The Price of Collusion in Series-Parallel Networks*

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Abstract. We study the quality of equilibrium in atomic splittable routing games. We show that in single-source single-sink games on series-parallel graphs, the *price of collusion* — the ratio of the total delay of atomic Nash equilibrium to the Wardrop equilibrium — is at most 1. This proves that the existing bounds on the price of anarchy for Wardrop equilibria carry over to atomic splittable routing games in this setting.

1 Introduction

In a *routing game*, players have a fixed amount of flow which they route in a network [15, 17, 23]. The flow on any edge in the network faces a delay, and the delay on an edge is a function of the total flow on that edge. We look at routing games in which each player routes flow to minimize his own delay, where a player's delay is the sum over edges of the product of his flow on the edge and the delay of the edge. This objective measures the average delay of his flow and is commonly used in traffic planning [10] and network routing [15].

Routing games are used to model traffic congestion on roads, overlay routing on the Internet, transportation of freight, and scheduling tasks on machines. Players in these games can be of two types, depending on the amount of flow they control. Nonatomic players control only a negligible amount of flow, while atomic players control a larger, non-negligible amount of flow. Further, atomic players may or may not be able to split their flow along different paths. Depending on the players, three types of routing games are: games with (i) nonatomic players, (ii) atomic players who pick a single path to route their flow, and (iii) atomic players who can split their flow along several paths. These are *nonatomic* [20, 21, 23], *atomic unsplittable* [3, 9] and *atomic splittable* [7, 15, 18] routing games respectively. We study atomic splittable routing games in this work. These games are less well-understood than either nonatomic or atomic unsplittable routing games. One significant challenge here is that, unlike most other routing games, each player has an infinite strategy space. Further, unlike nonatomic routing games, the players are asymmetric since each player has different flow value.

An equilibrium flow in a routing game is a flow where no single player can change his flow pattern and reduce his delay. Equilibria are of interest since they are a stable outcome of games. In both atomic splittable and nonatomic routing games, equilibria exist under mild assumptions on the delay functions [4, 16]. We refer to equilibria in atomic splittable games as *Nash* equilibria and in nonatomic games as *Wardrop* equilibria [23]. While the Wardrop equilibrium is known to be essentially unique [23], atomic splittable games can have multiple equilibria [5].

One measure of the quality of a flow is the *total delay* of the flow: the sum over all edges of the product of the flow on the edge and the induced delay on the edge. For routing games, one conern

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is the degradation in the quality of equilibrium flow caused by lack of central coordination. This is measured by the *price of anarchy* of a routing game, defined as the ratio of the total delay of worst-case equilibrium in a routing game to the total delay of the flow that minimizes the total delay. Tight bounds on the price of anarchy are known for nonatomic routing games [19], and are extensively studied in various settings [7, 8, 19, 20, 18, 21, 22]. In [12], Hayrapetyan et al. consider the total delay of nonatomic routing games when nonatomic players form cost-sharing coalitions. These coalitions behave as atomic splittable players. Hayrapetyan et al. introduce the notion of *price of collusion* as a measure of the price of forming coalitions. For an atomic splittable routing game the price of collusion is defined as the ratio of the total delay of the worst Nash equilibrium to the Wardrop equilibrium. Together, a bound α on the price of anarchy for nonatomic routing games and a bound β on the price of collusion for an atomic splittable routing game, imply the price of anarchy for the atomic splittable routing game is bounded by $\alpha\beta$.

For atomic splittable routing games, bounds on the price of anarchy are obtained in [7, 11]. These bounds do not match the best known lower bounds. Bounds on the price of collusion in general also remains an open problem. Previously, the price of collusion has been shown to be 1 only in the following special cases: in the graph consisting of parallel links [12]; when the players are symmetric, i.e. each player has the same flow value and the same source and sink [7]; and when all delay functions are monomials of a fixed degree [2]. Conversely, if there are multiple sources and sinks, the total delay of Nash equilibrium can be worse than the Wardrop equilibrium of equal flow value, i.e., the price of collusion can exceed 1, even with linear delays [6, 7].

Our Contribution. Let $\mathcal{C}$ denote the class of differentiable nondecreasing convex functions. We prove the following theorem for atomic splittable routing games.

Theorem 1. *In single source-destination routing games on series-parallel graphs with delay functions drawn from the class $\mathcal{C}$, the price of collusion is 1.*

We first consider the case when all delays are affine. We show that in the case of affine delays in the setting described above, the total delay at equilibrium is largest when the players are symmetric, i.e. all players have the same flow value (Section 3). To do this, we first show that the equilibrium flow for a player i remains unchanged if we modify the game by changing slightly the value of flow of any player with larger flow value than player i. Then starting from a game with symmetric players, we show that if one moves flow from a player i evenly to all players with higher flow value the cost of the corresponding equilibrium flow never increases. Since it is known that the price of collusion is 1 if the players are symmetric [7], this shows that the bound extends to arbitrary players with affine delays.

In Section 4, we extend the result for general convex delays, by showing that the worst case price of collusion is obtained when the delays are affine.

In contrast to Theorem 1 which presents a bound on the price of collusion, we also present a new bound on the price of anarchy of atomic splittable routing games in series-parallel graphs.

Theorem 2. *In single source-destination routing games on series-parallel graphs, the price of anarchy is bounded by k, the number of players.*

This bound was proven earlier for parallel links in [11]. For nonatomic routing games bounds on the price of anarchy depend on the delay functions in the graph and, in the case of polynomial delays, the price of anarchy is bounded by $O(d/\log d)$. These bounds are known to be tight even on simple graphs consisting of 2 parallel links [19]. Theorem 2 improves on the bounds obtained by Theorem 1 when $k \leq d/\log d$. All missing proofs are contained in the appendix.

2 Preliminaries

Let $G = (V, E)$ be a directed graph, with two special vertices s and t called the *source* and *sink*. The vector f, indexed by edges $e \in E$, is defined as a *flow* of *value* v if the following conditions are satisfied.

$$\sum_w f_{uw} - \sum_w f_{wu} = 0, \quad \forall u \in V - \{s,t\} \tag{1}$$

$$\sum_w f_{sw} - \sum_w f_{ws} = v \tag{2}$$

$$f_e \geq 0, \quad \forall e \in E \;.$$

Here f_{uw} represents the flow on arc (u, w). If there are several flows f^1, f^2, $\cdots$, f^k, we define $f := (f^1, f^2, \cdots, f^k)$ and f^{-i} is the vector of the flows except f^i. In this case the flow on an edge $f_e = \sum_{i=1}^k f_e^i$.

Let $\mathcal{C}$ be the class of differentiable nondecreasing convex functions. Each edge e is associated with a delay function $l_e : \mathcal{R}^+ \rightarrow \mathcal{R}$ drawn from $\mathcal{C}$. Note that we allow delay functions to be negative. For a given flow f, the *induced delay* on edge e is $l_e(f_e)$. We define the *total delay* on an edge e as the product of the flow on the edge and the induced delay $C_e(f_e) := f_e l_e(f_e)$. The *marginal delay* on an edge e is the rate of change of the total delay: $L_e(f_e) := f_e l'_e(f_e) + l_e(f_e)$. The total delay of a flow f is $C(f) = \sum_{e \in E} f_e l_e(f_e)$.

An *atomic splittable routing game* is a tuple $(G, \boldsymbol{v}, \boldsymbol{l}, s, t)$ where $\boldsymbol{l}$ is a vector of delay functions for edges in G and $\boldsymbol{v} = (v^1, v^2, \cdots, v^k)$ is a tuple indicating the flow value of the players from 1 to k. We always assume that the players are indexed by the order of decreasing flow value, hence $v^1 \geq v^2 \cdots \geq v^k$. All players have source s and destination t. Player i has a strategy space consisting of all possible s-t flows of volume v^i. Let $(f^1, f^2, \cdots, f^k)$ be a strategy vector. Player i incurs a delay $C_e^i(f_e^i, f_e) := f_e^i l_e(f_e)$ on each edge e, and his objective is to minimize his delay $C^i(f) := \sum_{e \in E} C_e^i(f_e^i, f_e)$. A set of players are *symmetric* if each player has the same flow value.

A flow is a *Nash equilibrium* if no player can unilaterally alter his flow and reduce his delay. Formally,

Definition 3 (Nash Equilibrium). *In an atomic splittable routing game, flow f is a* Nash equilibrium *if and only if for every player i and every s-t flow g of volume v^i, $C^i(f^i, f^{-i}) \leq C^i(g, f^{-i})$.*

For player i, the marginal delay on edge e is defined as the rate of change of his delay on the edge $L_e^i(f_e^i, f_e) := l_e(f_e) + f_e^i l'_e(f_e)$. For any s-t path p, the marginal delay on path p is defined as the rate of change of total delay of player i when he adds flow along the edges of the path: $L_p^i(f) := \sum_{e \in p} L_e^i(f_e^i, f_e)$. The following lemma follows from Karush-Kuhn-Tucker optimality conditions for convex programs [14] applied to player i's minimization problem.

Lemma 4. *Flow f is a Nash equilibrium flow if and only if for any player i and any two directed paths p and q between the same pair of vertices such that on all edges $e \in p$, $f_e^i > 0$, then $L_p^i(f) \leq L_q^i(f)$.*

By Lemma 4, at equilibrium the marginal delay of a player is the same on any s-t path on every edge of which he has positive flow. For a player i, the *marginal delay* is $L^i(f) := L_p^i(f)$, where p is any s-t path on which player i has positive flow on every edge.

For a given flow f and for every player i, we let $E^i(f) = \{e | f_e^i > 0\}$. $\mathcal{P}^i$ is the set of all directed s-t paths p on which for every $e \in p$, $f_e^i > 0$. We will use $e \in \mathcal{P}^i$ to mean that the edge e is in some path $p \in \mathcal{P}^i$; then $e \in \mathcal{P}^i \Leftrightarrow e \in E^i$. Let p be a directed simple s-t path. A *path flow* on path p is a directed flow on p of value f_p. A *cycle flow* along cycle C is a directed flow along C of value f_C. Any flow f can be decomposed into a set of directed path flows and directed cycle flows $\{f_p\}_{p \in \mathcal{P}} \cup \{f_c\}_{c \in C}$, [1]. This is a *flow decomposition* of f. Directed cycle flows cannot exist in atomic splittable or nonatomic games (this follows easily from Lemma 4). Thus, f^i in these games can be expressed as a set of path flows $\{f_p^i\}_{p \in \mathcal{P}^i}$ such that $f_e^i = \sum_{p \in \mathcal{P}^i : e \in p} f_p^i$. This is a *path flow decomposition* of the given flow. A *generalized path flow decomposition* is a flow decomposition along paths where we allow the path flows to be negative.

Series-Parallel Graphs. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and vertices $v_1 \in V_1$, $v_2 \in V_2$, the operation merge(v_1, v_2) creates a new graph $G' = (V' = V_1 \cup V_2, E' = E_1 \cup E_2)$, replaces v_1 and v_2 in V' with a single vertex v and replaces each edge $e = (u, w) \in E'$ incident to v_1 or v_2 by an edge incident to v, directed in the same way as the original edge.

Definition 5. *A tuple (G, s, t) is* series-parallel *if G is a single edge $e = (s, t)$, or is obtained by a series or parallel composition of two series-parallel graphs (G_1, s_1, t_1) and (G_2, s_2, t_2). Nodes s and t are* terminals *of G.*
(i) Parallel Composition*: $s = merge(s_1, s_2)$, $t = merge(t_1, t_2)$,*
(ii) Series Composition*: $s := s_1$, $t := t_2$, $v = merge(s_2, t_1)$.*

In directed series-parallel graphs, all edges are directed from the source to the destination and the graph is acyclic in the directed edges. This is without loss of generality, since any edge not on an s-t path is not used in an equilibrium flow, and no flow is sent along a directed cycle. The following lemma describes a basic property of flows in a directed series-parallel graph.

Lemma 6. *Let $G = (V, E)$ be a directed series-parallel graph with terminals s and t. Let h be an s-t flow of value $|h|$, and c is a function defined on the edges of the graph G. (i) If $\sum_{e \in p} c(e) \geq \kappa$ on every s-t path p, then $\sum_{e \in E} c(e) h_e \geq \kappa |h|$. (ii) If $\sum_{e \in p} c(e) = \kappa$ on every s-t paths p then $\sum_{e \in E} c(e) h_e = \kappa |h|$.*

Vectors and matrices in the paper, except for flow vectors, will be referred to using boldface. **1** and **0** refer to the column vectors consisting of all ones and all zeros respectively. When the size of the vector or matrix is not clear from context, we use a subscript to denote it, e.g. $\mathbf{1}_n$.

Uniqueness of Equilibrium Flow. The equilibria in atomic splittable and nonatomic routing games are known to be unique for affine delays, up to induced delays on the edges (this is true for a larger class of delays [4], [16], but here we only need affine delays). Although there may be multiple equilibrium flows, in each of these flows the delay on an edge remains unchanged. If the delay functions are strictly increasing, then the flow on each edge is uniquely determined. However with constant delays, for two parallel links between s and t with the same constant delay on each edge, any valid flow is an equilibrium flow. In this paper, we assume only that the delay functions are differentiable, nondecreasing and convex, hence we allow edges to have constant delays. We instead assume that in the graph, between any pair of vertices, there is at most one path on which all edges have constant delay. This does not affect the generality of our results. In graphs without this restriction there are Nash and Wardrop equilibrium flows in which for every pair of vertices, there is at most one constant delay path which has flow in either equilibrium. To see this, consider any equilibrium flow in a general graph. For every pair of vertices with more than one constant delay path between them, only the minimum delay path will be used at equilibrium. If there are multiple minimum constant delay paths, we can shift all the flow onto a single path; this does not affect the marginal delay of any player on any path, and hence the flow is still an equilibrium flow.

Lemma 7. *For atomic splittable and nonatomic routing games on series-parallel networks with affine delays and at most one path between any pair of vertices with constant delays on all edges, the equilibrium flow is unique.*

For technical reasons, for proving Theorem 1 we also require that every s-t path in the graph have at least one edge with strictly increasing delay. We modify the graph in the following way: we add a single edge e in series with graph G, with delay function $l_e(x) = x$. It is easy to see that for any flow, this increases the total delay by exactly v^2 where v is the value of the flow, and does not change the value of flow on any edge at equilibrium. In addition, if the price of collusion in the modified graph is less than one, then the price of collusion in the original graph is also less than one. The proof of Theorem 2 does not use this assumption.

3 Equilibria with Affine Delays

In this section we prove Theorem 1 where all delays are affine functions of the form $l_e(x) = a_e x + b_e$. Our main result in this section is:

Theorem 8. *In a series-parallel graph with affine delay functions, the total delay of a Nash equilibrium is bounded by that of a Wardrop equilibrium of the same total flow value.*

We first present the high-level ideas of our proof. Given a series-parallel graph G, terminals s and t, and edge delay functions $\boldsymbol{l}$, let $f(\cdot) : \mathbb{R}_+^k \to \mathbb{R}_+^{m \times k}$ denote the function mapping a vector of flow values to the equilibrium flow in the atomic splittable routing game. By Lemma 7, the equilibrium flow is unique and hence the function $f(\cdot)$ is well-defined. Let $(G, \boldsymbol{u}, \boldsymbol{l}, s, t)$ be an atomic splittable routing game. Our proof consists of the following three steps:

Step 1. Start with $v^i = \sum_{j=1}^k u^j/k$ for each player i, i.e. the players are symmetric.
Step 2. Gradually adjust the flow values $\boldsymbol{v}$ of the k players so that the total delay of the equilibrium flow $f(\boldsymbol{v})$ is monotonically nonincreasing.
Step 3. Stop the flow redistribution process when for each i, $v^i = u^i$.

In step 1, we make use of a result of Cominetti et al. [7].

Lemma 9. [7] *Let $(G, \boldsymbol{v}, \boldsymbol{l}, s, t)$ denote an atomic splittable routing game with k symmetric players. Let g be a Wardrop equilibrium of the same flow value $\sum_{i=1}^k v^i$. Then $C(f(\boldsymbol{v})) \leq C(g)$.*

Step 2 is the heart of our proof. The flow redistribution works as follows. Let v^i denote the current flow value of player i. Initially, each player i has $v^i = \sum_{j=1}^k u^j/k$. Consider each player in turn from k to 1. We decrease the flow of the kth player and give it *evenly* to the first $k-1$ players until $v^k = u^k$. Similarly, when we consider the rth player, for any $r < k$, we decrease v^r and give the flow evenly to the first $r-1$ players until $v^r = u^r$. Throughout the following discussion and proofs, player r refers specifically to the player whose flow value is currently being decreased in our flow redistribution process.

Our flow redistribution strategy traces out a curve $\mathcal{S}$ in $\mathbb{R}_+^k$, where points in the curve correspond to flow value vectors $\boldsymbol{v}$.

Lemma 10. *For all $e \in E$, $i \in [k]$, the function $f(\boldsymbol{v})$ is continuous and piece-wise linear along the curve $\mathcal{S}$, with breakpoints occurring where the set of edges used by any player changes.*

In what follows, we consider expressions of the form $\frac{\partial J(f(\boldsymbol{v}))}{\partial v^i}$, where J is some differentiable function defined on a flow (e.g., the total delay, or the marginal delay along a path). The expression $\frac{\partial J(f(\boldsymbol{v}))}{\partial v^i}$ considers the change in the function $J(\cdot)$ evaluated at the equilibrium flow, as the flow value of player i changes by an infinitesimal amount, keeping the flow values of the other players constant. Though $f(\boldsymbol{v})$ is not differentiable at all points in $\mathcal{S}$, $\mathcal{S}$ is continous. Therefore, it suffices to look at the intervals between these breakpoints of $\mathcal{S}$. In the rest of the paper, we confine our attention to these intervals.

We show that when the flow values are adjusted as described, the total delay is monotonically nonincreasing.

Lemma 11. *In a series-parallel graph, suppose that $v^1 = v^2 = \cdots = v^{r-1} \geq v^r \geq \cdots \geq v^k$. If $i < r$, then $\frac{\partial C(f(\boldsymbol{v}))}{\partial v^i} \leq \frac{\partial C(f(\boldsymbol{v}))}{\partial v^r}$.*

Proof of Theorem 8. By Lemma 9, the equilibrium flow in Step 1 has total delay at most the delay of the Wardrop equilibrium. We show below that during step 2, $C(f(\boldsymbol{v}))$ does not increase. Since the total volume of flow remains fixed, the Wardrop equilibrium is unchanged throughout. Thus, the price of collusion does not increase above 1, and hence the final equilibrium flow when $\boldsymbol{v} = \boldsymbol{u}$ also has this property.

Let $\boldsymbol{v}$ be the current flow values of the players. Since $C(f(\boldsymbol{v}))$ is a continuous function of $\boldsymbol{v}$ (Lemma 10), it is sufficient to show that the $C(f(\boldsymbol{v}))$ does not increase between breakpoints. Define $\boldsymbol{x}$ as follows: $\boldsymbol{x}^r = -1$; $\boldsymbol{x}^i = 0$, if $i > r$; and $\boldsymbol{x}^i = \frac{1}{r-1}$, if $1 \leq i < r$. The vector $\boldsymbol{x}$ is the rate of change of $\boldsymbol{v}$ when we decrease the flow of player r in Step 2. Thus, using Lemma 11, the change in total delay between two breakpoints in $\mathcal{S}$ satisfies

$$\lim_{\delta \to 0} \frac{C(f(\boldsymbol{v} + \delta \boldsymbol{x})) - C(f(\boldsymbol{v}))}{\delta} = -\frac{\partial C(f(\boldsymbol{v}))}{\partial v^r} + \sum_{i=1}^{r-1} \frac{\partial C(f(\boldsymbol{v}))}{\partial v^i} \frac{1}{r-1} \leq 0 .$$

□

The proof of Lemma 11 is described in Section 3.2. Here we highlight the main ideas. To simplify notation, when the vector of flow values is clear from the context, we use f instead of $f(\boldsymbol{v})$ to denote the equilibrium flow.

By chain rule, we have that $\frac{C(f)}{\partial v^i} = \sum_{e \in E} \frac{\partial L_e(f_e)}{\partial f_e} \frac{\partial f_e}{\partial v^i}$. The exact expressions of $\frac{\partial C(f)}{\partial v^i}$, for $1 \leq i \leq r$, are given in Lemmas 18 and 19 in Section 3.2. Our derivations use the fact that it is possible to simplify the expression $\frac{\partial f_e}{\partial v^i}$ using the following "nesting property" of a series-parallel graph.

Definition 12. *A graph G with delay functions $\boldsymbol{l}$, source s, and destination t satisfies the* nesting property *if all atomic splittable routing games on G satisfy the following condition: for any players i and j with flow values v^i and v^j, $v^i > v^j$ if and only if on every edge $e \in E$, for the equilibrium flow f, either $f_e^i = f_e^j = 0$ or $f_e^i > f_e^j$.*

Lemma 13 ([5]). *A series-parallel graph satisfies the nesting property for any choice of non-decreasing, convex delay functions.*

If a graph satisfies the nesting property, symmetric players have identical flows at equilibrium. When the flow value of player r is decreased in Step 2, the first $r-1$ players are symmetric.

Thus, by Lemma 13, these players have identical flows at equilibrium. Hence, for any player $i < r$, $f_e^i = f_e^1$ and $L_e^i(f_e^i, f_e) = L_e^1(f_e^1, f_e)$ for any edge e. With affine delays, the nesting property has the following implication.

Lemma 14 (Frozen Lemma). *Let f be an equilibrium flow in an atomic splittable routing game $(G,\boldsymbol{v},\boldsymbol{l},s,t)$ with affine delays on the edges, and assume that the nesting property holds for $(G,\boldsymbol{l},s,t)$. Then for all players j, $j \neq i$ with $E^j(f) \subseteq E^i(f)$ and all edges e, $\dfrac{\partial f_e^j}{\partial v^i} = 0$.*

The frozen lemma has two important implications for our proof. Firstly, in Step 2, players $r+1, \cdots, k$ will not change their flow at equilibrium. Secondly, this implies a simple expression for $\frac{\partial f_e}{\partial v^i}$, $1 \leq i \leq r$,

$$\frac{\partial f_e}{\partial v^r} = \sum_{i=1}^{k} \frac{\partial f_e^i}{\partial v^r} = (r-1)\frac{\partial f_e^1}{\partial v^r} + \frac{\partial f_e^r}{\partial v^r} \ . \tag{3}$$

$$\frac{\partial f_e}{\partial v^i} = \sum_{i=1}^{k} \frac{\partial f_e^i}{\partial v^i} = \frac{\partial f_e^i}{\partial v^i}, \quad \forall i < r \ . \tag{4}$$

3.1 Proof of Lemma 14 (Frozen Lemma)

By Lemma 10, we can assume that f is between the breakpoints of $\mathcal{S}$ and is thus differentiable.

Lemma 15. *If player h has positive flow on every edge of two directed paths p and q between the same pair of vertices, then $\frac{\partial L_p^h(f)}{\partial v^i} = \frac{\partial L_q^h(f)}{\partial v^i}$.*

Proof. Since f is an equilibrium, Lemma 4 implies that $L_p^h(f) = L_q^h(f)$. Differentiation of the two quantities are the same since f is maintained as an equilibrium. □

Lemma 16. *Let G be a directed acyclic graph. For an atomic splittable routing game $(G,\boldsymbol{v},\boldsymbol{l},s,t)$ with equilibrium flow f, let c and κ be defined as in Lemma 6. Then $\sum_{e\in E} c(e)\frac{\partial f_e^i(\boldsymbol{v})}{\partial v^j} = \kappa$ if $i = j$, and is zero otherwise.*

Proof. Define $\boldsymbol{x}$ as follows: $\boldsymbol{x}^j = 1$ and $\boldsymbol{x}^i = 0$ for $j \neq i$. Then

$$\sum_{e\in E} c(e)\frac{\partial f_e^i(\boldsymbol{v})}{\partial v^j} = \sum_{e\in E} c(e)\left(\lim_{\delta\to 0}\frac{f_e^i(\boldsymbol{v}+\delta\boldsymbol{x}) - f_e^i(\boldsymbol{v})}{\delta}\right)$$
$$= \lim_{\delta\to 0}\frac{\sum_{e\in E} c(e)(f_e^i(\boldsymbol{v}+\delta\boldsymbol{x}) - f_e^i(\boldsymbol{v}))}{\delta} \ ,$$

where the second equality is due to the fact that $f_e^i(\cdot)$ is differentiable.

For any two s-t flows f^i, g^i, it follows from Lemma 6 that $\sum_{e\in E} c(e)(f_e^i - g_e^i) = \kappa(|f^i| - |g^i|)$. If $i \neq j$ then $|f^i(\boldsymbol{v}+\delta\boldsymbol{x})| = |f^i(\boldsymbol{v})|$, hence $\sum_{e\in E} c(e)(f_e^i(\boldsymbol{v}+\delta\boldsymbol{x}) - f_e^i(\boldsymbol{v})) = 0$. If $i = j$, then $|f^i(\boldsymbol{v}+\delta\boldsymbol{x})| - |f^i(\boldsymbol{v})| = \delta$, implying that $\sum_{e\in E} c(e)(f_e^i(\boldsymbol{v}+\delta\boldsymbol{x}) - f_e^i(\boldsymbol{v})) = \kappa\delta$. The proof follows. □

Proof of Lemma 14. We prove by induction on the decreasing order of the index of j. We make use of the folllowing claim.

Claim 17 *Let $S^j = \{h : E^h(f) \supseteq E^j(f)\}$. For player j and an s-t path p on which j has positive flow,*

$$|S^j|\frac{\partial L_p^j(f)}{\partial v^i} - \sum_{h\in S^j\setminus\{j\}} \frac{\partial L_p^h(f)}{\partial v^i} = (|S^j|+1)\sum\nolimits_{e\in p} a_e \frac{\partial f_e^j}{\partial v^i} + \sum\nolimits_{e\in p} a_e \frac{\partial \sum_{h:E^h(f)\subset E^j(f)} f_e^h}{\partial v^i} .$$

Proof. Given players i and h,

$$\frac{\partial L_p^h(f)}{\partial v^i} = \sum_{e\in p} a_e \frac{\partial (f_e + f_e^h)}{\partial v^i} . \tag{5}$$

Summing (5) over all players h in $S^j\setminus\{j\}$ and subtract it from $|S^j|$ times (5) for player j gives the proof. □

Let $G^j = (V, E^j)$. By definition, all players $h \in S^j$ have flow on every s-t path in this graph. Lemma 15 implies that for any s-t paths p, q in G^j and any player $h \in S^j$, $\frac{\partial L_p^h(f)}{\partial v^i} = \frac{\partial L_q^h(f)}{\partial v^i}$. The expression on the left hand side of Claim 17 is thus equal for any path $p \in \mathcal{P}^j$, and therefore so is the expression on the right.

For the base case $j = k$, the set $\{h : E^h(f) \subset E^j(f)\}$ is empty. Hence, the second term on the right of Claim 17 is zero, and by the previous discussion, the quantity $\sum_{e\in p} a_e \frac{\partial f_e^k}{\partial v^i}$ is equal for any path $p \in \mathcal{P}^k$. Define $c(e) = a_e \frac{\partial f_e^k}{\partial v^i}$ for each $e \in E^k$ and $\kappa = \sum_{e\in p} a_e \frac{\partial f_e^k}{\partial v^i}$ for any s-t path p in G^k. By Lemma 16, $\sum_{e\in E^j(f)} c(e)\frac{\partial f_e^k}{\partial v^i} = \sum_{e\in E^j(f)} a_e \left(\frac{\partial f_e^k}{\partial v^i}\right)^2 = 0$. Hence, $\frac{\partial f_e^k}{\partial v^i} = 0$, $\forall e \in E$.

For the induction step $j < k$, due to the inductive hypothesis, $\frac{\partial f_e^h}{\partial v^i} = 0$ for $h > j$. Since by the nesting property if $E^h(f) \subset E^j(f)$ then $h > j$, the second term on the right of Claim 17 is again zero. By the same argument as in the base case, $\frac{\partial f_e^j}{\partial v^i} = 0$, for each $e \in E$, proving the lemma. □

3.2 Proof of Lemma 11

An unstated assumption for all lemmas in this section is that the nesting property holds. For the proof of Lemma 11, our first step is to express the rate of change of total delay in terms of the rate of change of marginal delay of the players, as the flow value of player r is being decreased. The next lemma gives this expression for the first $r-1$ players.

Lemma 18. *For $f = f(\boldsymbol{v})$, and for each $i < r$, $\frac{\partial C(f)}{\partial v^i} = L^i(f) + \frac{\partial L^i(f)}{\partial v^i}\frac{\sum_{j=2}^k v^j}{2}$.*

Proof. For any player j, the set of edges used by player j is a subset of the edges used by player $i < r$, since player i has the largest flow value and we assume that the nesting property holds. Hence, the total delay at equilibrium $C(f) = \sum_{e\in E^i(f)} C_e(f_e)$.

$$\frac{\partial C(f)}{\partial v^i} = \sum_{e\in E^i(f)} \frac{\partial C_e(f_e)}{\partial f_e}\frac{\partial f_e}{\partial v^i} = \sum_{e\in E^i(f)} (2a_e f_e + b_e)\frac{\partial f_e}{\partial v^i}$$

$$= \sum_{e\in E^i(f)} \frac{\partial f_e}{\partial v^i}\left(L_e^i(f_e^i, f_e) + a_e \sum_{j\neq i} f_e^j\right) . \tag{6}$$

By Lemma 16 with $c(e) = L^i_e(f^i_e, f_e)$ and $\kappa = L^i(f)$, $\sum_{e\in E^i} L^i_e(f^i_e, f_e)\frac{\partial f_e}{\partial v^i} = L^i(f)$. Thus, $\dfrac{\partial C(f)}{\partial v^i}$ $= L^i(f) + \sum_{j\neq i}\sum_{e\in E^i} a_e f^j_e \dfrac{\partial f_e}{\partial v^i}$.

By (4), we have that $a_e\frac{\partial f_e}{\partial v^i} = \frac{1}{2}a_e\frac{\partial(f_e+f^i_e)}{\partial v^i} = \frac{1}{2}\frac{\partial L^i_e(f^i_e,f_e)}{\partial v^i}$. It follows that

$$\begin{aligned}\frac{\partial C(f)}{\partial v^i} &= L^i(f) + \frac{1}{2}\sum_{j\neq i}\sum_{e\in E^i} f^j_e \frac{\partial L^i_e(f^i_e, f_e)}{\partial v^i}\\ &= L^i(f) + \frac{1}{2}\sum_{j\neq i}\sum_{e\in E^i}\sum_{q\in\mathcal{P}^i:e\in q} f^j_q \frac{\partial L^i_e(f^i_e, f_e)}{\partial v^i} \quad ,\end{aligned}$$

where the last equality is because for any player j, $f^j_e = \sum_{q\in\mathcal{P}^j:e\in q} f^j_q = \sum_{q\in\mathcal{P}^i:e\in q} f^j_q$, and the nesting property. Reversing the order of summation and observing that $\sum_{e\in p:p\in\mathcal{P}^i}\frac{\partial L^i_e(f^i_e,f_e)}{\partial v^i} = \frac{\partial L^i(f)}{\partial v^i}$ and $v^i = v^1$, we have the required expression. □

We obtain a similar expression for $\frac{\partial C(f)}{\partial v^r}$.

Lemma 19. *Let $f = f(\boldsymbol{v})$. For player r whose flow value decreases in Step 2,*

$$\begin{aligned}\frac{\partial C(f)}{\partial v^r} &= L^1(f) + \frac{r-1}{r+1}\left(\frac{\partial L^1(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right) + \frac{1}{r+1}\left(\frac{\partial L^r(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right)\\ &\quad + (r-2)\left(\sum_{e\in E^1} a_e f^1_e \frac{\partial f_e}{\partial v^r}\right) \quad . \end{aligned} \tag{7}$$

Let $\mathcal{P}$ denote the set of all s-t paths in G, and for equilibrium flow f, let $\{f^i_p\}_{p\in\mathcal{P},i\in[k]}$ denote a path flow decomposition of f. For players $i, j \in [r]$ with player r defined as in the flow redistribution, we will be interested in the rate of change of marginal delay of player i along an s-t path p as the value of flow controlled by player j changes. Given a decomposition $\{f^i_p\}_{p\in\mathcal{P},i\in[k]}$ along paths of the equilibrium flow, this rate of change can be expressed as

$$\begin{aligned}\frac{\partial L^i_p(f)}{\partial v^j} &= \sum_{e\in p} a_e \frac{\partial(f_e + f^i_e)}{\partial v^j} = \sum_{e\in p} a_e \sum_{q\in\mathcal{P}:e\in q}\frac{\partial(f_q + f^i_q)}{\partial v^j}\\ &= \sum_{q\in\mathcal{P}}\frac{\partial(f_q + f^i_q)}{\partial v^j}\sum_{e\in q\cap p} a_e \quad . \end{aligned}\tag{8}$$

Let $u_{pq} = \sum_{e\in p\cap q} a_e$ for any paths $p, q \in \mathcal{P}$ and the matrix $\mathbf{U}$ is defined as the matrix of size $|\mathcal{P}| \times |\mathcal{P}|$ with entries $[u_{pq}]_{p,q\in\mathcal{P}}$.

Lemma 20. *For an equilibrium flow f, there exists a generalized path flow decomposition $\{f^i_p\}_{p\in\overline{\mathcal{P}}^i,i\in[k]}$ so that $\overline{\mathcal{P}}^i \subseteq \mathcal{P}^i$ for all $i \in [k]$ and $\overline{\mathcal{P}}^1 \supseteq \overline{\mathcal{P}}^2 \supseteq \cdots \overline{\mathcal{P}}^k$. Moreover, each of the submatrices $\mathbf{U}^i = [u_{pq}]_{p,q\in\overline{\mathcal{P}}^i}$ of $\mathbf{U}$ is invertible, $\forall i \in [k]$.*

Since $\overline{\mathcal{P}}^i \subseteq \overline{\mathcal{P}}^{i-1}$, we can arrange the rows and columns of $\mathbf{U}$ so that $\mathbf{U}^i$ is a leading principal submatrix of $\mathbf{U}$ for every player i.

Since matrix $\mathbf{U}^i$ is invertible, we define $\mathbf{W}^i = \mathbf{U}^{-1}$. For a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we use $\mathbf{A}_p$ to refer to the pth row vector and a_{pq} to refer to the entry in the pth row and qth column. We define $\|\mathbf{A}\| = \sum_{i\in[m],j\in[n]} a_{ij}$.

Lemma 21. *For equilibrium flow f and sets $\overline{\mathcal{P}}^i \subseteq \mathcal{P}$ as described in Lemma 20, for all players $i \in [k]$, $\|\mathbf{W}^i\| \geq \|\mathbf{W}^{i+1}\|$ and $\|\mathbf{W}^k\| > 0$.*

The next lemma gives the rate of change of marginal delay at equilibrium.

Lemma 22. *For player r defined as in the flow redistribution process and any player $i < r$, for $f = f(\boldsymbol{v})$,*

(i) $\dfrac{\partial L^i(f(\boldsymbol{v}))}{\partial v^i} = \dfrac{2}{\|\mathbf{W}^i\|}$,

(ii) $\dfrac{\partial L^i(f)}{\partial v^r} = \dfrac{1}{\|\mathbf{W}^i\|}$,

(iii) $\dfrac{\partial L^r(f)}{\partial v^r} = \dfrac{r+1}{r}\dfrac{1}{\|\mathbf{W}^r\|} + \dfrac{r-1}{r}\dfrac{1}{\|\mathbf{W}^1\|}$.

If we have just two players, it follows by substituting $i = 1$ and $r = 2$ and the expressions from Lemma 22 into Lemma 18 and Lemma 19 that $\frac{\partial C(f)}{\partial v^2} - \frac{\partial C(f)}{\partial v^1} = \frac{1}{2}v^2\left(\frac{1}{\|\mathbf{W}^2\|} - \frac{1}{\|\mathbf{W}^1\|}\right)$. By Lemma 21, $\|\mathbf{W}^1\| \geq \|\mathbf{W}^2\|$, and hence $\frac{\partial C(f)}{\partial v^2} - \frac{\partial C(f)}{\partial v^1} \geq 0$, proving Lemma 11 for the case of two players. However, if we have more than two players, when $r \neq 2$ the fourth term on the right hand side of (7) has nonzero contribution. Calculating this term is complicated. However, we show the following inequality for this expression.

Lemma 23. *For $f = f(v)$ and the player r as defined in the flow redistribution process, $\sum_{e\in E^1} a_e f_e^1 \dfrac{\partial f_e}{\partial v^r} \geq \dfrac{v^1}{\|\mathbf{W}^1\|} - \dfrac{v^r}{r}\left(\dfrac{1}{\|\mathbf{W}^1\|} - \dfrac{1}{\|\mathbf{W}^r\|}\right)$.*

Proof of Lemma 11. For any player $i < r$, substituting the expression for $\frac{\partial L^i(f)}{\partial v^i}$ from Lemma 22 into Lemma 18, and observing that $L^i(f) = L^1(f)$ and $\|\mathbf{W}^i\| = \|\mathbf{W}^1\|$ since the flow of the first $r-1$ players is identical,

$$\frac{\partial C(f)}{\partial v^i} \;=\; L^1(f) + \frac{\sum_{j=2}^k v^j}{\|\mathbf{W}^1\|} \;. \tag{9}$$

Similarly, substituting from Lemmas 22 and 23 into Lemma 19 and simplifying,

$$\frac{\partial C(f)}{\partial v^r} \;\geq\; L^1(f) + \frac{\sum_{i=2}^k v^i}{\|\mathbf{W}^1\|} + \frac{1}{r}\left(\frac{1}{\|\mathbf{W}^r\|} - \frac{1}{\|\mathbf{W}^1\|}\right)\left(\sum_{i=2}^k v^i - (r-2)(v^1 - v^r)\right) \;. \tag{10}$$

We subtract (9) from (10) to obtain, for any player $i < r$,

$$\frac{\partial C(f)}{\partial v^r} - \frac{\partial C(f)}{\partial v^i} \geq \frac{1}{r}\left(\frac{1}{\|\mathbf{W}^r\|} - \frac{1}{\|\mathbf{W}^1\|}\right)\left(\sum_{i=2}^{k} v^i - (r-2)(v^1 - v^r)\right) \quad . \tag{11}$$

From Lemma 21 we know that $\|\mathbf{W}^1\| \geq \|\mathbf{W}^r\|$. Also, $\sum_{i=2}^{k} v^i = (r-2)v^1 + \sum_{i=r}^{k} v^i \geq (r-2)(v^1 - v^r)$. Hence, the expression on the right of (11) is nonnegative, completing the proof. □

4 Convex delays on series-parallel graphs

Let $\mathcal{C}$ denote the class of continuous, differentiable, nondecreasing and convex functions. In this section we prove the following result.

Theorem 24. *The price of collusion on a series-parallel graph with delay functions taken from the set $\mathcal{C}$ is at most the price of collusion with linear delay functions.*

This theorem combined with Theorem 8, suffices to prove Theorem 1. The following lemma is proved by Milchtaich.[3]

Lemma 25 ([13]). *Let $(G,v,\boldsymbol{l},s,t)$ and $(G,\tilde{v},\tilde{\boldsymbol{l}},s,t)$ be nonatomic routing games on a directed series-parallel graph with terminals s and t, where $v \geq \tilde{v}$, and $\forall x \in \mathbb{R}^+$ and $e \in E$, $l_e(x) \geq \tilde{l}_e(x)$. Let f and $\tilde{f}$ be equilibrium flows for the games with delays l and $\tilde{l}$ respectively. Then $C(f) \geq \tilde{C}(\tilde{f})$.*

We now use Lemma 25 to prove Theorem 24.

Proof of Theorem 24. Given a series-parallel graph G with delay functions $\boldsymbol{l}$ taken from $\mathcal{C}$, let g denote the atomic equilibrium flow and f denote the nonatomic equilibrium. We define a set of linear delay functions $\tilde{\boldsymbol{l}}$ as follows. For an edge, $\tilde{l}_e(x) = a_e x + b_e$, where $a_e = \left.\frac{\partial l_e(f_e)}{\partial f_e}\right|_{f_e = g_e}$ and $b_e = l_e(g_e) - a_e g_e$. Hence, the delay function $\tilde{l}_e$ is the tangent to the original delay function at the atomic equilibrium flow. Note that a convex continuous differentiable function lies above all of its tangents.

Let $\tilde{g}$ and $\tilde{f}$ denote the atomic and nonatomic equilibrium flows respectively with delay functions $\tilde{\boldsymbol{l}}$. Then by the definition of $\tilde{\boldsymbol{l}}$, $\tilde{g} = g$ and $\tilde{\boldsymbol{l}}(\tilde{g}) = \boldsymbol{l}(g)$. Hence, $\tilde{C}(\tilde{g}) = C(g)$. Further, by Lemma 25, $C(f) \geq \tilde{C}(\tilde{f})$. Since $\frac{C(g)}{C(f)} \leq \frac{\tilde{C}(\tilde{g})}{\tilde{C}(\tilde{f})}$, the proof follows.

5 Total Delay without the Nesting Property

If the nesting property does not hold, the total delay can increase as we decrease the flow of a smaller player and increase the flow of a larger player, thus causing our flow redistribution strategy presented in Section 3.2 to break down. An example of this is given in the appendix.

[3] Milchtaich in fact shows the same result for *undirected* series-parallel graphs. In our context, every simple s-t path in the underlying undirected graph is also an s-t path in the directed graph G.

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A Appendix

A.1 Proof of Lemma 6

Proof. Let $\mathcal{P}$ denote the set of simple directed s-t path in G; since we are considering only series-parallel graphs, there are no directed cycles. Consider a path flow decomposition $\{h_p\}_{p\in\mathcal{P}}$ for h. For any edge e on an s-t path, $h_e = \sum_{p\in P:e\in p} h_p$. Thus,

$$\sum_{e \in E} c(e) h_e \;=\; \sum_{e \in E} c(e) \sum_{p \in \mathcal{P} : e \in p} h_p \;=\; \sum_{p \in \mathcal{P}} h_p \sum_{e \in p} c(e) \;\; .$$

The proof for both (i) and (ii) follows immediately. □

A.2 Proof of Lemma 7

Claim 26 *Let (G, s, t) be an undirected series-parallel graph. If f is a circulation or an s-t flow, then any directed cycle C of f must be contained in a series-parallel subgraph (G', s', t') of G and must pass through s' and t'.*

Proof. We prove by induction on the size of the graph G. In the base case G is a single edge and the claim is obviously true. For the induction step, first suppose G is a series-composition of G_1 and G_2. As f restricted to G_1 or G_2 is also a flow or a circulation, we can apply induction hypothesis. Suppose that G is a parallel-composition of G_1 and G_2. A directed cycle C of f either uses edges in both G_1 and G_2, or only those in either one of them. In the former case, C must use both s and t. For the latter, observe that f restricted to G_1 or G_2 is also a circulation or a s-t flow (or a t-s flow). Thus induction hypothesis gives the proof. □

Proof of Lemma 7. We prove by contradiction. Suppose that there are two equilibrium flows $f \neq g$. Then $f - g$ is a circulation in the underlying *undirected* graph G. Moreover, $f_e - g_e \neq 0$ only if $a_e \neq 0$, since equilibrium flows are unique up to induced delay on the edges. By Claim 26, any directed cycle C in $f - g$ must be contained in a series-parallel subgraph (G', s', t') of G and C passes through s' and t'. Since C uses only edges with constant delays, there are two disjoint directed paths from s' to t' with constant delays, a contradiction. □

A.3 Proof of Lemma 10

As before, function $f(\cdot) : \mathbb{R}_+^k \rightarrow \mathbb{R}_+^{m \times k}$ is the function from the vector of flow values $\boldsymbol{v} \in \mathbb{R}_+^k$ to the unique equilibrium flow for $\boldsymbol{v}$. Functions $f^i(\cdot)$ and $f_e^i(\cdot)$ are the projections of $f(\cdot)$ for player i, and for player i on edge e respectively.

Our flow redistribution strategy traces out a curve $\mathcal{S}$ in $\mathbb{R}_+^k$, where coordinate i corresponds to the flow value of player i. $\mathcal{S}$ is then piecewise linear, and for each fixed value of r, when flow is being redistributed from player r to players $1, \ldots, r-1$, the curve $\mathcal{S}$ is linear. Define a *linear interval* of $\mathcal{S}$ to be the set of points in $\mathcal{S}$ when a fixed player r is decreasing in flow value. Then the following claim shows that in a linear interval of $\mathcal{S}$, where the set of paths any player uses also does not change, the function $f(\cdot)$ is linear.

Claim 27 *Let $\boldsymbol{u}$, $\boldsymbol{w}$ be points in a linear interval of $\mathcal{S}$ such that for every player i the set of edges on which player i has strictly positive flow is the same in both $f(\boldsymbol{u})$ and $f(\boldsymbol{w})$. Assume $u^r < w^r$. Then for any $\boldsymbol{v} = \lambda \boldsymbol{u} + (1 - \lambda)\boldsymbol{w}$ where $\lambda \in [0, 1]$, $f(\boldsymbol{v}) = \lambda f(\boldsymbol{u}) + (1 - \lambda) f(\boldsymbol{w})$.*

Proof. Let $g = \lambda f(\boldsymbol{u}) + (1 - \lambda) f(\boldsymbol{w})$. For any edge e and any player i, $L_e^i(g) = a_e(g_e + g_e^i) + b_e = a_e(\lambda(f_e^i(\boldsymbol{u}) + f_e(\boldsymbol{u})) + (1-\lambda)(f_e^i(\boldsymbol{w}) + f_e(\boldsymbol{w}))) + b_e = \lambda(a_e(f_e^i(\boldsymbol{u}) + f_e(\boldsymbol{u})) + b_e) + (1-\lambda)(a_e(f_e^i(\boldsymbol{w}) + f_e(\boldsymbol{w})) + b_e) = \lambda L_e^i(f(\boldsymbol{u})) + (1 - \lambda)(L_e^i(f(\boldsymbol{w})))$. Thus for any s-t path p, $L_p^i(g) = \lambda L_p^i(f(\boldsymbol{u})) + (1 - \lambda) L_p^i(f(\boldsymbol{w}))$.

To show that g is an equilibrium flow, we need to show that for every player i and any s-t paths p, q so that $g_e^i > 0$ for every edge $e \in p$, $L_p^i(g) \leq L_q^i(g)$. Given player i, and s-t paths p, q, if $g_e^i > 0$ then both $f_e^i(\boldsymbol{u})$ and $f_e^i(\boldsymbol{w})$ are > 0, since we are assuming that the set of edges on which player i has strictly positive flow is the same in both $f(\boldsymbol{u})$ and $f(\boldsymbol{w})$. Hence for both $\boldsymbol{u}$ and $\boldsymbol{w}$, the marginal delay along path p is at most that along path q; then this must be true for a linear combination as well, and hence g must be an equilibrium flow. □

We define the set $R = 2^{[k]\times E}$ as follows: the set R contains all possible *sets of pairs* (i, e), where i is a player and e is an edge. Hence, R contains all possible ways in which the players can put their flow on the edges. In particular, any flow $f = \{f^i\}_{i\in[k]}$ can be mapped to an element of R by considering all pairs (i, e) such that player i has strictly positive flow on edge e in f, and mapping f to the set containing all such pairs; this set is by definition an element of R.

Define the function $\gamma : \mathcal{R}_+^k \to R$ as the mapping just described from the equilibrium flow given by the input vector of flow values to an element of R.

Corollary 28. *The set of points in a linear interval of $\mathcal{S}$ where the function γ is discontinuous, is finite.*

Proof. For a given linear interval of $\mathcal{S}$, let T denote the set of points where the function γ is discontinuous, i.e., where the set of edges used by any player in $f(\cdot)$ changes. We show that the size of the set T is bounded by $2|R|$. Assume the set T contains three points $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ which map to the same point in T, i.e., $\gamma(\boldsymbol{v}_1) = \gamma(\boldsymbol{v}_2) = \gamma(\boldsymbol{v}_3)$. But since we are considering a linear interval, one of these points must lie between the others, and by Claim 27 the set of edges used by any player at the point in between cannot change. Hence the point in the middle cannot be in set T. It follows that at most two points in T can be mapped to same point in R, and hence $|T| \leq 2|R|$. □

Proof of Lemma 10. It follows from Claim 27 that if there are discontinuities in function $f(\cdot)$, then they occur where the set of edges used by any player changes and that in between these points, the function $f(\cdot)$ is linear. We show that $f(\cdot)$ is also continuous.

Let $\boldsymbol{u}$ be a point of discontinuity for function $f(\cdot)$. In order to use Claim 27, we assume $\mathbf{x}^+$ and $\mathbf{x}^-$ to be vectors in the direction of $\mathcal{S}$ at $\boldsymbol{u}$, so that $\boldsymbol{u} + \epsilon\boldsymbol{x}^+$ and $\boldsymbol{u} - \epsilon\boldsymbol{x}^-$ are points in $\mathcal{S}$, for small enough ϵ. We show the left-continuity of $f(\cdot)$; right-continuity follows from the same arguments.

By Corollary 28, in a small neighbourhood of $\boldsymbol{u}$, the number of points where any player changes the set of edges it uses is finite. Let $\boldsymbol{w}'$ be the closest point to $\boldsymbol{u}$ where this happens, then picking $\boldsymbol{w}$ slightly closer to $\boldsymbol{u}$, the set of edges used by any player does not change in the interval $[\boldsymbol{w}, \boldsymbol{u})$. Then by Claim 27, the limit of the sequence $\lim_{\epsilon\to 0} f(\boldsymbol{w} + \epsilon\boldsymbol{x}^+)$ is well-defined. Let $f_{\boldsymbol{u}} = \lim_{\epsilon\to 0} f(\boldsymbol{u} + \epsilon\boldsymbol{x}^+)$. Since the set of flows is a closed set, $f_{\boldsymbol{u}}$ is a flow. We will show that $f_{\boldsymbol{u}} = f(\boldsymbol{u})$, by showing that $f_{\boldsymbol{u}}$ is an equilibrium flow; then by Lemma 7, $f_{\boldsymbol{u}} = f(\boldsymbol{u})$.

Consider two s-t paths p, q, such that $f_{\boldsymbol{u}}$ has strictly positive flow on every edge of p and $L_p(f_{\boldsymbol{u}}) > L_q(f_{\boldsymbol{u}})$. But then by continuity, these inequalities should hold even if we change $\boldsymbol{u}$ slightly; i.e. there exists $\epsilon > 0$ s.t. $f(\boldsymbol{u} + \epsilon\boldsymbol{x}^+)$ has strictly positive flow on every edge of p and $L_p(\boldsymbol{u} + \epsilon\boldsymbol{x}^+) > L_q(\boldsymbol{u} + \epsilon\boldsymbol{x}^+)$. By definition of $f(\cdot)$ and characterization of an equilibrium flow, this cannot be. Hence $f_{\boldsymbol{u}}$ must be an equilibrium flow. Since we know the equilibrium flow is unique by Lemma 7, $f_{\boldsymbol{u}} = f(\boldsymbol{u})$, completing the proof. □

A.4 Proof of Lemma 19

Proof of Lemma 19. Proceeding as in the proof of Lemma 18, the rate of change of cost as the value of flow of player r changes can be written as

$$\frac{\partial C(f)}{\partial v^r} \;=\; L^1(f) + \sum_{i>1}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;\; .$$

Since $f_e^i = f_e^1$, for all players $i < r$,

$$\frac{\partial C(f)}{\partial v^r} \;=\; L^1(f) + (r-2)\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} + \sum_{i=r}^{k}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;\; . \tag{12}$$

We now concentrate on the third term on the right hand side of (12), i.e. the expression $\sum_{i=r}^k \sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r}$. Since the players $i \geq r$ do not have flow on edges $e \notin E^r$, we can sum over edges $e \in E^r$ instead of E^1. Also, by Lemma 14, $\frac{\partial f_e}{\partial v^r} = \sum_{j=1}^r \frac{\partial f_e^j}{\partial v^r}$, and hence $(r+1)\frac{\partial f_e}{\partial v^r} = \sum_{j=1}^r \frac{\partial (f_e + f_e^j)}{\partial v^r}$. Now

$$\sum_{i=r}^{k}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;=\; \frac{1}{r+1}\sum_{i=r}^{k}\sum_{e\in E^r} a_e f_e^i \sum_{j=1}^{r}\frac{\partial (f_e + f_e^j)}{\partial v^r} \;\; .$$

For player i, let $\{f_p^i\}_{p\in\mathcal{P}^i}$ be an arbitrary path flow decomposition of the equilibrium flow. Since $a_e \frac{\partial (f_e+f_e^j)}{\partial v^r} = \frac{\partial L_e^j(f)}{\partial v^r}$, and for any player $i \geq r$, $f_e^i = \sum_{p\in\mathcal{P}^r : e\in p} f_p^i$:

$$\sum_{i=r}^{k}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;=\; \frac{1}{r+1}\sum_{i=r}^{k}\sum_{j=1}^{r}\sum_{e\in E^r}\frac{\partial L_e^j(f)}{\partial v^r}\sum_{p\in\mathcal{P}^r : e\in p} f_p^i \;=\; \frac{1}{r+1}\sum_{i=r}^{k}\sum_{j=1}^{r}\sum_{p\in\mathcal{P}^r} f_p^i \frac{\partial L_p^j(f)}{\partial v^r} \;\; ,$$

where the last equality is obtained by simply reversing the order of summation and observing that $\sum_{e\in p} \frac{\partial L_e^j(f)}{\partial v^r} = \frac{\partial L_p^j(f)}{\partial v^r}$.

For any player $j \leq r$, the rate of change of marginal delay on any path $p \in \mathcal{P}^r$ is the same, and hence $\frac{\partial L_p^j(f)}{\partial v^r} = \frac{\partial L^j(f)}{\partial v^r}$. Then

$$\begin{aligned}
\sum_{i=r}^{k}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;&=\; \frac{1}{r+1}\left(\sum_{j=1}^{r-1}\frac{\partial L^j(f)}{\partial v^r}\sum_{i=r}^{k}\sum_{p\in\mathcal{P}^r} f_p^i + \frac{\partial L^r(f)}{\partial v^r}\sum_{i=r}^{k}\sum_{p\in\mathcal{P}^r} f_p^i\right) \\
&=\; \frac{1}{r+1}\left(\sum_{j=1}^{r-1}\frac{\partial L^j(f)}{\partial v^r}\sum_{i=r}^{k} v^i + \frac{\partial L^r((f)}{\partial v^r}\sum_{i=r}^{k} v^i\right) .
\end{aligned}$$

Since all players $j < r$ have the same flow at equilibrium,

$$\sum_{i=r}^{k}\sum_{e\in E^1} a_e f_e^i \frac{\partial f_e}{\partial v^r} \;=\; \frac{1}{r+1}\left((r-1)\frac{\partial L^1(f)}{\partial v^r}\sum_{i=r}^{k} v^i + \frac{\partial L^r(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right) .$$

Substituting into (12),

$$\frac{\partial C(f)}{\partial v^r} \;=\; L^1(f) + (r-2)\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} + \frac{1}{r+1}\left((r-1)\frac{\partial L^1(f)}{\partial v^r}\sum_{i=r}^{k} v^i + \frac{\partial L^r(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right),$$

completing the proof. □

A.5 Proof of Lemma 20 and Lemma 21

As earlier, the players are ordered according to decreasing flow value so that player 1 has the largest flow value and player k has the smallest. By Lemma 13, $E^k \subseteq E^{k-1} \subseteq \cdots \subseteq E^1$ and hence $\mathcal{P}^k \subseteq \mathcal{P}^{k-1} \subseteq \cdots \subseteq \mathcal{P}^1$.

Given a set of s-t paths $\mathcal{P}'$, we define an edge-path incidence matrix $\mathbf{Y}'$ as the matrix with rows corresponding to edges $e \in E$ and columns corresponding to paths $p \in \mathcal{P}'$. Entry $y_{e,p} = \sqrt{a_e}$ if edge e is in path p, and is 0 otherwise.

Claim 29 *Let $\mathbf{Y}'$ be the edge-path incidence matrix for a given set of s-t paths $\mathcal{P}'$. Any column vector $\lambda = [\lambda_p]_{p \in \mathcal{P}'}$ in the null space of $\mathbf{Y}'$ is a generalized path flow decomposition of a flow of value 0, with $\lambda_e = \sum_{p \in \mathcal{P}' : e \in p} \lambda_p = 0 \; \forall e \in E$.*

Proof. Since $\mathbf{Y}'\lambda = \mathbf{0}$, for all edges $e \in E$, $\sum_{p \in \mathcal{P}'} \lambda_p y_{e,p} = \sqrt{a_e} \sum_{p \in \mathcal{P}' : e \in p} \lambda_p = \sqrt{a_e}\lambda_e = 0$. Hence on every edge, either $a_e = 0$ or $\lambda_e = 0$. Recall that we modified our graph by adding an initial edge e' in series with the graph, with delay function x. Since all paths must use edge e', and $a_{e'} > 0$, $0 = \lambda_{e'} = \sum_{p \in \mathcal{P}'} \lambda_p$. If on any edge $e \in E$, $\lambda_e \neq 0$, then there is another flow h of value 0, $h : h_e = 0 \; \forall e \in E$. Then there is a cycle C on every edge of which $\lambda_e \neq h_e$. Since every cycle in the graph has an edge with $a_e > 0$, there is an edge $e \in E$ such that $\lambda_e \neq 0$ and $a_e \neq 0$. This is a contradiction, and hence $\lambda = h$. □

Proof of Lemma 20. Each s-t path p corresponds to a vector $\mathbf{z}^p$ of size $|E|$ and with entries $z_e^p = \sqrt{a_e}$ if $e \in p$, and 0 otherwise. The paths $\overline{\mathcal{P}}^i$ and the generalized path flow decomposition $\{f_p^i\}_{p \in \overline{\mathcal{P}}^i}$ are constructed by induction as follows. Let $\mathcal{P}^{k+1} = \overline{\mathcal{P}}^{k+1} = \emptyset$, then in the base case, the set of vectors $\mathbf{z}^p$ corresponding to paths in $\overline{\mathcal{P}}^{k+1}$ are a basis for the set $\{\mathbf{z}^p : p \in \mathcal{P}^{k+1}\}$. Given a basis $\{\mathbf{z}^p : p \in \overline{\mathcal{P}}^{i+1}\}$ for the set $\{\mathbf{z}^p : p \in \mathcal{P}^{i+1}\}$, we can extend the basis to a larger basis for the set $\{\mathbf{z}^p : p \in \mathcal{P}^i\}$. We define $\overline{\mathcal{P}}^i$ is the set of paths in this extended basis. By definition, $\overline{\mathcal{P}}^i \supseteq \overline{\mathcal{P}}^{i+1}$.

The generalized path flow decomposition $\{f_p^i\}_{p \in \overline{\mathcal{P}}^i}$ is obtained as follows. Let $\{g_p^i\}_{p \in \mathcal{P}^i}$ be an arbitrary flow decomposition, and initially let $f_p^i = g_p^i$ for paths $p \in \overline{\mathcal{P}}^i$. For each path $q \in \mathcal{P}^i \setminus \overline{\mathcal{P}}^i$, since $\{\mathbf{z}^p : p \in \overline{\mathcal{P}}^i\}$ is a basis for the set $\{\mathbf{z}^p : p \in \mathcal{P}^i\}$, there exist multipliers λ such that $\sum_{p \in \overline{\mathcal{P}}^i \cup \{q\}} \lambda_p \mathbf{z}^p = 0$ and $\lambda_q = 1$. Note that λ is a vector in the nullspace of the matrix with columns $\{\mathbf{z}^p : p \in \overline{\mathcal{P}}^i \cup \{q\}\}$. Modify the generalized path flow decomposition in the following way: for each path $p \in \overline{\mathcal{P}}^i$, $f_p^i = f_p^i - \lambda_p g_q^i$. By Claim 29, $\lambda_e = \sum_{p \in \overline{\mathcal{P}}^i \cup \{q\}} \lambda_p = 0$ on any edge e, hence it is easy to check that by modifying f_p^i, the flow on any edge remains unchanged. Carrying out this modification for every path $p \in \mathcal{P}^i \setminus \overline{\mathcal{P}}^i$ gives us the required generalized path decomposition. Further, since the set of vectors $\{\mathbf{z}^p : p \in \overline{\mathcal{P}}^1\}$ forms a basis and is hence linearly independent, the matrix $\mathbf{Y}^i$ with columns corresponding to this set of vectors has full rank, and hence the matrix $U^i = \mathbf{Y}^{i^T}\mathbf{Y}^i$ is invertible. □

For any player i, let $\mathbf{Y}^i$ be the edge-path incidence matrix for the edge set E and path set $\overline{\mathcal{P}}^i$. Then $\mathbf{U}^i = \mathbf{Y}^{i^T}\mathbf{Y}^i$.

Proof of Lemma 21. Since $\mathbf{U}^i = \mathbf{Y}^{i^T}\mathbf{Y}^i$ and $\mathbf{Y}^i$ has full rank, $\mathbf{U}^i$ is symmetric and positive definite. Thus its inverse $\mathbf{W}^i$ is as well, and $\|\mathbf{W}^k\| = \mathbf{1}^T\mathbf{W}^k\mathbf{1} > 0$.

Now we prove that $\|\mathbf{W}^{i-1}\| \geq \|\mathbf{W}^i\|$ for any player i. We assume for the proof that $\overline{\mathcal{P}}^{i-1} = \overline{\mathcal{P}}^i \cup \{q\}$, i.e., there is a single path q in $\overline{\mathcal{P}}^{i-1}$, $\notin \overline{\mathcal{P}}^i$. The proof easily extends to the more general case. Let $\mathbf{Y}^{i-1} = [\mathbf{Y}^i \ \mathbf{z}^i]$. Thus

$$\mathbf{U}^{i-1} = \begin{bmatrix} {\mathbf{Y}^i}^T\mathbf{Y}^i & {\mathbf{Y}^i}^T\mathbf{z}_i \\ {\mathbf{z}^i}^T\mathbf{Y}^i & {\mathbf{z}^i}^T\mathbf{z}^i & . \end{bmatrix}$$

The following expression for $\mathbf{W}^{i-1}$ is easily checked.

$$\begin{aligned} \mathbf{W}^{i-1} &= \begin{bmatrix} \mathbf{W}^i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{({\mathbf{z}^i}^T\mathbf{z}^i - {\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i)} \begin{bmatrix} \mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i{\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i & -\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i \\ -{\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{W}^i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \frac{1}{({\mathbf{z}^i}^T\mathbf{z}^i - {\mathbf{z}^i}^T\mathbf{Y}^{i+1}\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i)} \begin{bmatrix} -\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i \\ 1 \end{bmatrix} \begin{bmatrix} -{\mathbf{z}^i}^T\mathbf{Y}^{i+1}\mathbf{W}^i & 1 \end{bmatrix} \end{aligned}$$

Let $\mathbf{s}$ denote the row vector $[-{\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i \ 1]$. Then

$$\|\mathbf{W}^{i-1}\| = \|\mathbf{W}^i\| + \frac{1}{({\mathbf{z}^i}^T\mathbf{z}^i - {\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i)}\|\mathbf{s}^T\mathbf{s}\|$$

Since $\|\mathbf{s}^T\mathbf{s}\| = (\|s\|)^2 \geq 0$ for any vector $\mathbf{s}$, we show that $({\mathbf{z}^i}^T\mathbf{z}^i - {\mathbf{z}^i}^T\mathbf{Y}^i\mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i) \geq 0$, and this completes the proof. Let $\mathbf{x} = \mathbf{W}^i{\mathbf{Y}^i}^T\mathbf{z}^i$; then since $(\mathbf{z}^i - \mathbf{Y}^i\mathbf{x})^T(\mathbf{z}^i - \mathbf{Y}^i\mathbf{x}) \geq 0$, we have ${\mathbf{z}^i}^T(\mathbf{z}^i - \mathbf{Y}^i\mathbf{x}) \geq \mathbf{x}^T{\mathbf{Y}^i}^T(\mathbf{z}^i - \mathbf{Y}^i\mathbf{x}) = 0$. □

A.6 Proof of Lemma 22

Proof of Lemma 22. For equilibrium flow f, let $\{f_p^i\}_{p\in\overline{\mathcal{P}}^i}$ be the generalized path flow decomposition given by Lemma 20. For any path $p \in \overline{\mathcal{P}}^i$, it follows from (8) and the definition of u_{pq} that $\frac{\partial L^i(f)}{\partial v^i} = \frac{\partial L_p^i(f)}{\partial v^i} = \sum_{q\in\mathcal{P}} u_{pq}\frac{\partial(f_q+f_q^i)}{\partial v^i}$. By (4), $\frac{\partial(f_q+f_q^i)}{\partial v^i} = 2\frac{\partial f_q^i}{\partial v^i}$. It follows that $\dfrac{\partial L^i(f)}{\partial v^i} = 2\sum_{q\in\mathcal{P}} u_{pq}\dfrac{\partial f_q^i}{\partial v^i} = 2\sum_{q\in\overline{\mathcal{P}}^i} u_{pq}\dfrac{\partial f_q^i}{\partial v^i}$ since $f_q^i = 0$ for paths $q \notin \overline{\mathcal{P}}^i$. We write this as a matrix equation, where each row corresponds to a path in $\overline{\mathcal{P}}^i$.

$$\frac{\partial L^i(f)}{\partial v^i}\mathbf{1}_{|\overline{\mathcal{P}}^i|} = 2\,\mathbf{U}^i \begin{bmatrix} \dfrac{\partial f_q^i}{\partial v^i} \end{bmatrix}_{q\in\overline{\mathcal{P}}^i} .$$

By Lemma 20, the matrix $\mathbf{U}^i$ is invertible, with $\mathbf{W}^i = (\mathbf{U}^i)^{-1}$. Multiplying both sides by $\mathbf{1}^T\mathbf{W}^i$ yields $\frac{\partial L^i(f)}{\partial v^i}\|\mathbf{W}^i\| = 2\frac{\partial v^i}{\partial v^i} = 2$, proving (i).

By Lemma 15, for player $i < r$ on any path $p \in \overline{\mathcal{P}}^i$, $\frac{\partial L_p^i(f)}{\partial v^r} = \frac{\partial L^i(f)}{\partial v^r}$. Further,

$$\frac{\partial L^i(f)}{\partial v^r} \;=\; \sum_{e\in p} a_e \frac{\partial (f_e+f_e^i)}{\partial v^r} \;=\; \sum_{q\in\overline{\mathcal{P}}^i} u_{pq} \frac{\partial (f_q+f_q^i)}{\partial v^r}$$

since $f_e^i = \sum_{q\in\mathcal{P}^i : e\in q} f_q^i$, and by reversing the order of summation. This is true for all paths $p \in \overline{\mathcal{P}}^i$, giving us $|\overline{\mathcal{P}}^i|$ equalities for $\frac{\partial L^i(f)}{\partial v^r}$, which we can solve to obtain its value. Formally, write the equalities in matrix form as

$$\frac{\partial L^i(f)}{\partial v^r}\, \mathbf{1}_{|\overline{\mathcal{P}}^i|} \;=\; \mathbf{U}^i \left[\frac{\partial (f_q+f_q^i)}{\partial v^r}\right]_{q\in\mathcal{P}^i} . \tag{13}$$

Left-multiplying both sides by $\mathbf{1}^T\mathbf{W}^i$,

$$\frac{\partial L^i(f)}{\partial v^r}\, \|\mathbf{W}^i\| \;=\; \sum_{q\in\overline{\mathcal{P}}^i} \frac{\partial (f_q+f_q^i)}{\partial v^r} \;=\; 1,$$

since $\overline{\mathcal{P}}^j \subseteq \overline{\mathcal{P}}^i$ for any player j by construction, and the rate of change of total flow value is 1. This proves (ii).

For the third part consider paths $p \in \overline{\mathcal{P}}^r$. By Lemma 15, for player r, $\frac{\partial L_p^r(f)}{\partial v^r} = \frac{\partial L^r(f)}{\partial v^r}$. As above, the rate of change is

$$\frac{\partial L^r(f)}{\partial v^r} = \sum_{q\in\overline{\mathcal{P}}^i} u_{pq} \frac{\partial (f_q+f_q^r)}{\partial v^r} . \tag{14}$$

Our first step is to show a relation between $\frac{\partial L^r(f)}{\partial v^r}$ and $\frac{\partial L^1(f)}{\partial v^r}$. Using Lemma 14 and the fact that and all players $i < r$ have the same flow at equilibrium, for any path $q \in \overline{\mathcal{P}}^1$,

$$\begin{aligned}\frac{\partial (f_q+f_q^r)}{\partial v^r} &= (r-1)\frac{\partial f_q^1}{\partial v^r} + 2\frac{\partial f_q^r}{\partial v^r}\\ &= \frac{r-1}{r}\left(\frac{\partial (r f_q^1+f_q^r)}{\partial v^r}\right) + \frac{r+1}{r}\frac{\partial f_q^r}{\partial v^r} .\end{aligned}$$

For the same reasons, $\frac{\partial (f_q+f_q^1)}{\partial v^r} = \frac{\partial (r f_q^1+f_q^r)}{\partial v^r}$. Using this in the above equation and substituting into (14) yields

$$\begin{aligned}\frac{\partial L^r(f)}{\partial v^r} &= \frac{r-1}{r}\sum_{q\in\overline{\mathcal{P}}^1} u_{pq}\frac{\partial (f_q+f_q^1)}{\partial v^r} + \frac{r+1}{r}\sum_{q\in\overline{\mathcal{P}}^r} u_{pq}\frac{\partial f_q^r}{\partial v^r}\\ &= \frac{r-1}{r}\frac{\partial L^1(f)}{\partial v^r} + \frac{r+1}{r}\sum_{q\in\overline{\mathcal{P}}^i} u_{pq}\frac{\partial f_q^r}{\partial v^r} ,\end{aligned}$$

where the second inequality uses (13) for $i = 1$. Since $\frac{\partial f_q^r}{\partial v^r} = 0$ on any path $q \notin \overline{\mathcal{P}}^r$, the summation in the second term needs to be done only over paths $q \in \overline{\mathcal{P}}^r$. For all $p \in \overline{\mathcal{P}}^r$ we have the system of equations

$$\frac{\partial L^r(f)}{\partial v^r}\mathbf{1}_{|\overline{\mathcal{P}}^r|} \;=\; \frac{r-1}{r}\frac{\partial L^1(f)}{\partial v^r}\mathbf{1}_{|\mathcal{P}^r|} + \frac{r+1}{r}\mathbf{U}^r\left[\frac{\partial f_q^r}{\partial v^r}\right]_{|\mathcal{P}^r|} \quad . \tag{15}$$

Left-multiplying both sides by $\mathbf{1}^T\mathbf{W}^r$ yields

$$\frac{\partial L^r(f)}{\partial v^r}\|\mathbf{W}^r\| \;=\; \frac{r-1}{r}\frac{\partial L^1(f)}{\partial v^r}\|\mathbf{W}^r\| + \frac{r+1}{r}\sum_{q\in\overline{\mathcal{P}}^r}\frac{\partial f_q^r}{\partial v^r} \quad .$$

Dividing by $\|\mathbf{W}^r\|$ and using $\frac{\partial L^1(f)}{\partial v^r} = \frac{1}{\|\mathbf{W}^1\|}$ from (i) gives us the desired expression. □

A.7 Proof of Lemma 23

Before we prove Lemma 23, we show the following two claims. We assume for this subsection that for equilibrium flow f, $\{f_p^i\}_{p\in\overline{\mathcal{P}}^i}$ is the generalized path flow decomposition given by Lemma 20.

Claim 30 *(i) For any path* $p \in \overline{\mathcal{P}}^r$, $\frac{\partial f_p}{\partial v^r} \;=\; \frac{r-1}{r}\frac{\|\mathbf{W}_p^1\|}{\|\mathbf{W}^1\|} + \frac{1}{r}\frac{\|\mathbf{W}_p^r\|}{\|\mathbf{W}^r\|}$. *(ii) For any path* $p \notin \overline{\mathcal{P}}^r$, $\frac{\partial f_p}{\partial v^r} \;=\; \frac{r-1}{r}\frac{\|\mathbf{W}_p^1\|}{\|\mathbf{W}^1\|}$.

Proof. Left-multiply both sides of (15) by $\mathbf{W}^r$ to obtain, for each path $p \in \overline{\mathcal{P}}^r$,

$$\frac{\partial L^r(f)}{\partial v^r}\|\mathbf{W}_p^r\| \;=\; \frac{r-1}{r}\frac{\partial L^1(f)}{\partial v^r}\|\mathbf{W}_p^r\| + \frac{r+1}{r}\frac{\partial f_p^r}{\partial v^r} \quad .$$

Substituting the values of $\frac{\partial L^r(f)}{\partial v^r}$ and $\frac{\partial L^1(f)}{\partial v^r}$ from Lemma 22 and simplifying,

$$\frac{\partial f_p^r}{\partial v^r} \;=\; \frac{\|\mathbf{W}_p^r\|}{\|\mathbf{W}^r\|} \quad . \tag{16}$$

By Lemma 14, $\frac{\partial f_p}{\partial v^r} = \sum_{i=1}^r \frac{\partial f_p^i}{\partial v^r} = (r-1)\frac{\partial f_p^1}{\partial v^r} + \frac{\partial f_p^r}{\partial v^r}$. Left-multiplying both sides of (13) for $i = 1$ by $\mathbf{W}^1$ yields for each path $p \in \overline{\mathcal{P}}^1$,

$$\frac{\partial L^1(f)}{\partial v^r}\|\mathbf{W}_p^1\| \;=\; \frac{\partial(f_p + f_p^1)}{\partial v^r} \;=\; r\frac{\partial f_p^1}{\partial v^r} + \frac{\partial f_p^r}{\partial v^r} \quad . \tag{17}$$

since as just discussed $\frac{\partial(f_p+f_p^1)}{\partial v^r} = r\frac{\partial f_p^1}{\partial v^r} + \frac{\partial f_p^r}{\partial v^r}$. For paths $p \notin \overline{\mathcal{P}}^r$, $\frac{\partial f_p^r}{\partial v^r} = 0$. Substituting in the value of $\frac{\partial L^1(f)}{\partial v^r}$ from Lemma 22 into 17) yields

$$\frac{\partial f_p^1}{\partial v^r} \;=\; \frac{1}{r}\frac{\|\mathbf{W}_p^1\|}{\|\mathbf{W}^1\|} \quad . \tag{18}$$

Thus for $p \in \overline{\mathcal{P}}^1 \setminus \overline{\mathcal{P}}^r$, $\frac{\partial f_p}{\partial v^r} = (r-1)\frac{\partial f_p^1}{\partial v^r} = \frac{r-1}{r}\frac{\|\mathbf{W}_p^1\|}{\|\mathbf{W}^1\|}$. For a path $p \in \overline{\mathcal{P}}^r$, we add $\frac{r-1}{r}$ times (17) to $\frac{1}{r}$ times (16) to obtain

$$\frac{\partial f_p}{\partial v^r} = (r-1)\frac{\partial f_p^1}{\partial v^r} + \frac{\partial f_p^r}{\partial v^r} = \frac{r-1}{r}\frac{\|\mathbf{W}_p^1\|}{\|\mathbf{W}^1\|} + \frac{1}{r}\frac{\|\mathbf{W}_p^r\|}{\|\mathbf{W}^r\|} \ ,$$

where the first equality follows from Lemma 14. □

Claim 31 *At equilibrium,* $\dfrac{v^1 - v^r}{\|\mathbf{W}^1\|} \leq L^1(f) - L^r(f)$.

Proof. For each path $p \in \overline{\mathcal{P}}^1$, $L_p^1(f) - L_p^r(f) = \sum_{e\in p} a_e(f_e^1 - f_e^r) = \sum_{q\in\overline{\mathcal{P}}^1} u_{pq}(f_q^1 - f_q^r)$. Hence,

$$\begin{aligned}(L^1(f) - L^r(f)) + (L^r(f) - L_p^r(f)) &= \sum_{e\in p} a_e(f_e^1 - f_e^r) \\ &= \sum_{q\in\overline{\mathcal{P}}^1} u_{pq}(f_q^1 - f_q^r) \ . \end{aligned} \tag{19}$$

The system of equations for all $p \in \overline{\mathcal{P}}^1$ is

$$\mathbf{1}_{|\overline{\mathcal{P}}^1|}(L^1(f) - L^r(f)) - \left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1} = \mathbf{U}^1\left[f_p^1 - f_q^r\right]_{p\in\mathcal{P}^1} . \tag{20}$$

Left multiplying both sides of (20) by $\mathbf{1}^T\mathbf{W}^1$ yields

$$\|\mathbf{W}^1\|(L^1(f) - L^r(f)) - \sum_{p\in\overline{\mathcal{P}}^1}(L_p^r(f) - L^r(f))\|\mathbf{W}_p^1\| = v^1 - v^r. \tag{21}$$

We show that the second term on the left is nonnegative, which finishes the proof. Left-multiply both sides of (20) by the row vector $\left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1}^T \mathbf{W}^1$ to obtain

$$\begin{aligned}\left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1}^T \mathbf{W}^1\mathbf{1}_{|\overline{\mathcal{P}}^1|}(L^1(f) - L^r(f)) \ - \ \left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1}^T \mathbf{W}^1\left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1} \\ = \left[L_p^r(f) - L^r(f)\right]_{p\in\overline{\mathcal{P}}^1}^T\left[f_p^1 - f_p^r\right]_{p\in\overline{\mathcal{P}}^1} .\end{aligned}$$

Since $\mathbf{W}^1$ is positive definite, this implies

$$\begin{aligned}(L^1(f) - L^r(f))\sum_{p\in\overline{\mathcal{P}}^1}(L_p^r(f) - L^r(f))\|\mathbf{W}_p^1\| &\geq \sum_{p\in\overline{\mathcal{P}}^1}(L_p^r(f) - L^r(f))(f_p^1 - f_p^r) \\ &= \sum_{p\in\overline{\mathcal{P}}^1}(f_p^1 - f_p^r)\sum_{e\in p} L_e^r(f) + L^r(f)\sum_{p\in\overline{\mathcal{P}}^1}(f_p^1 - f_p^r) \\ &= \sum_{e\in\overline{\mathcal{P}}^1} L_e^r(f)\sum_{p\in\overline{\mathcal{P}}^1 : e\in p}(f_p^1 - f_p^r) + L^r(f)(v^1 - v^r) \\ &= \sum_{e\in\overline{\mathcal{P}}^1} L_e^r(f)(f_e^1 - f_e^r) + L^r(f)(v^1 - v^r) \\ &\geq 0 \ .\end{aligned}$$

where the last inequality follows from Lemma 6 and observing that $e \in \overline{\mathcal{P}}^1 \Leftrightarrow e \in E^1$. Since $L^1(f) \geq L^r(f)$, this shows $\sum_{p\in\overline{\mathcal{P}}^1}(L^r_p(f) - L^r(f))\|\mathbf{W}^1_p\| \geq 0$. □

Proof of Lemma 23. We first put the expression in terms of path flows:

$$\begin{aligned}
\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} &= \sum_{e\in E^1} a_e f_e^1 \sum_{p\in\overline{\mathcal{P}}^1:e\in p} \frac{\partial f_p}{\partial v^r} = \sum_{p\in\overline{\mathcal{P}}^1} \frac{\partial f_p}{\partial v^r} \sum_{e\in p} a_e f_e^1 \\
&= \sum_{p\in\overline{\mathcal{P}}^1} \frac{\partial f_p}{\partial v^r} \sum_{e\in p} \sum_{q\in\overline{\mathcal{P}}^1:e\in q} a_e f_q^1 = \sum_{p\in\overline{\mathcal{P}}^1} \frac{\partial f_p}{\partial v^r} \sum_{q\in\overline{\mathcal{P}}^1} \sum_{e\in p\cap q} a_e f_q^1 \\
&= \sum_{p\in\overline{\mathcal{P}}^1} \sum_{q\in\overline{\mathcal{P}}^1} \frac{\partial f_p}{\partial v^r} u_{pq} f_q^1 \ .
\end{aligned}$$

Substituting the value of $\frac{\partial f_p}{\partial v^r}$ from Claim 30,

$$\begin{aligned}
\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} &= \sum_{p\in\overline{\mathcal{P}}^r} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \left(\frac{r-1}{r} \frac{\|\mathbf{W}^1_p\|}{\|\mathbf{W}^1\|} + \frac{1}{r} \frac{\|\mathbf{W}^r_p\|}{\|\mathbf{W}^r\|} \right) + \sum_{p\in\overline{\mathcal{P}}^1\setminus\overline{\mathcal{P}}^r} \sum_{q\in\mathcal{P}^1} u_{pq} f_q^1 \frac{r-1}{r} \frac{\|\mathbf{W}^1_p\|}{\|\mathbf{W}^1\|} \\
&= \sum_{p\in\overline{\mathcal{P}}^r} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \frac{1}{r} \frac{\|\mathbf{W}^r_p\|}{\|\mathbf{W}^r\|} + \sum_{p\in\overline{\mathcal{P}}^1} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \frac{r-1}{r} \frac{\|\mathbf{W}^1_p\|}{\|\mathbf{W}^1\|} \ .
\end{aligned}$$

Since $\mathbf{U}^1\mathbf{W}^1 = \mathbf{I}$, for all paths $q \in \overline{\mathcal{P}}^1$, $\sum_{p\in\overline{\mathcal{P}}^1} u_{pq}\|\mathbf{W}^1_p\| = 1$. Hence,

$$\begin{aligned}
\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} &= \sum_{p\in\overline{\mathcal{P}}^r} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \frac{1}{r} \frac{\|\mathbf{W}^r_p\|}{\|\mathbf{W}^r\|} + \frac{r-1}{r} \sum_{q\in\overline{\mathcal{P}}^1} f_q^1 \frac{1}{\|\mathbf{W}^1\|} \\
&= \sum_{p\in\overline{\mathcal{P}}^r} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \frac{1}{r} \frac{\|\mathbf{W}^r_p\|}{\|\mathbf{W}^r\|} + \frac{r-1}{r} \frac{v^1}{\|\mathbf{W}^1\|} \qquad (22)
\end{aligned}$$

We will now show that $\sum_{p\in\overline{\mathcal{P}}^r} \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 \|\mathbf{W}^r_p\| = v^r + \|\mathbf{W}^r\|(L^1(f) - L^r(f))$. For any path $p \in \overline{\mathcal{P}}^r$, $L^1(f) - L^r(f) = \sum_{e\in p} a_e(f_e^1 - f_e^r)$. Since $f_e^1 = \sum_{q\in\overline{\mathcal{P}}^1:e\in q} f_q^1$ and $f_e^r = \sum_{q\in\overline{\mathcal{P}}^r:e\in q} f_q^r$, we can rewrite this in terms of paths as

$$L^1(f) - L^r(f) = \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f_q^1 - \sum_{q\in\overline{\mathcal{P}}^r} u_{pq} f_q^r \qquad \forall p \in \overline{\mathcal{P}}^r.$$

Since this is true for all paths $p \in \overline{\mathcal{P}}^r$, we can rewrite this as the following matrix equation. Let $\mathbf{U}^{r,1}$ be the submatrix of $\mathbf{U}^1$ with rows corresponding to paths in $\overline{\mathcal{P}}^r$ and columns corresponding to paths in $\overline{\mathcal{P}}^1$.

$$\mathbf{1}_{|\overline{\mathcal{P}}^r|}(L^1(f) - L^r(f)) = \mathbf{U}^{r,1} \left[f_q^1\right]_{q\in\mathcal{P}^1} - \mathbf{U}^r \left[f_q^r\right]_{q\in\overline{\mathcal{P}}^r} \ . \qquad (23)$$

Left-multiplying both sides of (23) by $\mathbf{1}^T\mathbf{W}^r$,

$$\|\mathbf{W}^r\|(L^1(f) - L^r(f)) \;=\; \sum_{p\in\overline{\mathcal{P}}^r} \|\mathbf{W}^r_p\| \sum_{q\in\overline{\mathcal{P}}^1} u_{pq} f^1_q - v^r \;\; . \tag{24}$$

Combining (22) and (24) and Claim 31 yields

$$\begin{aligned}\sum_{e\in E^1} a_e f^1_e \frac{\partial f_e}{\partial v^r} \;&\geq\; \frac{1}{r}\left(\frac{v^1 - v^r}{\|\mathbf{W}^1\|} + \frac{v^r}{\|\mathbf{W}^r\|}\right) + \frac{r-1}{r}\frac{v^1}{\|\mathbf{W}^1\|} \\ &=\; \frac{v^1}{\|\mathbf{W}^1\|} - \frac{v^r}{r}\left(\frac{1}{\|\mathbf{W}^1\|} - \frac{1}{\|\mathbf{W}^r\|}\right).\end{aligned}$$

□

A.8 An Example When the Nesting Property Fails

We give a simple example (Figure 2) of two players with linear delays in a Braess graph to show this. In the example, the total delay at Nash equilibrium when the players have unequal flow values exceeds the total delay when the flow value of the players is equal, therefore at some point during flow redistribution the cost must increase.

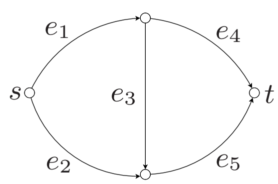

Fig. 1. The Braess Graph

		$v^1 = 0.5,\ v^2 = 0.5$		$v^1 = 0.8,\ v^2 = 0.2$	
Edge	Delay function	f^1_e	f^2_e	f^1_e	f^2_e
1	$7x$	0.2576	0.2576	0.4	0.1455
2	$2x + 6$	0.2424	0.2424	0.4	0.0545
3	$x + 2$	0.0152	0.0152	0	0.0909
4	$2x + 6$	0.2424	0.2424	0.4	0.0545
5	$7x$	0.2576	0.2576	0.4	0.1455
Total delay		10.5354		10.6364	

Fig. 2. An example where the flow redistribution strategy does not work

A.9 Proof of Theorem 2

Our proof relies on the following lemma.

Lemma 32. *In a series-parallel graph with s-t flows f and g, if $|f| > |g|$ then there exists another s-t flow h of value $|f| - |g|$, so that on every edge $e \in E$ with $h_e > 0$,*

$$g_e + h_e \leq f_e. \tag{25}$$

Proof of Theorem 2: Let f be the equilibrium flow of k players and o be the optimal flow of the same value. By Lemma 32, for player i, there exists a different strategy $\tilde{f}^i$ so that $\tilde{f}^i_e + f^{-i}_e \leq o_e$, if $\tilde{f}^i_e > 0$. Since f is a Nash equilibrium, player i cannot change his strategy from f^i to $\tilde{f}^i$ to decrease his delay. Therefore,

$$C^i(f) = C^i(f^i, f^{-i}) \leq C^i(\tilde{f}^i, f^{-i}) = \sum_{e \in G} \tilde{f}_e l_e(\tilde{f}^i_e + f^{-i}_e) \leq \sum_{e \in G} o_e l_e(o_e) = C(o).$$

Summing the above inequality over all k players gives us the bound on the price of anarchy claimed. □

Proof of Lemma 32: We prove by induction on the size of the series-parallel graph G. In the base case G is a single edge and the lemma is trivial. For the induction step, assume that G is constructed from two subgraphs (G_1, s_1, t_1) and (G_2, s_2, t_2).

Suppose that G is a series-composition of G_1 and G_2 by merging t_1 and s_2. By induction hypothesis, there exists a flow h^1 from s_1 to t_1 and a flow h^2 from s_2 to t_2, both of which are of values $|f| - |g|$, so that for all edges $e \in G_1 \cup G_2$, (25) holds. The concatenation of h^1 and h^2 gives another flow of value $|f| - |g|$, and still satisfies (25).

Next suppose that G is a parallel composition of G_1 and G_2 by merging s_1 and s_2 and by merging t_1 and t_2. We denote the sub-flow of f going through G_i as $f_{|G_i}$. If $v(f|_{G_1}) \geq v(g|_{G_1})$ and $v(f|_{G_2}) \geq v(g|_{G_2})$, then by induction hypothesis, there exists a flow h^1 of value $v(f|_{G_1}) - v(g|_{G_1})$ in G_1 and another flow h^2 of value $v(f|_{G_2}) \geq v(g|_{G_2})$ in G_2 so that (25) holds. It is easy to see that $h^1 + h^2$ is the flow we want.

We are left with the case that $v(f|_{G_1}) \leq v(g|_{G_1})$ and $v(f|_{G_2}) > v(f|_{G_2})$. By induction hypothesis, there exists a flow h^2 of value $v(f|_{G_2}) - v(g|_{G_2})$ and it satisfies (25) in G_2. We now create another flow h from h^2 by scaling:

$$h = \frac{|f| - |g|}{v(f|_{G_2}) - v(g|_{G_2})} h^2.$$

(Note that $|f| - |g| \leq v(f|_{G_2}) - v(g|_{G_2})$.) It is clear that h is also a flow and its value is $|f| - |g|$. Furthermore, for $e \in G_1$, $h_e = 0$ and (25) holds trivially, and for $e \in G_2$, if $h_e > 0$, we have $g_e \geq f_e + h^2_e \geq f_e + h_e$, still satisfying (25). □