt{k}} - K_1 \frac{\log k}{\sqrt{k}} \end{aligned} \quad (\text{D.36})$$

where (D.34) for some scalar a is obtained by applying a Taylor series expansion to $\frac{1}{\sqrt{\text{Var}[\bar{d}_{\bar{\mathbf{Z}}^*}(\mathbf{S}|\bar{\mathbf{X}})|\bar{\mathbf{X}}]}}$ in the neighborhood of typical $P_{\bar{\mathbf{X}}}$, i.e. those types corresponding to $x^k \in \mathcal{T}_k$, and (D.35) invokes (A.32) with $\xi \sim \frac{\log k}{\sqrt{k}}$ because $a_k = O(\sqrt{k \log k})$ for typical $P_{\bar{\mathbf{X}}}$.

- (D.27) holds because due to Taylor's theorem, there exists $c > 0$ such that

$$\mathbb{R}_{\bar{\mathbf{X}}}(d) \geq \mathbb{R}_{\mathbf{X}}(d) + \sum_{a \in \mathcal{A}} (P_{\bar{\mathbf{X}}}(a) - P_{\mathbf{X}}(a)) \dot{\mathbb{R}}_{\mathbf{X}}(a) - c |P_{\bar{\mathbf{X}}} - P_{\mathbf{X}}|^2 \quad (\text{D.37})$$

$$= \mathbb{R}_{\mathbf{X}}(d) + \frac{1}{k} \sum_{i=1}^k \dot{\mathbb{R}}_{\mathbf{X}}(X_i) - \mathbb{E}[\dot{\mathbb{R}}_{\mathbf{X}}(\mathbf{X})] - c |P_{\bar{\mathbf{X}}} - P_{\mathbf{X}}|^2 \quad (\text{D.38})$$

$$= \frac{1}{k} \sum_{i=1}^k j_{\mathbf{X}}(X_i, d) - c |P_{\bar{\mathbf{X}}} - P_{\mathbf{X}}|^2 \quad (\text{D.39})$$

$$\geq \frac{1}{k} \sum_{i=1}^k j_{\mathbf{X}}(X_i, d) - c \log k \quad (\text{D.40})$$

where (D.39) uses (2.16), and (D.40) is by the definition (D.7) of the typical set of x^k 's.

- (D.29) is by Lemma D.1.
- (D.30) applies the Berry-Esséen theorem to the sequence of i.i.d. random variables W_i^* whose Berry-Esséen ratio is denoted by B^* .

The result now follows by letting

$$\epsilon_k = \epsilon + \frac{K + 2B + B^* + |\mathcal{A}| + 1 + K_1 \log k}{\sqrt{k}} \quad (\text{D.41})$$

in (D.2).

D.2 Proof of the achievability part of Theorem 4.6

The proof consists of an asymptotic analysis of the bound in Theorem 4.5 with a careful choice of auxiliary parameters so that only the first term in (4.49) survives.

Let $P_{\bar{Z}^k} = P_{Z^{k*}} = P_{Z^*} \times \dots \times P_{Z^*}$, where Z^* achieves $\mathbb{R}_X(d)$, and let $P_{\bar{X}} = P_{\bar{X}^k} = P_{\bar{X}} \times \dots \times P_{\bar{X}}$, where $P_{\bar{X}}$ is the measure on \mathcal{X} generated by the empirical distribution of $x^k \in \mathcal{X}^k$:

$$P_{\bar{X}}(a) = \frac{1}{k} \sum_{i=1}^k 1\{x_i = a\} \quad (\text{D.42})$$

We let $\mathcal{T} = \mathcal{T}_k$ in (D.7) so that by Lemma D.1

$$P_{X^k}(\mathcal{T}_k^c) \leq \frac{|\mathcal{A}|}{\sqrt{k}} \quad (\text{D.43})$$

so we will concern ourselves only with typical x^k .

Let $P_{Z^*|\bar{X}}$ be the transition probability kernel that achieves $\mathbb{R}_{\bar{X}, Z^*}(d)$, and let $P_{Z^{*k}|X^k} = P_{Z^*|\bar{X}} \times \dots \times P_{Z^*|\bar{X}}$. Let $P_{Z^k|X^k}$ be uniform on the conditional type which is closest to (in terms of Euclidean distance)

$$P_{Z|X=x}(z) = \frac{P_{Z^*}(z) \exp(-\lambda \bar{d}(x, z))}{\mathbb{E}[\exp(-\lambda \bar{d}(x, Z^*))]} \quad (\text{D.44})$$

(cf. (4.16)) where

$$\lambda = \lambda_{\bar{X}} = -\mathbb{R}'_{\bar{X}, Z^*}(d - \xi) \quad (\text{D.45})$$

$$\xi = \sqrt{\frac{a \log k}{k}} \quad (\text{D.46})$$

for some $0 < a < 1$, so that (4.48) holds, and

$$\mathbb{E}[\bar{d}(\bar{X}, Z)] = d - \xi \quad (\text{D.47})$$

where $P_{\bar{X}} \rightarrow P_{Z|X} \rightarrow P_Z$.

It follows by the Berry-Esséen Theorem that

$$\mathbb{P} \left[\sum_{i=1}^k \bar{d}_Z(S_i|X_i = x_i) > kd|X^k = x^k \right] \leq \frac{1}{\sqrt{2\pi a k^a \log k}} + \frac{B}{\sqrt{k}} \quad (\text{D.48})$$

where B is the maximum (over $x^k \in \mathcal{T}_k$) of the Berry-Esséen ratios for $\bar{d}_Z(S_i|X_i = x_i)$. Also by the Berry-Esséen theorem, we have

$$\mathbb{P} \left[kd - \tau \leq \sum_{i=1}^k \bar{d}_Z(S_i|X_i = x_i) \leq kd|X^k = x^k \right] \geq \frac{b}{\sqrt{k}} \quad (\text{D.49})$$

for some $b > 0$, k large enough and $\tau = (2B + b)\sqrt{2\pi k^a}$.

We now proceed to evaluate the first term in (4.49).

$$D(P_{Z^k \| X^k = x^k} \| P_{Z^*} \times \dots \times P_{Z^*}) \leq kD(Z \| Z^*) + kH(Z) - kH(Z|\bar{X}) + |\mathcal{A}||\hat{\mathcal{S}}| \log(k+1) \quad (\text{D.50})$$

$$= kD(P_{Z|X} \| P_{Z^*} | P_{\bar{X}}) + |\mathcal{A}||\hat{\mathcal{S}}| \log(k+1) \quad (\text{D.51})$$

where to obtain (D.60) we used the type counting lemma [46, Lemma 2.6]. Therefore

$$g(s^k, x^k) \triangleq D(P_{Z^k \| X^k = x^k} \| P_{Z^*} \times \dots \times P_{Z^*}) + k\lambda_{\bar{X}} \bar{d}_{Z^k}(s^k|x^k) - k\lambda_{\bar{X}} d \quad (\text{D.52})$$

$$= kD(P_{Z|X} \| P_{Z^*} | P_{\bar{X}}) + \lambda_{\bar{X}} \sum_{i=1}^k \mathbb{E} [\bar{d}_Z(s_i|\bar{X})] - k\lambda_{\bar{X}} d + |\mathcal{A}||\hat{\mathcal{S}}| \log(k+1) \quad (\text{D.53})$$

$$= \mathbb{E} [J_{Z^*}(\bar{X}, \lambda_{\bar{X}})] - k\lambda_{\bar{X}} d + k\lambda_{\bar{X}} \sum_{i=1}^k \mathbb{E} [\bar{d}_Z(s_i|\bar{X})] - \lambda_{\bar{X}} \mathbb{E} [\bar{d}_Z(S|\bar{X})] + |\mathcal{A}||\hat{\mathcal{S}}| \log(k+1) \quad (\text{D.54})$$

$$\begin{aligned} &\leq k\mathbb{E} [J_{Z^*}(\bar{X}, \lambda_{\bar{X}, Z^*}^*)] - k\lambda_{\bar{X}, Z^*}^* d + \lambda_{\bar{X}} \sum_{i=1}^k \mathbb{E} [\bar{d}_Z(s_i|\bar{X})] - \lambda_{\bar{X}} \mathbb{E} [\bar{d}_Z(S|\bar{X})] \\ &\quad + |\mathcal{A}||\hat{\mathcal{S}}| \log(k+1) + L \log k \end{aligned} \quad (\text{D.55})$$

where to show (D.55) recall that by the assumption $\mathbb{R}_{\bar{X}, Z^*}(d)$ is twice continuously differentiable, so there exists $a > 0$ such that

$$\lambda - \lambda_{\bar{X}, Z^*}^* = \mathbb{R}'_{\bar{X}, Z^*}(d - \xi) - \mathbb{R}'_{\bar{X}, Z^*}(d) \quad (\text{D.56})$$

$$\leq a\xi \quad (\text{D.57})$$

Since $\lambda_{\bar{X}, Z^*}^*$ is a maximizer of $\mathbb{E} [J_{Z^*}(\bar{X}, \lambda)] - \lambda d$ (see (B.58) in Appendix B.5.A)

$$\frac{\partial}{\partial \lambda} \mathbb{E} [J_{Z^*}(\bar{X}, \lambda)] |_{\lambda=\lambda_{\bar{X}, Z^*}^*} = d \quad (\text{D.58})$$

the first term in the Taylor series expansion of $\mathbb{E} [J_{Z^*}(\bar{X}, \lambda)] - \lambda d$ in the vicinity of $\lambda_{\bar{X}, Z^*}^*$ vanishes, and we conclude that there exists L such that

$$\mathbb{E} [J_{Z^*}(\bar{X}, \lambda)] - \lambda d \geq \mathbb{E} [J_{Z^*}(\bar{X}, \lambda_{\bar{X}, Z^*}^*)] - \lambda_{\bar{X}, Z^*}^* d - L\xi^2 \quad (\text{D.59})$$

Moreover, according to Lemma B.4, there exist $C_2, K_2 > 0$ such that

$$\mathbb{P} \left[\sum_{i=1}^k (J_{Z^*}(X_i, \lambda_{\bar{X}, Z^*}) - \lambda_{\bar{X}, Z^*} d) \leq \sum_{i=1}^k J_X(X_i, d) + C_2 \log k \right] > 1 - \frac{K_2}{\sqrt{k}} \quad (\text{D.60})$$

The cdf of the sum of the zero-mean random variables $\lambda_{\bar{X}} (\bar{d}_Z(S_i|X_i = x_i) - \mathbb{E} [\bar{d}_Z(S|X_i)|X_i = x_i])$ is bounded for each $x^k \in \mathcal{T}_k$ as in the proof of (D.26), leading to the conclusion that there exists $K_1 > 0$ such that for k large enough

$$\begin{aligned} & \mathbb{P} \left[\sum_{i=1}^k J_{Z^*}(X_i, \lambda_{\bar{X}, Z}) - k\lambda_{\bar{X}, Z} d + \lambda_{\bar{X}} \sum_{i=1}^k (\bar{d}_Z(S_i|X_i) - \mathbb{E} [\bar{d}_Z(S|X_i)|X_i]) \leq \sum_{i=1}^k \tilde{J}_{S, X}(S_i, X_i, d) + C_2 \log k \right] \\ & > 1 - \frac{K_1 \log k + K_2 + |\mathcal{A}|}{\sqrt{k}} \end{aligned} \quad (\text{D.61})$$

It follows using (D.55) and (D.61) that

$$\mathbb{P} [g(S^k, X^k) > \log \gamma - \log \beta - k\lambda_0 \delta] \leq \mathbb{P} \left[\sum_{i=1}^k \tilde{J}_{S, X}(S_i, X_i, d) > \log \gamma - \Delta_k \right] + \frac{K_1 \log k + K_2 + |\mathcal{A}|}{\sqrt{k}} \quad (\text{D.62})$$

where $\lambda_0 = \max_{x^k \in \mathcal{T}_k} \lambda_{\bar{X}, Z^*}$ and

$$\Delta_k = \log \beta + k\lambda_0 \delta + |\mathcal{A}| |\hat{\mathcal{S}}| \log(k+1) + L \log k + C_2 \log k \quad (\text{D.63})$$

We now weaken the bound in Theorem 4.5 by choosing

$$\beta = \frac{\sqrt{k}}{b} \quad (\text{D.64})$$

$$\delta = \frac{\tau}{k} \quad (\text{D.65})$$

$$\log \gamma = \log M - \log \log_e k + \log 2 \quad (\text{D.66})$$

where b is that in (D.49) and $\tau > 0$ is that in (D.49). Letting

$$\log M = kR(d) + \sqrt{k\tilde{\mathcal{V}}(d)Q^{-1}(\epsilon_k)} + \Delta_k \quad (\text{D.67})$$

$$\epsilon_k = \epsilon - \frac{K_1 \log k + K_2 + B + \tilde{B} + |\mathcal{A}| + 1}{\sqrt{k}} \quad (\text{D.68})$$

where \tilde{B} is the Berry-Esséen ratio for $\tilde{j}_{S,X}(S_i, X_i, d)$, and applying (D.48), (D.49) and (D.62), we conclude using Theorem 4.5 that there exists an (M, d, ϵ') code with M in (D.67) satisfying

$$\begin{aligned} \epsilon' &\leq \mathbb{P} \left[\sum_{i=1}^k \tilde{j}_{S,X}(S_i, X_i, d) > kR(d) + \sqrt{k\tilde{\mathcal{V}}(d)Q^{-1}(\epsilon_k)} \right] \\ &\quad + \frac{K_1 \log k + K_2 + B + \tilde{B} + |\mathcal{A}| + 1}{\sqrt{k}} \end{aligned} \quad (\text{D.69})$$

$$\leq \epsilon \quad (\text{D.70})$$

where (D.70) is by the Berry-Esséen bound.

D.3 Proof of Theorem 4.9

Converse. The proof of the converse part follows the Gaussian approximation analysis of the converse bound in Theorem 4.7. Let $i = k\frac{\delta}{2} + k\Delta_1$ and $j = k\delta - k\Delta_2$. Using Stirling's approximation for the binomial sum (B.147), after applying a Taylor series expansion we have

$$2^{-(k-j)} \left\langle \begin{matrix} k-j \\ \lfloor nd-i \rfloor \end{matrix} \right\rangle = \frac{C(\Delta)}{\sqrt{k}} \exp \{-k g(\Delta_1, \Delta_2)\} \quad (\text{D.71})$$

where $C(\Delta)$ is such that there exist positive constants \underline{C} , \bar{C} , ξ such that $\underline{C} \leq C(\Delta) \leq \bar{C}$ for all $|\Delta| \leq \xi$, and the twice differentiable function $g(\Delta_1, \Delta_2)$ can be written as

$$g(\Delta_1, \Delta_2) = R(d) + a_1\Delta_1 + a_2\Delta_2 + O(|\Delta|^2) \quad (\text{D.72})$$

$$a_1 = \log \frac{1 - d - \frac{\delta}{2}}{d - \frac{\delta}{2}} = \lambda^* \quad (\text{D.73})$$

$$a_2 = \log \frac{2(1 - d - \frac{\delta}{2})}{1 - \delta} = \log \frac{2}{1 + \exp(-\lambda^*)} \quad (\text{D.74})$$

It follows from (D.72) that $g(\Delta_1, \Delta_2)$ is increasing in $a_1\Delta_1 + a_2\Delta_2 \in (-\underline{b}, \bar{b})$ for some constants $\underline{b}, \bar{b} > 0$ (obviously, we can choose \underline{b}, \bar{b} small enough in order for $\underline{C} \leq C(\Delta) \leq \bar{C}$ to hold). In the sequel, we will represent the probabilities in the right side of (4.80) via a sequence of i.i.d. random variables W_1, \dots, W_k with common distribution

$$W = \begin{cases} a_1 & \text{w.p. } \frac{\delta}{2} \\ a_2 & \text{w.p. } 1 - \delta \\ 0 & \text{otherwise} \end{cases} \quad (\text{D.75})$$

Note that

$$\mathbb{E}[W] = \frac{a_1\delta}{2} + a_2(1 - \delta) \quad (\text{D.76})$$

$$\text{Var}[W] = \delta(1 - \delta) \left(a_2 - \frac{a_1}{2} \right)^2 + \frac{\delta a_1^2}{4} = V(d) \quad (\text{D.77})$$

and the third central moment of W is finite, so that B_k in (2.159) is a finite constant. Let

$$b_k = \sqrt{\frac{V(d)}{k}} Q^{-1}(\epsilon_k) \quad (\text{D.78})$$

$$\epsilon_k = \left(1 - \frac{\bar{C}}{\sqrt{k}} \right)^{-1} \epsilon + \frac{2B_k}{\sqrt{k}} + \sqrt{\frac{V(d)}{2\pi k}} \frac{1}{b} e^{-k \frac{\epsilon^2}{2V(d)}} \quad (\text{D.79})$$

$$R = \min_{\substack{\Delta_1, \Delta_2: \\ b_k \leq a_1\Delta_1 + a_2\Delta_2 \leq \bar{b}}} g(\Delta_1, \Delta_2) \quad (\text{D.80})$$

$$= R(d) + b_k + O(b_k^2) \quad (\text{D.81})$$

With $M = \exp(nR)$, since $R \leq g(\Delta_1, \Delta_2)$ for all $a_1\Delta_1 + a_2\Delta_2 \in [b_k, \bar{b}]$, for such (Δ_1, Δ_2) it holds that

$$\left[1 - \frac{\bar{C}}{\sqrt{k}} M \exp\{-k g(\Delta_1, \Delta_2)\} \right]^+ \geq 1 - \frac{\bar{C}}{\sqrt{k}} \quad (\text{D.82})$$

Denoting the random variables

$$N(x) = \frac{1}{k} \sum_{i=1}^k 1\{W_i = x\} \quad (\text{D.83})$$

$$G_k = k g\left(N(a_1) - \frac{\delta}{2}, N(a_2) - 1 + \delta\right) \quad (\text{D.84})$$

and using (D.71) to express the probability in the right side of (4.80) in terms of W_1, \dots, W_k , we conclude that the excess-distortion probability is lower bounded by

$$\mathbb{E} \left[\left(1 - \frac{\bar{C}}{\sqrt{k}} \exp \{ \log M - G_k \} \right)^+ \right] \geq \left(1 - \frac{\bar{C}}{\sqrt{k}} \right) \mathbb{P} \left[b_k \leq \sum_{i=1}^k W_i - k \mathbb{E}[W] < \bar{b} \right] \quad (\text{D.85})$$

$$\geq \left(1 - \frac{\bar{C}}{\sqrt{k}} \right) \left(\epsilon_k - \frac{2B_k}{\sqrt{k}} - \sqrt{\frac{V(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2V(d)}} \right) \quad (\text{D.86})$$

$$= \epsilon \quad (\text{D.87})$$

where (D.85) follows from (D.82), and (D.86) follows from the Berry-Esséen inequality (2.155) and (A.28), and (D.87) is equivalent to (D.79). \square

Achievability. We now proceed to the Gaussian approximation analysis of the achievability bound in Theorem 4.8. Let

$$b_k = \sqrt{\frac{V(d)}{k}} Q^{-1}(\epsilon_k) \quad (\text{D.88})$$

$$\epsilon_k = \epsilon - \frac{2B_k}{\sqrt{k}} - \sqrt{\frac{V(d)}{2\pi k}} \frac{1}{\bar{b}} e^{-k \frac{\bar{b}^2}{2V(d)}} - \frac{1}{\sqrt{k}} \quad (\text{D.89})$$

$$\log M = k \min_{\substack{\Delta_1, \Delta_2: \\ b_k \leq a_1 \Delta_1 + a_2 \Delta_2 \leq \bar{b}}} g(\Delta_1, \Delta_2) + \frac{1}{2} \log k + \log \left(\frac{\log_e k}{2\bar{C}} \right) \quad (\text{D.90})$$

$$= kR(d) + \sqrt{nV(d)} Q^{-1}(\epsilon) + \frac{1}{2} \log k + \log \log k + O(1) \quad (\text{D.91})$$

where $g(\Delta_1, \Delta_2)$ is defined in (D.71), and (D.91) follows from (D.72) and a Taylor series expansion of $Q^{-1}(\cdot)$. Using (D.71) and $(1-x)^M \leq e^{-Mx}$ to weaken the right side of (4.84) and expressing the resulting probability in terms of i.i.d. random variables W_1, \dots, W_k with common distribution (D.75), we conclude that the excess-distortion probability is upper bounded by (recall notation

(D.84))

$$\begin{aligned} \mathbb{E} \left[e^{-\frac{C}{\sqrt{k}} \exp\{\log M - G_k\}} \right] &\leq \mathbb{P} \left[\sum_{i=1}^k W_i \geq k\mathbb{E}[W] + kb_k \right] + \mathbb{P} \left[\sum_{i=1}^k W_i \leq k\mathbb{E}[W] - kb \right] \\ &\quad + \mathbb{E} \left[e^{-\frac{C}{\sqrt{k}} \exp\{\log M - G_k\}} \mathbf{1} \left\{ kb < \sum_{i=1}^k W_i - k\mathbb{E}[W] < kb_k \right\} \right] \end{aligned} \quad (\text{D.92})$$

$$\leq \epsilon_k + \frac{B_k}{\sqrt{k}} + \frac{B_k}{\sqrt{k}} + \sqrt{\frac{V(d)}{2\pi k}} \frac{1}{b} e^{-k \frac{b^2}{2V(d)}} + \frac{1}{\sqrt{k}} \quad (\text{D.93})$$

$$= \epsilon \quad (\text{D.94})$$

where the probabilities are upper bounded by the Berry-Esséen inequality (2.155) and (A.28), and the expectation is bounded using the fact that in $b < a_1\Delta_1 + a_2\Delta_2 < b_k$, the minimum difference between $\log M$ and $k g(\Delta_1, \Delta_2)$ is $\frac{1}{2} \log k + \log \left(\frac{\log_e k}{2C} \right)$. Finally, (D.94) is just (D.89). \square

Appendix E

Channels with cost constraints: proofs

E.1 Proof of Theorem 5.1

This proof is from [24]. Equality in (5.12) is a standard result in convex optimization (Lagrange duality). By the assumption, the supremum in the right side of (5.12) is attained by P_{X^*} , therefore $\mathbb{C}(\alpha)$ is equal to the right side of (5.14).

To show (5.13), fix $0 \leq \alpha \leq 1$. Denote

$$P_{\bar{X}} \rightarrow P_{Y|X} \rightarrow P_{\bar{Y}} \quad (\text{E.1})$$

$$P_{\hat{X}} = \alpha P_{\bar{X}} + (1 - \alpha) P_{X^*} \quad (\text{E.2})$$

$$P_{\hat{X}} \rightarrow P_{Y|X} \rightarrow P_{\hat{Y}} = \alpha P_{\bar{Y}} + (1 - \alpha) P_{Y^*} \quad (\text{E.3})$$

and write

$$\begin{aligned} & \alpha (\mathbb{E} [j_{X^*, Y^*}^*(X^*; Y^*, \beta)] - \mathbb{E} [j_{\bar{X}, \bar{Y}}^*(\bar{X}; \bar{Y}, \beta)]) + D(\hat{Y} \| Y^*) \\ &= \alpha D(P_{Y|X} \| P_{Y^*} | P_{X^*}) - \alpha D(P_{Y|X} \| P_{Y^*} | P_{\bar{X}}) + D(\hat{Y} \| Y^*) + \lambda^* \alpha \mathbb{E} [\mathbf{b}(\bar{X})] - \lambda^* \alpha \mathbb{E} [\mathbf{b}(X^*)] \end{aligned} \quad (\text{E.4})$$

$$= D(P_{Y|X} \| P_{Y^*} | P_{X^*}) + D(P_{Y|X} \| P_{\hat{Y}} | P_{\hat{X}}) - \lambda^* \mathbb{E} [\mathbf{b}(X^*)] + \lambda^* \mathbb{E} [\mathbf{b}(\hat{X})] \quad (\text{E.5})$$

$$= \mathbb{E} [j_{X^*, Y^*}^*(X^*; Y^*, \beta)] - \mathbb{E} [j_{\hat{X}, \hat{Y}}^*(\hat{X}; \hat{Y}, \beta)] \quad (\text{E.6})$$

$$\geq 0 \quad (\text{E.7})$$

where (E.7) holds because X^* achieves the supremum in the right side of (5.12). Since $D(\hat{Y} \| Y^*) < \infty$ (e.g. [24]), Lemma A.1 implies that $D(\hat{Y} \| Y^*) = o(\alpha)$. Thus, supposing that $\mathbb{E} [j_{X;Y}^*(\bar{X}; \bar{Y}, \beta)] > \mathbb{E} [j_{X;Y}^*(X^*; Y^*, \beta)]$ would lead to a contradiction, since then the left side of (E.4) would be negative for sufficiently small α . We thus infer that (5.13) holds.

To show (5.15), denote $P_{\bar{X}} \rightarrow P_{Y|X} \rightarrow P_{\bar{Y}}$ and define the following function of a pair of probability distributions on \mathcal{X} :

$$F(P_X, P_{\bar{X}}) = \mathbb{E} [j_{\bar{X};\bar{Y}}(X; Y, \beta)] - D(X \| \bar{X}) \quad (\text{E.8})$$

$$= \mathbb{E} [j_{X;Y}(X; Y, \beta)] - D(X \| \bar{X}) + D(Y \| \bar{Y}) \quad (\text{E.9})$$

$$\leq \mathbb{E} [j_{X;Y}(X; Y, \beta)] \quad (\text{E.10})$$

where (E.10) holds by the data processing inequality for relative entropy. Since equality in (E.10) is achieved by $P_X = P_{\bar{X}}$, $\mathbb{C}(\beta)$ can be expressed as the double maximization

$$\mathbb{C}(\beta) = \max_{P_{\bar{X}}} \max_{P_X} F(P_X, P_{\bar{X}}) \quad (\text{E.11})$$

To solve the inner maximization in (E.11), we invoke Lemma A.2 with

$$g(x) = \mathbb{E} [j_{\bar{X};\bar{Y}}(x; Y, \beta) | X = x] \quad (\text{E.12})$$

to conclude that

$$\max_{P_{\bar{X}}} F(P_X, P_{\bar{X}}) = \log \mathbb{E} [\exp (\mathbb{E} [j_{\bar{X};\bar{Y}}(\bar{X}; \bar{Y}, \beta) | \bar{X}])] \quad (\text{E.13})$$

which in the special case $P_{\bar{X}} = P_{X^*}$ yields, using representation (E.11),

$$\mathbb{C}(\beta) \geq \log \mathbb{E} [\exp (\mathbb{E} [j_{X;Y}^*(X^*; Y, \beta) | X^*])] \quad (\text{E.14})$$

$$\geq \mathbb{E} [j_{X;Y}^*(X^*; Y^*, \beta)] \quad (\text{E.15})$$

$$= \mathbb{C}(\beta) \quad (\text{E.16})$$

where (E.15) applies Jensen's inequality to the strictly convex function $\exp(\cdot)$, and (E.16) holds by the assumption. We conclude that, in fact, (E.15) holds with equality, which implies that $\mathbb{E} [j_{X;Y}^*(X^*; Y, \beta) | X^*]$ is almost surely constant, thereby showing (5.15).

E.2 Proof of Corollary 5.2

To show (5.17), we invoke (5.5) to write, for any $x \in \mathcal{X}$,

$$\text{Var} [j_{X;Y}^*(X; Y, \beta) | X = x] = \text{Var} [i_{X;Y}^*(X; Y) - \lambda^* (\mathbf{b}(X) - \beta) | X = x] \quad (\text{E.17})$$

$$= \text{Var} [i_{X;Y}^*(X; Y) | X = x] \quad (\text{E.18})$$

To show (5.16), we invoke (5.15) to write

$$\begin{aligned} \mathbb{E} [\text{Var} [j_{X;Y}^*(X^*; Y^*, \beta) | X^*]] &= \mathbb{E} \left[(j_{X;Y}^*(X^*; Y^*, \beta))^2 \right] \\ &\quad - \mathbb{E} \left[(\mathbb{E} [j_{X;Y}^*(X^*; Y^*, \beta) | X^*])^2 \right] \end{aligned} \quad (\text{E.19})$$

$$= \mathbb{E} \left[(j_{X;Y}^*(X^*; Y^*, \beta))^2 \right] - \mathbb{C}^2(\beta) \quad (\text{E.20})$$

$$= \text{Var} [j_{X;Y}^*(X^*; Y^*, \beta)] \quad (\text{E.21})$$

E.3 Proof of the converse part of Theorem 5.5

Given a finite set \mathcal{A} , let \mathcal{P} be the set of all distributions on \mathcal{A} that satisfy the cost constraint,

$$\mathbb{E} [\mathbf{b}(\mathbf{X})] \leq \beta \quad (\text{E.22})$$

which is a convex set in $\mathbb{R}^{|\mathcal{A}|}$. We say that $x^n \in \mathcal{A}^n$ has type P_X if the number of times each letter $a \in \mathcal{A}$ is encountered in x^n is $nP_X(a)$. An n -type is a distribution whose masses are multiples of $\frac{1}{n}$.

We will weaken (5.24) by choosing $P_{\bar{Y}^n}$ to be the following convex combination of non-product distributions (cf. [99]):

$$P_{\bar{Y}^n}(y^n) = \frac{1}{A} \sum_{\mathbf{k} \in \mathcal{K}} \exp(-|\mathbf{k}|^2) \prod_{i=1}^n P_{Y|\mathbf{K}=\mathbf{k}}(y_i) \quad (\text{E.23})$$

where $\{P_{Y|\mathbf{K}=\mathbf{k}}, \mathbf{k} \in \mathcal{K}\}$ are defined as follows, for some $c > 0$,

$$P_{Y|\mathbf{K}=\mathbf{k}}(y) = P_{Y^*}(y) + \frac{k_y}{\sqrt{nc}} \quad (\text{E.24})$$

$$\mathcal{K} = \left\{ \mathbf{k} \in \mathbb{Z}^{|\mathcal{B}|} : \sum_{y \in \mathcal{B}} k_y = 0, -P_{Y^*}(y) + \frac{1}{\sqrt{nc}} \leq \frac{k_y}{\sqrt{nc}} \leq 1 - P_{Y^*}(y) \right\} \quad (\text{E.25})$$

$$A = \sum_{\mathbf{k} \in \mathcal{K}} \exp(-|\mathbf{k}|^2) < \infty \quad (\text{E.26})$$

Denote by $P_{\Pi(Y)}$ the minimum Euclidean distance approximation of an arbitrary $P_Y \in \mathcal{Q}$, where \mathcal{Q} is the set of distributions on the channel output alphabet \mathcal{B} , in the set $\{P_{Y|\mathbf{K}=\mathbf{k}} : \mathbf{k} \in \mathcal{K}\}$:

$$P_{\Pi(Y)} = P_{Y|\mathbf{K}=\mathbf{k}^*} \text{ where } \mathbf{k}^* = \arg \min_{\mathbf{k} \in \mathcal{K}} |P_Y - P_{Y|\mathbf{K}=\mathbf{k}}| \quad (\text{E.27})$$

The quality of approximation (E.27) is governed by

$$|P_{\Pi(Y)} - P_Y| \leq \sqrt{\frac{|\mathcal{B}|(|\mathcal{B}| - 1)}{nc}} \quad (\text{E.28})$$

For an arbitrary $x^n \in \mathcal{A}^n$, let $\text{type}(x^n) = P_X \rightarrow P_{Y|X} \rightarrow P_Y$. Lower-bounding the sum in (E.23) by the term containing $P_{\Pi(Y)}$, we have

$$\mathcal{J}_{Y^n|X^n \parallel \bar{Y}^n}(x^n; y^n, \beta) \leq \sum_{i=1}^n \mathcal{J}_{Y|X \parallel \Pi(Y)}(x_i, y_i, \beta) + nc |P_{\Pi(Y)} - P_{Y^*}|^2 + A \quad (\text{E.29})$$

Applying (E.23) and (E.29) to loosen (5.24), we conclude by Theorem 5.3 that, as long as an (n, M, ϵ') code exists, for an arbitrary $\gamma > 0$,

$$\epsilon' \geq \min_{x^n \in \mathcal{A}^n} \mathbb{P} \left[\sum_{i=1}^n W_i \leq \log M - \gamma - A |Z = \text{type}(x^n)| \right] - \exp(-\gamma) \quad (\text{E.30})$$

where

$$W_i = \mathcal{J}_{Y|X \parallel \Pi(Y)}(x_i, Y_i, \beta) + c |P_{\Pi(Y)} - P_{Y^*}|^2 \quad (\text{E.31})$$

$$Z = \text{type}(X^n) \quad (\text{E.32})$$

where Y_i is distributed according to $P_{Y|X=x_i}$. To evaluate the minimum on the right side of (E.30), we will apply Theorem A.6 with W_i in (E.31).

Define the following functions $\mathcal{P} \times \mathcal{Q} \mapsto \mathbb{R}_+$:

$$\mu(P_X, P_Y) = \mathbb{E} \left[j_{Y|X||\bar{Y}}(X; Y, \beta) \right] + c |P_Y - P_{Y^*}|^2 \quad (\text{E.33})$$

$$V(P_X, P_Y) = \mathbb{E} \left[\text{Var} \left[j_{Y|X||\bar{Y}}(X; Y, \beta) \mid X \right] \right] \quad (\text{E.34})$$

$$T(P_X, P_Y) = \mathbb{E} \left[\left| j_{Y|X||\bar{Y}}(X; Y, \beta) - \mathbb{E} \left[j_{Y|X||\bar{Y}}(X; Y, \beta) \mid X \right] \right|^3 \right] \quad (\text{E.35})$$

where the expectations are with respect to $P_{Y|X}P_X$. Denote by $P_{\hat{X}}$ the minimum Euclidean distance approximation of P_X in the set of n -types, that is,

$$P_{\hat{X}} = \arg \min_{\substack{P \in \mathcal{P}: \\ P \text{ is an } n\text{-type}}} |P_X - P| \quad (\text{E.36})$$

The accuracy of approximation in (E.36) is controlled by the following inequality.

$$|P_X - P_{\hat{X}}| \leq \frac{\sqrt{|\mathcal{A}|(|\mathcal{A}| - 1)}}{n} \quad (\text{E.37})$$

With the choice in (E.31) and (E.32) the functions (A.40)–(A.42) are particularized to the following mappings $\mathcal{P} \mapsto \mathbb{R}_+$:

$$\mu_n(P_X) = \mu \left(P_{\hat{X}}, P_{\Pi(\hat{Y})} \right) \quad (\text{E.38})$$

$$V_n(P_X) = V \left(P_{\hat{X}}, P_{\Pi(\hat{Y})} \right) \quad (\text{E.39})$$

$$T_n(P_X) = T \left(P_{\hat{X}}, P_{\Pi(\hat{Y})} \right) \quad (\text{E.40})$$

where $P_{\hat{X}} \rightarrow P_{Y|X} \rightarrow P_{\hat{Y}}$, and μ_n^*, V_n^* are

$$\mu_n^* = C(\beta) \quad (\text{E.41})$$

$$V_n^* = V(\beta) \quad (\text{E.42})$$

We perform the minimization on the right side of (E.30) separately for $\text{type}(x^n) \in \mathcal{P}_\delta^*$ and $\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*$, where

$$\mathcal{P}_\delta^* = \{P_X \in \mathcal{P} : |P_X - P_{X^*}| \leq \delta\} \quad (\text{E.43})$$

Assuming without loss of generality that all outputs in \mathcal{B} are accessible (which implies that $P_{Y^*}(\mathbf{y}) >$

0 for all $y \in \mathcal{B}$), we choose $\delta > 0$ so that

$$\min_{P_X \in \mathcal{P}_\delta^*} \min_{y \in \mathcal{B}} P_Y(y) = p_{\min} > 0 \quad (\text{E.44})$$

$$2 \min_{P_X \in \mathcal{P}_\delta^*} V(P_X) \geq V(\beta) \quad (\text{E.45})$$

To perform the minimization on the right side of (E.30) over \mathcal{P}_δ^* , we will invoke Theorem A.6 with $\mathcal{D} = \mathcal{P}_\delta^*$, the metric being the usual Euclidean distance between $|\mathcal{A}|$ -vectors. Let us check that the assumptions of Theorem A.6 are satisfied.

It is easy to verify directly that the functions $P_X \mapsto \mu(P_X, P_Y)$, $P_X \mapsto V(P_X, P_Y)$, $P_X \mapsto T(P_X, P_Y)$ are continuous (and therefore bounded) on \mathcal{P} and infinitely differentiable on \mathcal{P}_δ^* . Therefore, assumptions (A.46) and (A.47) of Theorem A.6 are met.

To verify that (A.43) holds, write

$$C(\beta) - \mu(P_{\hat{X}}, P_{\Pi(\hat{Y})}) = C(\beta) - \mu(P_X, P_Y) + \mu(P_X, P_Y) - \mu(P_{\hat{X}}, P_{\Pi(\hat{Y})}) \quad (\text{E.46})$$

$$\geq \ell'_1 \zeta^2 - c|P_Y - P_{Y^*}|^2 + \mu(P_X, P_Y) - \mu(P_{\hat{X}}, P_{\Pi(\hat{Y})}) \quad (\text{E.47})$$

$$\geq \ell_1 \zeta^2 + \mu(P_X, P_Y) - \mu(P_{\hat{X}}, P_{\Pi(\hat{Y})}) \quad (\text{E.48})$$

$$\geq \ell_1 \zeta^2 + \mu(P_{\hat{X}}, P_{\hat{Y}}) - \mu(P_{\hat{X}}, P_{\Pi(\hat{Y})}) - \frac{\ell'_3}{n} \quad (\text{E.49})$$

$$= \ell_1 \zeta^2 - D(\hat{Y} \parallel \Pi(\hat{Y})) + c|P_{\hat{Y}} - P_{Y^*}|^2 - c|P_{\Pi(\hat{Y})} - P_{Y^*}|^2 - \frac{\ell'_3}{n} \quad (\text{E.50})$$

$$\geq \ell_1 \zeta^2 + c|P_{\hat{Y}} - P_{Y^*}|^2 - c|P_{\Pi(\hat{Y})} - P_{Y^*}|^2 - \frac{\ell''_3}{n} \quad (\text{E.51})$$

$$\geq \ell_1 \zeta^2 - c|P_{\hat{Y}} - P_{\Pi(\hat{Y})}|^2 - 2c|P_{\hat{Y}} - P_{\Pi(\hat{Y})}||P_{\hat{Y}} - P_{Y^*}| - \frac{\ell''_3}{n} \quad (\text{E.52})$$

$$\geq \ell_1 \zeta^2 - \frac{\ell_2}{\sqrt{n}} \zeta - \frac{\ell_3}{n} \quad (\text{E.53})$$

where all constants ℓ are positive, and:

- (E.47) uses

$$\mathbb{E}[J_{X;Y}(X; Y, \beta)] \leq C(\beta) - \ell'_1 \zeta^2 \quad (\text{E.54})$$

shown in the same way as (C.50) invoking invoking (5.15) in lieu of the corresponding property for the conventional information density (C.7).

- In (E.48), we denoted

$$\ell_1 = \ell'_1 - c|\mathcal{A}| \quad (\text{E.55})$$

which can be made positive by choosing a small enough c , and used (E.37) and

$$|P_Y - P_{\tilde{Y}}| \leq |P_{Y|X}| |P_X - P_{\tilde{X}}| \quad (\text{E.56})$$

where $P_{\tilde{X}} \rightarrow P_{Y|X} \rightarrow P_{\tilde{Y}}$, and the spectral norm of $P_{Y|X}$ satisfies $|P_{Y|X}| \leq \sqrt{|\mathcal{A}|}$.

- (E.49) holds due to (E.37) and continuous differentiability of $P_X \mapsto \mu(P_X, P_Y)$, as the latter implies

$$|\mu(P_X, P_Y) - \mu(P_{\tilde{X}}, P_{\tilde{Y}})| \leq L |P_X - P_{\tilde{X}}| \quad (\text{E.57})$$

where $P_{\tilde{X}} \rightarrow P_{Y|X} \rightarrow P_{\tilde{Y}}$.

- (E.50) is equivalent to

$$\mathbb{E}[J_{X;Y}(X; Y, \beta)] = \mathbb{E}[J_{Y|X|\bar{Y}}(X; Y, \beta)] - D(Y\|\bar{Y}) \quad (\text{E.58})$$

- (E.51) uses (E.28), (E.44) and (A.7).
- (E.53) applies (E.28) and (E.56).

To establish (A.44), write

$$C(\beta) - D(P_{\hat{X}}, P_{\Pi(\hat{Y})}) \leq C(\beta) - \mathbb{E}[J_{Y|X|\Pi(\hat{Y})}(\hat{X}; \hat{Y}, \beta)] \quad (\text{E.59})$$

$$= C(\beta) - \mathbb{E}[J_{\hat{X};\hat{Y}}(\hat{X}; \hat{Y}, \beta)] - D(\hat{Y}\|\Pi(\hat{Y})) \quad (\text{E.60})$$

$$\leq C(\beta) - \mathbb{E}[J_{X;Y}(X; Y, \beta)] + \frac{L_1}{n} - D(\hat{Y}\|\Pi(\hat{Y})) \quad (\text{E.61})$$

$$\leq C(\beta) - \mathbb{E}[J_{X;Y}(X; Y, \beta)] + \frac{L_1}{n} \quad (\text{E.62})$$

where

- (E.61) uses continuous differentiability of $P_X \mapsto \mathbb{E}[J_{X;Y}(X; Y, \beta)]$ and (E.37);
- (E.62) applies $D(\hat{Y}\|\Pi(\hat{Y})) \geq 0$.

Substituting $X = X^*$ into (E.62), we obtain (A.44).

Finally, to verify (A.45), write

$$\begin{aligned} & \left| V \left(P_{\hat{X}}, P_{\Pi(\hat{Y})} \right) - V(\beta) \right| \\ & \leq |V(P_X, P_Y) - V(\beta)| + |V(P_X, P_Y) - V(P_{\hat{X}}, P_{\hat{Y}})| + \left| V \left(P_{\hat{X}}, P_{\Pi(\hat{Y})} \right) - V(P_{\hat{X}}, P_{\hat{Y}}) \right| \end{aligned} \quad (\text{E.63})$$

$$\leq F_1 |P_X - P_{X^*}| + F_2' |P_X - P_{\hat{X}}| + F_2'' \left| P_{\Pi(\hat{Y})} - P_{\hat{Y}} \right| \quad (\text{E.64})$$

$$\leq F_1 \zeta + \frac{F_2}{\sqrt{n}} \quad (\text{E.65})$$

where

- (E.64) uses continuous differentiability of $P_X \mapsto V(P_X, P_Y)$ (in \mathcal{P}_δ^*) and $P_Y \mapsto V(P_X, P_Y)$ (at any P_Y with $P_Y(Y) > 0$ a.s.).
- (E.65) applies (E.37) and (E.28).

Theorem A.6 is thereby applicable.

If $V(\beta) > 0$, letting

$$\gamma = \frac{1}{2} \log n \quad (\text{E.66})$$

$$\log M = nC(\beta) - \sqrt{nV(\beta)} Q^{-1} \left(\epsilon + \frac{K+1}{\sqrt{n}} \right) + \frac{1}{2} \log n + A \quad (\text{E.67})$$

where constant K is the same as in (A.48), we apply Theorem A.6.1 to conclude that the right side of (E.30) with minimization constrained to types in \mathcal{P}_δ^* is lower bounded by ϵ :

$$\min_{\text{type}(x^n) \in \mathcal{P}_\delta^*} \mathbb{P} \left[\sum_{i=1}^n W_i \leq \log M - \gamma - A | Z = \text{type}(x^n) \right] - \exp(-\gamma) \geq \epsilon \quad (\text{E.68})$$

If $V(\beta) = 0$, we fix $0 < \eta < 1 - \epsilon$ and let

$$\gamma = \log \frac{1}{\eta} \quad (\text{E.69})$$

$$\log M = nC(\beta) + \left(\frac{K}{1 - \epsilon - \eta} \right)^{\frac{2}{3}} n^{\frac{1}{3}} + \log \frac{1}{\eta} + A \quad (\text{E.70})$$

where $K > 0$ is that in (A.53). Applying Theorem A.6.4 with $\beta = \frac{1}{6}$, we conclude that (E.68) holds for the choice of M in (E.70) if $V(\beta) = 0$.

To evaluate the minimum over $\mathcal{P} \setminus \mathcal{P}_\delta^\star$ on the right side of (E.30), define

$$C(\beta) - \max_{P_X \in \mathcal{P} \setminus \mathcal{P}_\delta^\star} \mathbb{E}[j_{X;Y}(X; Y, \beta)] = 2\Delta > 0 \quad (\text{E.71})$$

and observe

$$D(P_X, P_{\Pi(Y)}) = \mathbb{E}[j_{X;Y}(X; Y, \beta)] + D(Y \parallel \Pi(Y)) + c|P_{\Pi(Y)} - P_{Y^\star}|^2 \quad (\text{E.72})$$

$$\leq \mathbb{E}[j_{X;Y}(X; Y, \beta)] + D(Y \parallel \Pi(Y)) + 4c \quad (\text{E.73})$$

$$\leq \mathbb{E}[j_{X;Y}(X; Y, \beta)] + \frac{|\mathcal{B}|(|\mathcal{B}| - 1) \log e}{\sqrt{nc}} + 4c \quad (\text{E.74})$$

where

- (E.73) holds because the Euclidean distance between two distributions satisfies

$$|P_Y - P_{Y^\star}| \leq 2 \quad (\text{E.75})$$

- (E.74) is due to (E.28), (A.10), and

$$\min_Y \min_{y \in \mathcal{B}} P_{\Pi(Y)}(y) \geq \frac{1}{\sqrt{nc}} \quad (\text{E.76})$$

which is a consequence of (E.25).

Therefore, choosing $c < \frac{\Delta}{4}$, we can ensure that for all n large enough,

$$C(\beta) - \max_{P_X \in \mathcal{P} \setminus \mathcal{P}_\delta^\star} \mu(P_X, P_{\Pi(Y)}) \geq \Delta > 0 \quad (\text{E.77})$$

Also, it is easy to show using (E.76) that there exists $a > 0$ such that

$$V(P_X, P_{\Pi(Y)}) \leq a \log^2 n \quad (\text{E.78})$$

By Chebyshev's inequality (see Lemma A.4.3), we have, for the choice of γ in (E.66) and M in

(E.67),

$$\begin{aligned} & \max_{\text{type}(x^n) \in \mathcal{P} \setminus \mathcal{P}_\delta^*} \mathbb{P} \left[\sum_{i=1}^n W_i > \log M - \gamma - A | Z = \text{type}(x^n) \right] \\ & \leq \mathbb{P} \left[\sum_{i=1}^n W_i - \mathbb{E}[W_i | Z = \text{type}(x^n)] > \frac{n\Delta}{2} | Z = \text{type}(x^n) \right] \end{aligned} \quad (\text{E.79})$$

$$\leq \frac{4a}{\Delta^2} \frac{\log^2 n}{n} \quad (\text{E.80})$$

Since for large enough n , $\frac{4a}{\Delta^2} \frac{\log^2 n}{n} < \frac{1}{\sqrt{n}}$, combining (E.68) and (E.80) concludes the proof.

E.4 Proof of the achievability part of Theorem 5.5

The proof consists of the asymptotic analysis of the following bound.

Theorem E.1 (Dependence Testing bound [3]). *There exists an (M, ϵ, β) code with*

$$\epsilon \leq \inf_{P_X} \mathbb{E} \left[\exp \left(- \left| \imath_{X;Y}(X;Y) - \log \frac{M-1}{2} \right|^+ \right) \right] \quad (\text{E.81})$$

where P_X is supported on $\mathbf{b}(X) \leq \beta$.

Let P_{X^n} be equiprobable on the set of sequences of type $P_{\hat{X}^*}$, where $P_{\hat{X}^*}$ is the minimum Euclidean distance approximation of P_{X^*} formally defined in (E.36). Let $P_{X^n} \rightarrow P_{Y^n|X^n} \rightarrow P_{Y^n}$, $P_{\hat{X}^*} \rightarrow P_{Y|X} \rightarrow P_{\hat{Y}^*}$, and $P_{\hat{Y}^{n*}} = P_{\hat{Y}^*} \times \dots \times P_{\hat{Y}^*}$.

The following lemma demonstrates that P_{Y^n} is close to $P_{\hat{Y}^{n*}}$.

Lemma E.2. *Almost surely, for n large enough and some constant c ,*

$$\imath_{Y^n \| \hat{Y}^{n*}}(Y^n) \leq \frac{1}{2} (|\text{supp}(P_{X^*})| - 1) \log n + c \quad (\text{E.82})$$

Proof. For a vector $\mathbf{k} = (k_1, \dots, k_{|\mathcal{B}|})$, denote the multinomial coefficient

$$\binom{n}{\mathbf{k}} = \frac{n!}{k_1! k_2! \dots k_{|\mathcal{B}|}!} \quad (\text{E.83})$$

By Stirling's approximation, the number of sequences of type $P_{\hat{X}^*}$ satisfies, for n large enough and some constant $c_1 > 0$

$$\binom{n}{n P_{\hat{X}^*}} \geq c_1 n^{-\frac{1}{2}(|\text{supp}(P_{X^*})| - 1)} \exp(n H(\hat{X}^*)) \quad (\text{E.84})$$

On the other hand, for all x^n of type $P_{\hat{X}^*}$,

$$P_{\hat{X}^{*n}}(x^n) = \exp\left(-nH(\hat{X}^*)\right) \quad (\text{E.85})$$

Assume without loss of generality that all outputs in \mathcal{B} are accessible, which implies that $P_{Y^*}(y) > 0$ for all $y \in \mathcal{B}$. Hence, the left side of (E.82) is almost surely finite, and for all $y^n \in \mathcal{Y}^n$ with nonzero probability according to P_{Y^n} ,

$$\frac{P_{Y^n}(y^n)}{P_{\hat{Y}^{*n}}(y^n)} = \frac{\binom{n}{nP_{\hat{X}^*}}^{-1} \sum^* P_{Y^n|X^n=x^n}(y^n)}{\sum_{x^n \in \mathcal{A}^n} P_{Y^n|X^n=x^n}(y^n) P_{\hat{X}^{*n}}(x^n)} \quad (\text{E.86})$$

$$\leq \frac{\binom{n}{nP_{\hat{X}^*}}^{-1} \sum^* P_{Y^n|X^n=x^n}(y^n)}{\sum^* P_{Y^n|X^n=x^n}(y^n) P_{\hat{X}^{*n}}(x^n)} \quad (\text{E.87})$$

$$= \frac{\binom{n}{nP_{\hat{X}^*}}^{-1} \sum^* P_{Y^n|X^n=x^n}(y^n)}{\exp\left(-nH(\hat{X}^*)\right) \sum^* P_{Y^n|X^n=x^n}(y^n)} \quad (\text{E.88})$$

$$= \binom{n}{nP_{\hat{X}^*}}^{-1} \exp\left(nH(\hat{X}^*)\right) \quad (\text{E.89})$$

$$\leq c_1 n^{\frac{1}{2}(|\text{supp}(P_{X^*})|-1)} \quad (\text{E.90})$$

where we abbreviated $\sum^* = \sum_{x^n: \text{type}(x^n)=P_{\hat{X}^*}}$. □

We first consider the case $V(\beta) > 0$. For c in (E.82), let

$$\log \frac{M-1}{2} = S_n - \frac{1}{2}(|\text{supp}(P_{X^*})| - 1) \log n - c \quad (\text{E.91})$$

$$S_n = n\mu_n - \sqrt{nV_n}Q^{-1}\left(\epsilon - 2\left(\frac{\log 2}{\sqrt{2\pi}} + \frac{2T_n}{\sqrt{nV_n}}\right)\frac{1}{\gamma\sqrt{nV_n}} - \frac{B_n}{\sqrt{n}}\right) \quad (\text{E.92})$$

where μ_n and V_n are those in (2.156) and (2.157), computed with $W_i = \iota_{\hat{X}^*, \hat{Y}^*}(x_i; Y_i)$, namely

$$\mu_n = \mathbb{E}\left[\iota_{\hat{X}^*, \hat{Y}^*}(\hat{X}^*; \hat{Y}^*)\right] \quad (\text{E.93})$$

$$V_n = \text{Var}\left[\iota_{\hat{X}^*, \hat{Y}^*}(\hat{X}^*; \hat{Y}^*)|\hat{X}^*\right] \quad (\text{E.94})$$

Since the functions $P_X \mapsto \mathbb{E}[\iota_{X,Y}(X; Y)]$ and $P_X \mapsto \text{Var}[\iota_{X,Y}(X; Y)|X]$ are continuously differentiable in a neighborhood of P_{X^*} in which $P_Y(Y) > 0$ a.s., there exist constants $L_1 \geq 0$, $F_1 \geq 0$ such that

$$|\mu_n - C(\beta)| \leq L_1 |P_{\hat{X}^*} - P_{X^*}| \quad (\text{E.95})$$

$$|V_n - V(\beta)| \leq F_1 |P_{\hat{X}^*} - P_{X^*}| \quad (\text{E.96})$$

where we used (5.17). Applying (E.37), we observe that the choice of $\log M$ in (E.91) satisfies (5.41), (5.44). Therefore, to prove the claim we need to show that the right side of (E.81) with the choice of M in (E.91) is upper bounded by ϵ .

Weakening (E.81) by choosing P_{X^n} equiprobable on the set of sequences of type $P_{\hat{X}^*}$, as above, we infer that an (M, ϵ', β) code exists with

$$\epsilon' \leq \mathbb{E} \left[\exp \left(- \left| \iota_{X^n; Y^n}(X^n; Y^n) - \log \frac{M-1}{2} \right|^+ \right) \right] \quad (\text{E.97})$$

$$\leq \mathbb{E} \left[\exp \left(- \left| \iota_{Y^n | X^n \parallel \hat{Y}^{n*}}(X^n; Y^n) - \iota_{Y^n \parallel \hat{Y}^{n*}}(Y^n) - \log \frac{M-1}{2} \right|^+ \right) \right] \quad (\text{E.98})$$

$$= \mathbb{E} \left[\exp \left(- \left| \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(X_i; Y_i) - \iota_{Y^n \parallel \hat{Y}^{n*}}(Y^n) - \log \frac{M-1}{2} \right|^+ \right) \right] \quad (\text{E.99})$$

$$\leq \mathbb{E} \left[\exp \left(- \left| \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(X_i; Y_i) - S_n \right|^+ \right) \right] \quad (\text{E.100})$$

$$= \mathbb{E} \left[\exp \left(- \left| \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(x_i; Y_i) - S_n \right|^+ \right) \right] \quad (\text{E.101})$$

$$\leq \exp(S_n) \mathbb{E} \left[\exp \left(- \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(x_i; Y_i) \right) 1 \left\{ \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(x_i; Y_i) > S_n \right\} \right] \\ + \mathbb{P} \left[\sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(x_i; Y_i) \leq S_n \right] \quad (\text{E.102})$$

$$\leq \epsilon \quad (\text{E.103})$$

where

- (E.100) applies Lemma E.2 and substitutes (E.91);
- (E.101) holds for any choice of x^n of type $P_{\hat{X}^*}$ because the (conditional on $X^n = x^n$) distribution of $\iota_{Y^n | X^n \parallel \hat{Y}^{n*}}(x^n; Y^n) = \sum_{i=1}^n \iota_{\hat{X}^*; \hat{Y}^*}(x_i; Y_i)$ depends the choice of x^n only through its type;
- (E.103) upper-bounds the first term using Lemma A.4.4, and the second term using Theorem 2.23.

If $V(\beta) = 0$, let S_n in (E.91) be

$$S_n = n\mu_n - 2\gamma \quad (\text{E.104})$$

and let $\gamma > 0$ be the solution to

$$\exp(-\gamma) + \frac{F_1 \sqrt{|\mathcal{A}|(|\mathcal{A}| - 1)}}{\gamma^2} = \epsilon \quad (\text{E.105})$$

where F_1 is that in (E.96). Note that such solution exists because the function in the left side of (E.105) is continuous on $(0, \infty)$, unbounded as $\gamma \rightarrow 0$ and vanishing as $\gamma \rightarrow \infty$. The reasoning up to (E.101) still applies, at which point we upper-bound the right-side of (E.101) in the following way:

$$\epsilon' \leq \exp(-\gamma) \mathbb{P} \left[\sum_{i=1}^n \iota_{\hat{\mathbf{X}}^*, \hat{\mathbf{Y}}^*}(x_i; Y_i) > S_n + \gamma \right] + \mathbb{P} \left[\sum_{i=1}^n \iota_{\hat{\mathbf{X}}^*, \hat{\mathbf{Y}}^*}(x_i; Y_i) \leq S_n + \gamma \right] \quad (\text{E.106})$$

$$\leq \exp(-\gamma) + \frac{nV_n}{\gamma^2} \quad (\text{E.107})$$

$$\leq \epsilon \quad (\text{E.108})$$

where

- (E.107) upper-bounds the second probability using Chebyshev's inequality;
- (E.108) uses $V(\alpha) = 0$, (E.37) and (E.96).

E.5 Proof of Theorem 5.5 under the assumptions of Remark 5.3

Under assumption (a), every (n, M, ϵ, β) code with a maximal power constraint can be converted to an $(n+1, M, \epsilon, \beta)$ code with an equal power constraint (i.e. equality in (5.22) is requested) by appending to each codeword a coordinate x_{n+1} with

$$\mathbf{b}(x_{n+1}) = \beta - \sum_{i=1}^n \mathbf{b}(x_i) \quad (\text{E.109})$$

Since $\sum_{i=1}^n \mathbf{b}(x_i) \leq \beta n$, the right side of (E.109) is no smaller than β , and so by assumption (a) a coordinate x_{n+1} satisfying (E.109) can be found. It follows that

$$M_{\text{eq}}^*(n, \epsilon, \beta) \leq M_{\text{max}}^*(n, \epsilon, \beta) \leq M_{\text{eq}}^*(n+1, \epsilon, \beta) \quad (\text{E.110})$$

where the subscript specifies the nature of the cost constraint. We thus may focus only on the codes with equal power constraint. To show that the capacity-cost function can be expressed as (5.48), write

$$C(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\substack{P_{X^n}: \\ \mathbf{b}_n(X^n) = \beta \text{ a.s.}}} I(X^n; Y^n) \quad (\text{E.111})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\substack{P_{X^n}: \\ \mathbf{b}_n(X^n) = \beta \text{ a.s.}}} \{D(P_{Y^n|X^n} \| P_{Y^{n*}} | P_{X^n}) - D(Y^n \| Y^{n*})\} \quad (\text{E.112})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\substack{P_{X^n}: \\ \mathbf{b}_n(X^n) = \beta \text{ a.s.}}} \{D(P_{Y^n|X^n=x^n} \| P_{Y^{n*}}) - D(Y^n \| Y^{n*})\} \quad (\text{E.113})$$

$$= D(P_{Y|X=\mathbf{x}} \| P_{Y^*}) - \lim_{n \rightarrow \infty} \min_{\substack{P_{X^n}: \\ \mathbf{b}_n(X^n) = \beta \text{ a.s.}}} \frac{1}{n} D(Y^n \| Y^{n*}) \quad (\text{E.114})$$

$$= D(P_{Y|X=\mathbf{x}} \| P_{Y^*}) \quad (\text{E.115})$$

where

- (E.113) holds for all x^n satisfying the equal power constraint due to assumption (b);
- (E.114) holds for all coordinates \mathbf{x} such that \mathbf{x} appears in some x^n with $\mathbf{b}_n(x^n) = \beta$;
- (E.115) invokes assumption (d) to calculate the limit.

Now, to show the converse part, we invoke (5.23) where the infimum is over all distributions supported on \mathcal{F} , and $P_{\bar{Y}^n} = P_{Y^*} \times \dots \times P_{Y^*}$, $\gamma = \frac{1}{2} \log n$. A simple application of the Berry-Esséen bound (Theorem 2.23) leads to the desired result.

To show the achievability part, we follow the proof in Appendix E.4, drawing the codewords from P_{X^n} appearing in assumption (d), replacing all minimum distance approximations by the true distributions, and replacing the right side of (E.82) by f_n .

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