Stabilizing Dynamical Systems with Fixed-Rate Feedback using Constrained Quantizers

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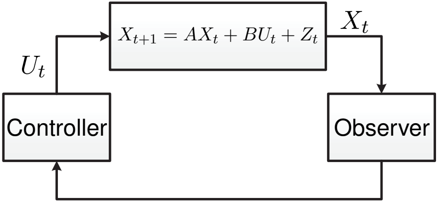
The model

- A linear dynamical system



- In practical scenarios, the observer and controller are not co-located
- The traditional observer and controller also serve as encoder-decoder
- The mappings are online (causal)

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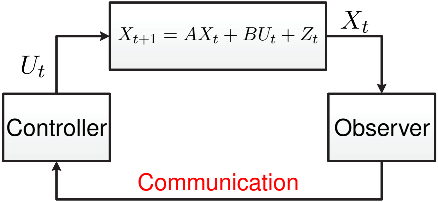

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Motivation and related works

Networked control settings:
- Wearable devices
- Remote sensors
- Drones

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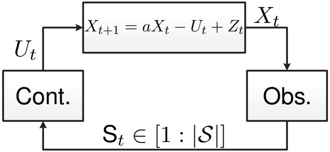
Control-Communication hybrid systems:

- Stabilizing dynamical systems using communication [Tatikonda, Sahai, Mitter 04], [Elia 04], [Borkar 97], [Silva, Derpich, Ostergaard, Encina 16], [Nair, Evans 04]
- Tradeoff between LQG cost and communication: [Tanaka, Kim, Parrilo, Mitter 17], [Fox, Tishby 16], [Khina, Nakahira, Su, Yildiz, Hassibi 18], [Tanaka, Esfahani, Mitter 17], [Sabga, Tian, Kostina, Hassibi 20]

The setting



- The noise Z_t has an α-bounded moment

$$\mathrm{E}[|Z_t|^\alpha] < \infty.$$

- The observer mapping: $X_1, \dots, X_t \to S_t, \quad S_t \in [1 : |\mathcal{S}|]$
- The controller mapping: $S_1, \dots, S_t \to U_t$
- The objective: to stabilize the system
  - A dynamical system is β-*stable* if there exists a sequence of observer-controller mappings such that

$$\limsup_{t\to\infty} \mathrm{E}[|X_t|^\beta] < \infty.$$

The converse

Theorem (Nair, Evans 04)

Any scheme which stabilizes a dynamical system with Gaussian noise satisfies

$$|\mathcal{S}| > a.$$

- Very simple converse noise for bounded moments [Kostina et al. 18]
- In [Nair et al. 04], the rate is achievable with variable-rate communication
- In the fixed-rate setting, $|\mathcal{S}|$ is an integer, implying

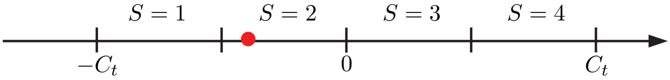
$$|\mathcal{S}^*| \geq \lfloor a \rfloor + 1$$

Zoom-in/zoom-out schemes

- Simple case - no noise:

$$X_{t+1} = aX_t - U_t$$

1. An interval $[-C_t, C_t]$ is known to the Obs. and the Cont.



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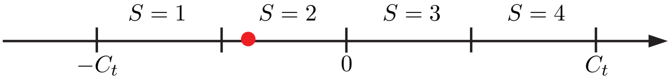

2. The Observer transmits the state quantization
3. The Cont. applies $U_t = a\hat{x}_t$ ($\hat{x}_t$ is the midpoint of the cell)

- The new state:

$$|X_{t+1}| = a|x_t - \hat{x}_t| \leq a\frac{C_t}{|\mathcal{S}|}$$

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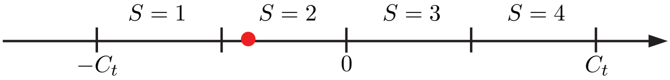

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4. If $a < |\mathcal{S}|$, the interval can be zoomed-in as $C_{t+1} = rC_t$ where r satisfies $\frac{a}{|\mathcal{S}|} \leq r < 1$)

Zoom-in/zoom-out schemes

- With noise:

$$X_{t+1} = aX_t - U_t + Z_t$$

- If the noise has bounded support $|Z_t| \leq \Delta$,

$$|X_{t+1}| = a|x_t - \hat{x}_t| + |Z_t| \leq a\frac{c}{|\mathcal{S}|} + \Delta$$

  and zoom-in (plus an additive term) is sufficient:

$$C_{t+1} = rC_t + \Delta$$

- For noise with unbounded support (e.g., Gaussian), the state may ”escape” the interval
  - Thus, we may need to zoom-out as $C_{t+1} = PC_t$ with $P > 1$
- The transition to zoom-out should be communicated!

How to communicate the transition

- The minimal rate to achieve is

$$|\mathcal{S}^*| = \lfloor a \rfloor + 1$$

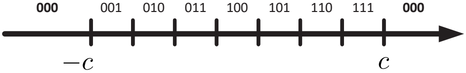
- If a symbol is dedicated for communicating transitions,

$$|\mathcal{S}| = \lfloor a \rfloor + 2$$

  is achievable (Yuksel 10)
- Optimal scheme with non-explicit coding parameters (Kostina, Peres, Ranade, Sellke 18)
- Our main idea is to encode the transition over several times using constrained quantizer
- Our main contribution is the *constrained quantizer*:
  - communicates the transitions with the optimal rate
  - Precise analysis leads to explicit scheme

The constrained quantizer

- Constrained coding (avoiding patterns) is popular in storage media
- For instance, the $(0, l-1)$-RLL constrained quantizer ($|\mathcal{S}| = 2$ and $l = 2, 3$)



- The $00..$ sequence is for transition to zoom-out
- We can always choose l such that there is a zoom-in, i.e.,

$$\frac{a^l}{(\lfloor a \rfloor + 1)^l - 1} < 1$$

The algorithm

Inputs: c_0, l, r, Δ, P

$[x, c] \leftarrow$ Zoom-Out(x, c_0, P)

procedure

  $s^l \leftarrow Q_C(x, c, l)$ ▷ *Quantization (every l times)*

  $S_i \leftarrow s_i$, for $i = 1, \ldots, l$ ▷ *Transmission*

  $U_i \leftarrow 0$, for $i = 1, \ldots, l-1$

  if $s^l \neq 0^l$ **then**

    $U_l \leftarrow U(S^l)$ ▷ *Control action*

    $c \leftarrow r \cdot c + \Delta$ ▷ *Interval update*

  else

    $[x, c] \leftarrow$ Zoom-Out(x, c, P)

  end if

end procedure

Main result

Theorem (Algorithm optimality)

Any dynamical system with $\mathrm{E}[|X_0|^\alpha] \leq \rho_\alpha$ *is* β*-stable, for* $\beta < \alpha$*, using the algorithm with* $|\mathcal{S}| = \lfloor a \rfloor + 1$ *if*

$$1 > r \geq \frac{a^l}{(\lfloor a \rfloor + 1)^l - 1}$$

$$P > a^{\frac{\alpha}{\lfloor a \rfloor (\alpha - \beta)}},$$

$$\Delta^\alpha > \left(\frac{\ln(P^\beta)}{1 - \frac{a^\alpha}{P^{\alpha-\beta}}} \frac{a^{\alpha l} \rho_\alpha}{(1-a)^\alpha} \right) 2^{\beta - 1}.$$

- We can always choose l s.t. $\frac{a^l}{(\lfloor a \rfloor + 1)^l - 1} < 1$

Proof idea

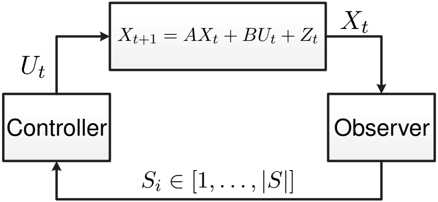
- Analysis of the states by the end of procedures
- Each procedure has a random duration
- The explicit parameters are due to the following upper bound

Lemma (Bounded sums-moment)

Let Z_i be random variables with $\mathrm{E}[|Z_i|^\alpha] \leq \rho_\alpha$ and $a > 1$. Then, for any $\beta \leq \alpha$,

$$\mathrm{E}\left[|\sum_{j=0}^{i} a^{-j} Z_j|^\beta \right] \leq \rho_\alpha \left(\frac{1 - a^{-i}}{1 - a^{-1}} \right)^\beta .$$

MIMO dynamical systems



- We assume that

$$A = \bigoplus_{i=1}^{\Lambda} J_i,$$

where J_i is a Jordan block with dimension m_i
- The pair (A, B) is controllable
- Define $a = \prod_{i=1}^{d} \max\{1, |\lambda_i|\}$. The converse gives

$$|\mathcal{S}^*| \geq \lfloor a \rfloor + 1$$

Solution to MIMO dynamical systems

- We use time-sharing between the Jordan blocks
- Each Jordan block will use our SISO algorithm for l_i transmissions (extension to Jordan blocks is trivial)
- One of the Jordan block will use the constrained quantizer

Lemma (Feasible time-sharing solution)

For any $\{\lambda_1, \dots, \lambda_\Lambda\}$ *with* $a = \prod_{i=1}^{\Lambda} |\lambda_i|$, *there exists a sequence* $\{l_i\}_{i=1}^{\Lambda}$ *with* $L = \sum_{i=1}^{\Lambda} l_i$ *such that*

$$|\lambda_i|^L \leq (\lfloor a \rfloor + 1)^{\frac{l_i}{m_i}} \tag{1}$$

for all $i = 1, \dots, \Lambda$, *and*

$$|\lambda_i|^L \leq (\lfloor a \rfloor + 1)^{\frac{l_i}{m_i}} - 1$$

for some i.

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Thank you very much!