Gaussian Multiple and Random Access in the Finite Blocklength Regime

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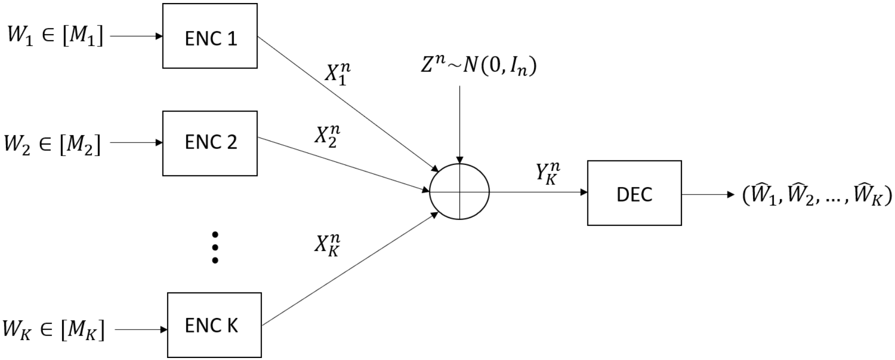
Talk Plan

We present two achievability results for

1. Gaussian Multiple Access Channel (MAC)
2. Gaussian Random Access Channel (RAC)

Gaussian Multiple Access Channel (MAC)



- Maximal power constraint on the codewords: $\|X_k^n\|^2 \leq nP_k$ for $k = 1, \ldots, K$
- Notation: $[M] = \{1, \ldots, M\}, x_{\mathcal{A}} = (x_a \colon a \in \mathcal{A})$

MAC Code Definition

Definition (K-transmitter MAC)

An $(n, M_1, \ldots, M_K, \epsilon, P_1, \ldots, P_K)$ code for the K-transmitter MAC consists of

- K encoding functions $\mathsf{f}_k : [M_k] \to \mathbb{R}^n, \quad k \in [K]$
- a decoding function $\mathsf{g} : \mathbb{R}^n \to [M_1] \times \cdots \times [M_K]$

with maximal power constraint

$$\|\mathsf{f}_k(m_k)\|^2 \leq nP_k \text{ for } m_k \in [M_k], k \in [K]$$

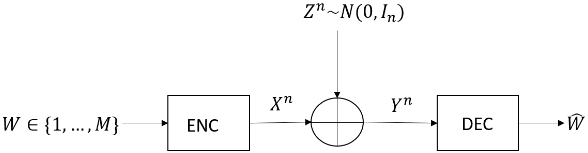
and

$$\frac{1}{\prod_{k=1}^{K} M_k} \sum_{m_{[K]} \in [M_1] \times \cdots \times [M_K]} \mathbb{P}\left[\mathsf{g}(Y_K^n) \neq m_{[K]} \mid X_k^n = \mathsf{f}_k(m_k) \, \forall k \in [K]\right] \leq \epsilon$$

average probability of error

Prior art: Point-to-point (P2P) Gaussian Channel ($K = 1$)

- Channel:



- $M^*(n, \epsilon, P) \triangleq \{\max M : \text{ an } (n, M, \epsilon, P) \text{ code exists.}\}$.

$$\log M^*(n, \epsilon, P) = nC(P) - \sqrt{nV(P)}Q^{-1}(\epsilon) + \frac{1}{2}\log n + O(1)$$

$C(P) = \frac{1}{2}\log(1+P)$ (capacity)

$V(P) = \frac{P(P+2)}{2(1+P)^2}$ (dispersion)

third-order term

Achievability ($\geq$): [Tan-Tomamichel 15']
Converse ($\leq$): [Polyanskiy et al. 10']

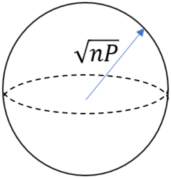
The Lesson from P2P Channel

We can achieve

$$\log M^*(n, \epsilon, P) = nC(P) - \sqrt{nV(P)}Q^{-1}(\epsilon) + \frac{1}{2}\log n + O(1)$$

by using

Uniformly distributed codewords over:



n dim power sphere

(Spherical codebook)

+ Maximum Likelihood (ML) Decoder

Motivation (MAC)

- We are interested in refining the achievable third-order term for the Gaussian MAC in the finite blocklength regime.
- For the point-to-point case, it is known that the third-order term $+1/2 \log n$ is optimal. We want to show that $+1/2 \log n\mathbf{1}$ is achievable for the Gaussian MAC.

Gaussian MAC - Main Result

Theorem

For any $\epsilon \in (0,1)$ *and any* $P_1, P_2 > 0$, *an* $(n, M_1, M_2, \epsilon, P_1, P_2)$ *code for the two-transmitter Gaussian MAC exists provided that*

$$\begin{bmatrix} \log M_1 \\ \log M_2 \\ \log M_1 M_2 \end{bmatrix} \in n\mathsf{C}(P_1, P_2) - \sqrt{n} Q_{\text{inv}}(\mathsf{V}(P_1, P_2), \epsilon) + \frac{1}{2} \log n \mathbf{1} + O(1)\mathbf{1}.$$

- $\mathsf{C}(P_1, P_2) = \begin{bmatrix} C(P_1) \\ C(P_2) \\ C(P_1 + P_2) \end{bmatrix}$ = capacity vector

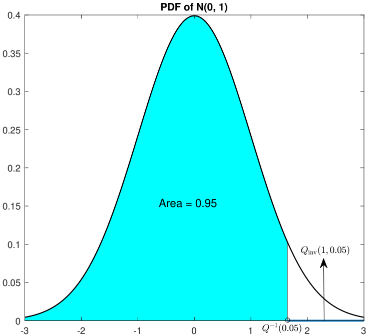
  $\mathsf{V}(P_1, P_2) = 3 \times 3$ positive-definite dispersion matrix
- $Q_{\text{inv}}(\mathsf{V}, \epsilon)$ = multidimensional counterpart of inverse Q-function

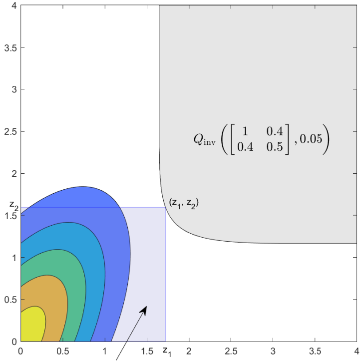
$$Q_{\text{inv}}(\mathsf{V}, \epsilon) \triangleq \left\{ \mathsf{z} \in \mathbb{R}^d : \mathbb{P}[\mathsf{Z} \leq \mathsf{z}] \geq 1 - \epsilon \right\}$$

where $\mathsf{Z} \sim \mathcal{N}(0, \mathsf{V})$ component-wise (referring to $\leq$)

What does $Q_{\text{inv}}(\mathsf{V}, \epsilon)$ look like?

$$Q_{\text{inv}}(1,\epsilon) \triangleq \{x : x \geq Q^{-1}(\epsilon)\} \qquad Q_{\text{inv}}(\mathsf{V},\epsilon) \triangleq \left\{\mathsf{z} \in \mathbb{R}^d : \mathbb{P}[\mathsf{Z} \leq \mathsf{z}] \geq 1-\epsilon\right\}$$

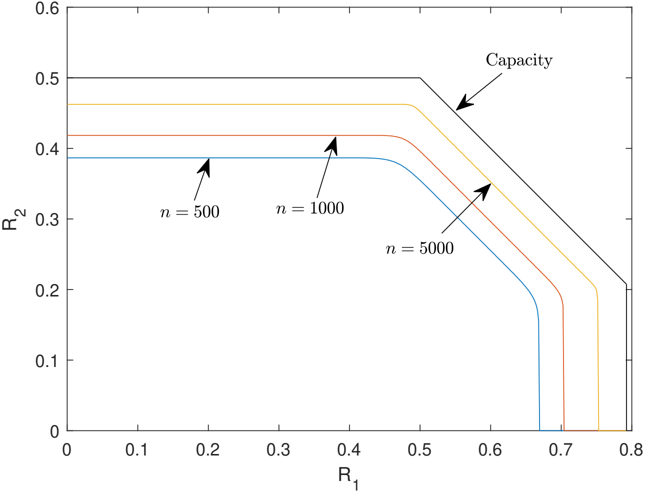




$$\mathbb{P}\left[\mathcal{N}(0,\mathsf{V}) \leq (z_1, z_2)\right] = 0.95$$

Example

Achievable region for $P_1 = 2, P_2 = 1$ and $\epsilon = 10^{-3}$:



Comparison with the literature

- Our third-order term improves!

$$n\mathsf{C}(P_1,P_2)-\sqrt{n}Q_{\mathrm{inv}}(\mathsf{V}(P_1,P_2),\epsilon)+\frac{1}{2}\log n\mathbb{1}+O(1)\mathbb{1}$$

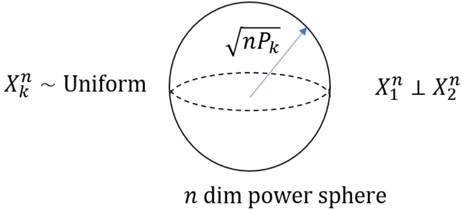
$> O\left(n^{1/4}\right)\mathbb{1}$ [MolavianJazi-Laneman 15']

$> O\left(n^{1/4}\log n\right)\mathbb{1}$ [Scarlett et al. 15']

- Proof techniques:
  - Our bound: Spherical codebook + Maximum-likelihood decoder
  - [MolavianJazi-Laneman 15'] : Spherical codebook + threshold decoder
  - [Scarlett et al. 15'] : Constant composition codes + Quantization

Encoding and decoding

- **Encoding**: independently generate M_k codewords for $k = 1, 2$:



(Spherical codebook)

[Shannon 49'] used spherical codebook to bound error exponent of the P2P Gaussian channel.

- **Decoding**: Mutual information density

$$\imath_{1,2}(x_1^n, x_2^n; y^n) \triangleq \log \frac{P_{Y_2^n|X_1^n,X_2^n}(y^n|x_1^n, x_2^n)}{P_{Y_2^n}(y^n)}$$

Maximum likelihood (ML) Decoder:

$$\mathsf{g}(y^n) = \arg \max_{m_1,m_2} \imath_{1,2}(\mathsf{f}_1(m_1), \mathsf{f}_2(m_2); y^n)$$

Main Tool: Random-Coding Union (RCU) Bound

P2P case: proved in [Polyanskiy et al. 10']
Using the ML decoder, for a general MAC:

Theorem (New RCU bound for MAC)

For arbitrary input distributions P_{X_1} and P_{X_2}, there exists a (M_1, M_2, ϵ)-MAC code such that

$$\epsilon \leq \mathbb{E}\Big[\min\Big\{1, (M_1 - 1)\mathbb{P}\left[\imath_1(\bar{X}_1; Y_2|X_2) \geq \imath_1(X_1; Y_2|X_2) \mid X_1, X_2, Y_2\right] + (M_2 - 1)\mathbb{P}\left[\imath_2(\bar{X}_2; Y_2|X_1) \geq \imath_2(X_2; Y_2|X_1) \mid X_1, X_2, Y_2\right] + (M_1 - 1)(M_2 - 1)\mathbb{P}\left[\imath_{1,2}(\bar{X}_1, \bar{X}_2; Y_2) \geq \imath_{1,2}(X_1, X_2; Y_2) \mid X_1, X_2, Y_2\right]\Big\}\Big],$$

where $P_{X_1,\bar{X}_1,X_2,\bar{X}_2,Y_2}(x_1, \bar{x}_1, x_2, \bar{x}_2, y) = P_{X_1}(x_1)P_{X_1}(\bar{x}_1)P_{X_2}(x_2)P_{X_2}(\bar{x}_2)P_{Y_2|X_1X_2}(y|x_1, x_2)$.

- Crucial in refining the third-order term to $\frac{1}{2}\log n$

Key Challenge

Modified mutual information density r.v.:

$$\tilde{\boldsymbol{\imath}}_2 \triangleq \begin{bmatrix} \tilde{\imath}_1(X_1^n; Y_2^n | X_2^n) \\ \tilde{\imath}_2(X_2^n; Y_2^n | X_1^n) \\ \tilde{\imath}_{1,2}(X_1^n, X_2^n; Y_2^n) \end{bmatrix} - n\mathsf{C}(P_1, P_2)$$

$$\tilde{\imath}_{1,2}(x_1^n, x_2^n; y^n) \triangleq \log \frac{P_{Y_2^n | X_1^n, X_2^n}(y^n | x_1^n, x_2^n)}{Q_{Y_2^n}(y^n)} \text{ with } Q_{Y_2^n} \sim \mathcal{N}(0, (1 + P_1 + P_2)\mathsf{I}_n)$$

Lemma (New Berry-Esséen type bound)

Let $\mathcal{D} \in \mathbb{R}^3$ be a convex, Borel measurable set and $\mathsf{Z} \sim \mathcal{N}(0, \mathsf{V}(\mathsf{P}_1, \mathsf{P}_2))$. Then

$$\left| \mathbb{P}\left[\frac{1}{\sqrt{n}} \tilde{\boldsymbol{\imath}}_2 \in \mathcal{D} \right] - \mathbb{P}\left[\mathsf{Z} \in \mathcal{D} \right] \right| \leq \frac{C_0}{\sqrt{n}}$$

- [MolavianJazi-Laneman 15', Prop. 1] showed a weaker upper bound with $O\left(\frac{1}{n^{1/4}}\right)$ using CLT for functions $\Longrightarrow$ affects the third-order term
- We use a different technique to prove this lemma.

Proof of Lemma

- Problem: We cannot use Berry-Esséen theorem directly since X_1^n and X_2^n are not i.i.d.
- Solution:
  - Conditional dist. $\tilde{\imath}_2 | \langle X_1^n, X_2^n \rangle = q$ is a sum of independent r.v.s
  - Apply the multidimensional Berry-Esséen theorem to that sum of independent vectors after conditioning on the inner product $\langle X_1^n, X_2^n \rangle$.
  - Then integrate the probabilities over q.

Extension to K-transmitter ($P_k = P,\ M_k = M\ \forall\, k \in [K]$)

Theorem

For any $\epsilon \in (0,1)$, and $P > 0$, an $(n, M\mathbf{1}, \epsilon, P\mathbf{1})$-MAC code for the K-transmitter Gaussian MAC exists provided that

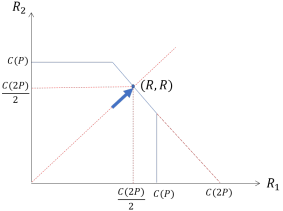
$$K \log M \le nC(KP) - \sqrt{n(V(KP) + V_{\mathrm{cr}}(K,P))}\,Q^{-1}(\epsilon) + \frac{1}{2}\log n + O(1).$$

$V_{\mathrm{cr}}(K,P)$ is the cross dispersion term

$$V_{\mathrm{cr}}(K,P) = \frac{K(K-1)P^2}{2(1+KP)^2}.$$

Message set size vector:

$$\begin{bmatrix} \log M_1 \\ \log M_2 \\ \vdots \\ \color{red}{\log(M_1 M_2 \cdots M_K)} \end{bmatrix} \in \mathbb{R}^{2^K - 1}$$



Talk Plan

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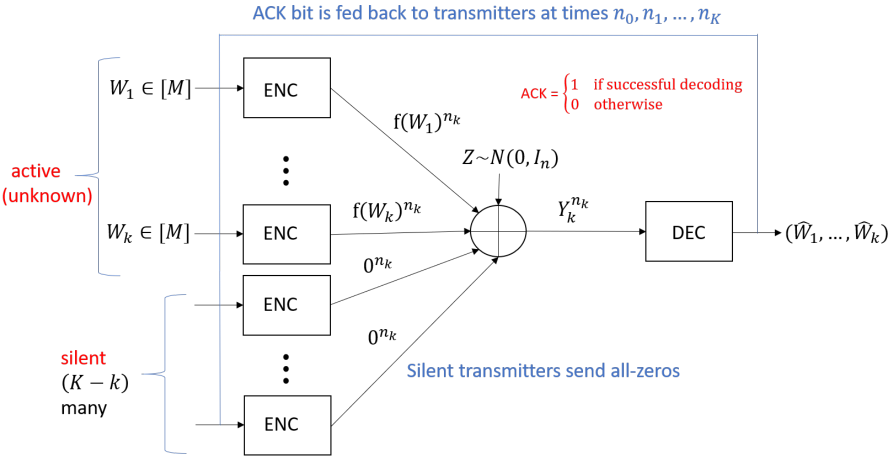
2. Gaussian Random Access Channel (RAC)

Random access

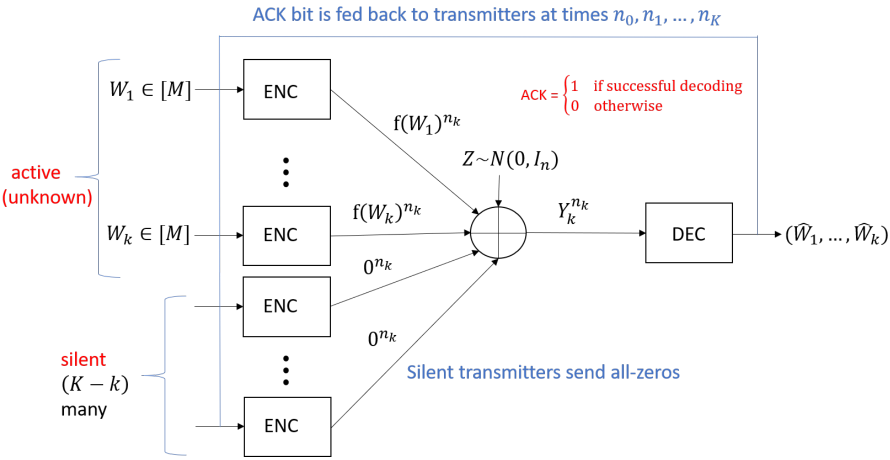
- Random access solutions such as ALOHA, treating interference as noise, or orthogonalization methods (TDMA/FDMA) perform poorly.
- We want to design a random access communication strategy that
  - does not require the knowledge of transmitter activity
  - and still does not cause a performance loss compared to k-MAC.

Rateless Gaussian RAC Communication



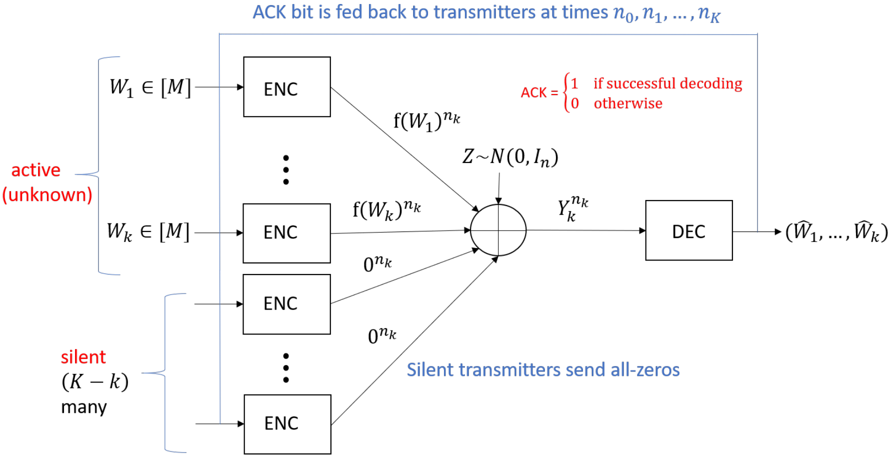
- There are K transmitters in total. A subset of those with size k are active.
- Nobody knows the active transmitters.
- No probability of being active is assigned to transmitters.

Rateless Gaussian RAC Communication



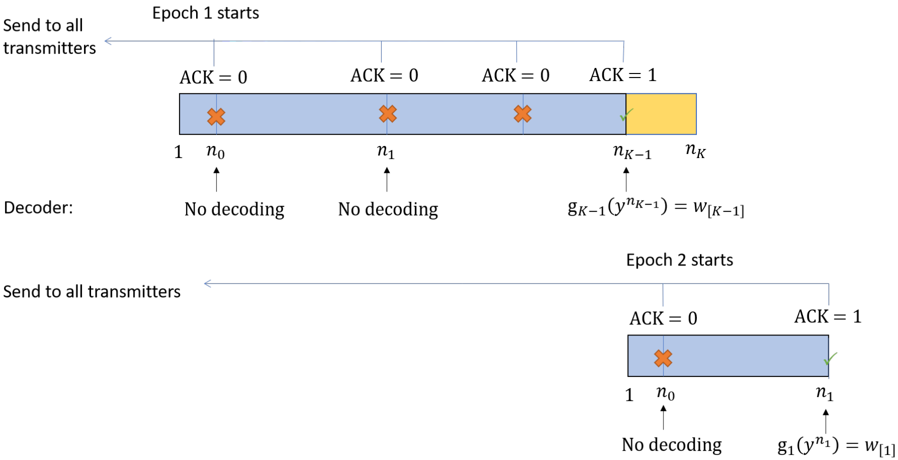
- Identical encoding and list decoding as in [Polyanskiy 17']
- Average probability of error $\leq \epsilon_k$ for $k = 0, \dots, K$
- New: Gaussian RAC, maximal power constraint: $\|f(m)^{n_k}\|^2 \leq n_k P$ for all k and m

Rateless Gaussian RAC Communication



- Rateless coding scheme that we defined in the context of DMCs [Effros, Kostina, Yavas, "Random access channel coding in the finite blocklength regime", 18']
- Predetermined decoding times: $n_0, \dots, n_K$

Communication Process



RAC Code Definition

Definition

An $\left(\{n_k, \epsilon_k\}_{k=0}^{K}, M, P\right)$-RAC consists of

- an encoder function f
- decoding functions $\{\mathsf{g}_k\}_{k=0}^{K}$

such that

- Maximal power constraints are satisfied:

$$\|\mathsf{f}(m)^{n_k}\|^2 \leq n_k P \text{ for } m \in \{1, \dots, M\}, k \in \{1, \dots, K\}$$

- and

$$\frac{1}{M^k} \sum_{m_{[k]} \in [M]^k} \mathbb{P}\left[\left\{ \bigcup_{t<k} \{\mathsf{g}_t(Y_k^{n_t}) \neq \mathsf{e}\} \right\} \bigcup \left\{ \mathsf{g}_k(Y_k^{n_k}) \overset{\pi}{\neq} m_{[k]} \right\} \middle| X_{[k]}^{n_k} = \mathsf{f}(m_{[k]})^{n_k} \right] \leq \epsilon_k$$

the average probability of error in decoding k messages at time n_k

Gaussian RAC - Main Result

Theorem

For any $K < \infty$, $\epsilon_k \in (0,1)$ *and any* $P > 0$, *an* $(M, \{(n_k, \epsilon_k)\}_{k=0}^{K}, P)$*-code for the Gaussian RAC exists provided that*

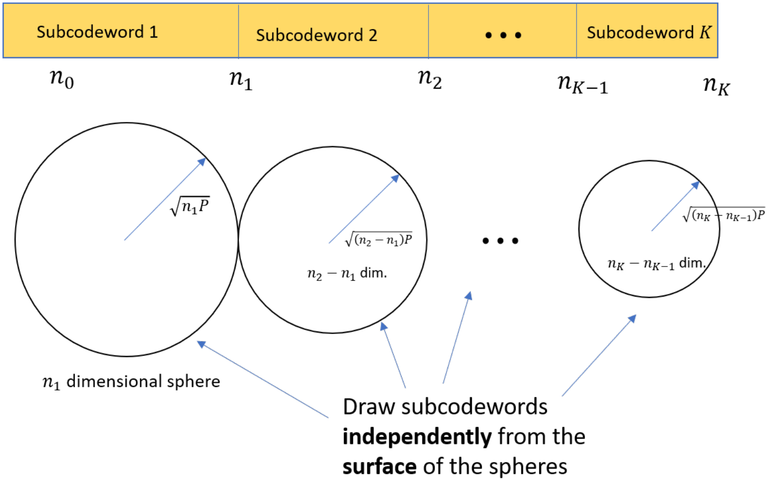
$$k \log M \leq n_k C(kP) - \sqrt{n_k (V(kP) + V_{\text{cr}}(k, P))} Q^{-1}(\epsilon_k) + \frac{1}{2} \log n_k + O(1)$$

for all $k \in [K]$, *for some positive constant* C.

- The same first, second, and third-order terms as in Gaussian MAC with known number of transmitters!

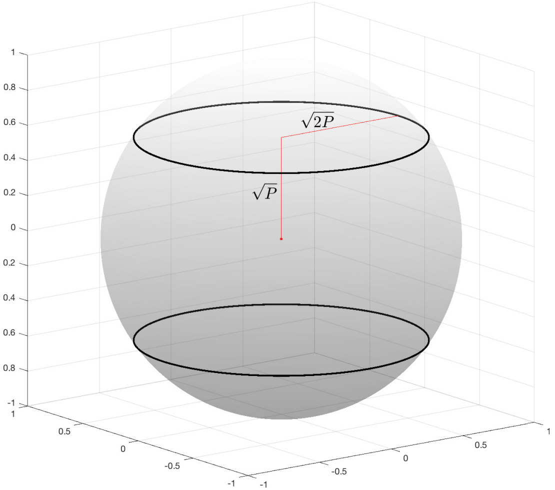
Gaussian RAC - Encoding

- To satisfy the maximal power constraints for all decoding times simultaneously, we set the input distribution as:



Feasible codeword set for Gaussian RAC

- $n_1 = 2, n_2 = 3, P = \frac{1}{3}$:[1]



[1] If we use this input dist. for the Gaussian MAC, we achieve the same first three order terms.

Gaussian RAC - Decoding

- Mutual information density for t transmitters:

$$\imath_{[t]}(x_{[t]}^{n_t}; y^{n_t}) \triangleq \log \frac{P_{Y_t^{n_t}|X_{[t]}^{n_t}}(y^{n_t}|x_{[t]}^{n_t})}{P_{Y_t^{n_t}}(y^{n_t})}$$

- Decoder output at time n_t is

$$\mathrm{g}_t(y^{n_t}) = \begin{cases} \arg\max_{m_{[t]}} \imath_{[t]}(\mathrm{f}(m_{[t]})^{n_t}; y^{n_t}) & \text{if } \left| \frac{1}{n_t} \|y^{n_t}\|^2 - (1 + tP) \right| \leq \lambda_t \\ \mathrm{e} & \text{otherwise} \end{cases}$$

If e, send ACK = 0 to request the next subcodeword of length $n_{t+1} - n_t$

Summary of the main theorems

- Gaussian MAC:
  - We refine the achievable third-order term to $1/2 \log n\mathbf{1}$ by using spherical codebook and ML decoder.
  - We derive a Berry-Esséen type bound for the spherical codebook.
- Gaussian RAC:
  - Our proposed rateless code performs as well in the first-, second-, and third-order terms as the best known communication scheme when the set of active transmitters is known.

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Thanks

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