The Price of Collusion in Series-Parallel Networks*

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Abstract. We study the quality of equilibrium in atomic splittable routing games. We show that in single-source single-sink games on series-parallel graphs, the *price of collusion* — the ratio of the total delay of atomic Nash equilibrium to the Wardrop equilibrium — is at most 1. This proves that the existing bounds on the price of anarchy for Wardrop equilibria carry over to atomic splittable routing games in this setting.

1 Introduction

In a *routing game*, players have a fixed amount of flow which they route in a network [16, 18, 24]. The flow on any edge in the network faces a delay, and the delay on an edge is a function of the total flow on that edge. We look at routing games in which each player routes flow to minimize his own delay, where a player's delay is the sum over edges of the product of his flow on the edge and the delay of the edge. This objective measures the average delay of his flow and is commonly used in traffic planning [11] and network routing [16].

Routing games are used to model traffic congestion on roads, overlay routing on the Internet, transportation of freight, and scheduling tasks on machines. Players in these games can be of two types, depending on the amount of flow they control. Nonatomic players control only a negligible amount of flow, while atomic players control a larger, non-negligible amount of flow. Further, atomic players may or may not be able to split their flow along different paths. Depending on the players, three types of routing games are: games with (i) nonatomic players, (ii) atomic players who pick a single path to route their flow, and (iii) atomic players who can split their flow along several paths. These are *nonatomic* [21, 22, 24], *atomic unsplittable* [3, 10] and *atomic splittable* [8, 16, 19] routing games respectively. We study atomic splittable routing games in this work. These games are less well-understood than either nonatomic or atomic unsplittable routing games. One significant challenge here is that, unlike most other routing games, each player has an infinite strategy space. Further, unlike nonatomic routing games, the players are asymmetric since each player has different flow value.

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An equilibrium flow in a routing game is a flow where no single player can change his flow pattern and reduce his delay. Equilibria are of interest since they are a stable outcome of games. In both atomic splittable and nonatomic routing games, equilibria exist under mild assumptions on the delay functions [4, 17]. We refer to equilibria in atomic splittable games as *Nash* equilibria and in nonatomic games as *Wardrop* equilibria [24]. While the Wardrop equilibrium is known to be essentially unique [24], atomic splittable games can have multiple equilibria [5].

One measure of the quality of a flow is the *total delay* of the flow: the sum over all edges of the product of the flow on the edge and the induced delay on the edge. For routing games, one conern is the degradation in the quality of equilibrium flow caused by lack of central coordination. This is measured by the *price of anarchy* of a routing game, defined as the ratio of the total delay of worst-case equilibrium in a routing game to the total delay of the flow that minimizes the total delay. Tight bounds on the price of anarchy are known for nonatomic routing games [20], and are extensively studied in various settings [8, 9, 20, 21, 19, 22, 23]. In [13], Hayrapetyan et al. consider the total delay of nonatomic routing games when nonatomic players form cost-sharing coalitions. These coalitions behave as atomic splittable players. Hayrapetyan et al. introduce the notion of *price of collusion* as a measure of the price of forming coalitions. For an atomic splittable routing game the price of collusion is defined as the ratio of the total delay of the worst Nash equilibrium to the Wardrop equilibrium. Together, a bound α on the price of anarchy for nonatomic routing games and a bound β on the price of collusion for an atomic splittable routing game, imply the price of anarchy for the atomic splittable routing game is bounded by $\alpha\beta$.

For atomic splittable routing games, bounds on the price of anarchy are obtained in [8, 12]. These bounds do not match the best known lower bounds. Bounds on the price of collusion in general also remains an open problem. Previously, the price of collusion has been shown to be 1 only in the following special cases: in the graph consisting of parallel links [13]; when the players are symmetric, i.e. each player has the same flow value and the same source and sink [8]; and when all delay functions are monomials of a fixed degree [2]. Conversely, if there are multiple sources and sinks, the total delay of Nash equilibrium can be worse than the Wardrop equilibrium of equal flow value, i.e., the price of collusion can exceed 1, even with linear delays [7, 8].

Our Contribution. Let $\mathcal{C}$ denote the class of differentiable nondecreasing convex functions. We prove the following theorem for atomic splittable routing games.

Theorem 1. *In single source-destination routing games on series-parallel graphs with delay functions drawn from the class $\mathcal{C}$, the price of collusion is 1.*

We first consider the case when all delays are affine. We show that in the case of affine delays in the setting described above, the total delay at equilibrium is largest when the players are symmetric, i.e. all players have the same flow value

(Section 3). To do this, we first show that the equilibrium flow for a player i remains unchanged if we modify the game by changing slightly the value of flow of any player with larger flow value than player i. Then starting from a game with symmetric players, we show that if one moves flow from a player i evenly to all players with higher flow value the cost of the corresponding equilibrium flow never increases. Since it is known that the price of collusion is 1 if the players are symmetric [8], this shows that the bound extends to arbitrary players with affine delays.

In Section 4, we extend the result for general convex delays, by showing that the worst case price of collusion is obtained when the delays are affine.

In contrast to Theorem 1 which presents a bound on the price of collusion, we also present a new bound on the price of anarchy of atomic splittable routing games in series-parallel graphs.

Theorem 2. *In single source-destination routing games on series-parallel graphs, the price of anarchy is bounded by k, the number of players.*

This bound was proven earlier for parallel links in [12]. For nonatomic routing games bounds on the price of anarchy depend on the delay functions in the graph and, in the case of polynomial delays, the price of anarchy is bounded by $O(d/\log d)$. These bounds are known to be tight even on simple graphs consisting of 2 parallel links [20]. Theorem 2 improves on the bounds obtained by Theorem 1 when $k \leq d/\log d$. All missing proofs are contained in the full version [6].

2 Preliminaries

Let $G = (V, E)$ be a directed graph, with two special vertices s and t called the *source* and *sink*. The vector f, indexed by edges $e \in E$, is defined as a *flow* of *value* v if the following conditions are satisfied.

$$\sum_w f_{uw} - \sum_w f_{wu} = 0, \quad \forall u \in V - \{s, t\} \tag{1}$$

$$\sum_w f_{sw} - \sum_w f_{ws} = v \tag{2}$$

$$f_e \geq 0, \quad \forall e \in E \ .$$

Here f_{uw} represents the flow on arc (u, w). If there are several flows f^1, f^2, $\cdots$, f^k, we define $f := (f^1, f^2, \cdots, f^k)$ and f^{-i} is the vector of the flows except f^i. In this case the flow on an edge $f_e = \sum_{i=1}^k f_e^i$.

Let $\mathcal{C}$ be the class of differentiable nondecreasing convex functions. Each edge e is associated with a delay function $l_e : \mathcal{R}^+ \to \mathcal{R}$ drawn from $\mathcal{C}$. Note that we allow delay functions to be negative. For a given flow f, the *induced delay* on edge e is $l_e(f_e)$. We define the *total delay* on an edge e as the product of the flow on the edge and the induced delay $C_e(f_e) := f_e l_e(f_e)$. The *marginal delay* on an edge e is the rate of change of the total delay: $L_e(f_e) := f_e l'_e(f_e) + l_e(f_e)$. The total delay of a flow f is $C(f) = \sum_{e \in E} f_e l_e(f_e)$.

An *atomic splittable routing game* is a tuple $(G,\boldsymbol{v},\boldsymbol{l},s,t)$ where $\boldsymbol{l}$ is a vector of delay functions for edges in G and $\boldsymbol{v} = (v^1,v^2,\cdots,v^k)$ is a tuple indicating the flow value of the players from 1 to k. We always assume that the players are indexed by the order of decreasing flow value, hence $v^1 \geq v^2 \cdots \geq v^k$. All players have source s and destination t. Player i has a strategy space consisting of all possible s-t flows of volume v^i. Let $(f^1, f^2, \cdots, f^k)$ be a strategy vector. Player i incurs a delay $C^i_e(f^i_e, f_e) := f^i_e l_e(f_e)$ on each edge e, and his objective is to minimize his delay $C^i(f) := \sum_{e \in E} C^i_e(f^i_e, f_e)$. A set of players are *symmetric* if each player has the same flow value.

A flow is a *Nash equilibrium* if no player can unilaterally alter his flow and reduce his delay. Formally,

Definition 3 (Nash Equilibrium). *In an atomic splittable routing game, flow f is a* Nash equilibrium *if and only if for every player i and every s-t flow g of volume v^i, $C^i(f^i, f^{-i}) \leq C^i(g, f^{-i})$.*

For player i, the marginal delay on edge e is defined as the rate of change of his delay on the edge $L^i_e(f^i_e, f_e) := l_e(f_e) + f^i_e l'_e(f_e)$. For any s-t path p, the marginal delay on path p is defined as the rate of change of total delay of player i when he adds flow along the edges of the path: $L^i_p(f) := \sum_{e \in p} L^i_e(f^i_e, f_e)$. The following lemma follows from Karush-Kuhn-Tucker optimality conditions for convex programs [15] applied to player i's minimization problem.

Lemma 4. *Flow f is a Nash equilibrium flow if and only if for any player i and any two directed paths p and q between the same pair of vertices such that on all edges $e \in p$, $f^i_e > 0$, then $L^i_p(f) \leq L^i_q(f)$.*

By Lemma 4, at equilibrium the marginal delay of a player is the same on any s-t path on every edge of which he has positive flow. For a player i, the *marginal delay* is $L^i(f) := L^i_p(f)$, where p is any s-t path on which player i has positive flow on every edge.

For a given flow f and for every player i, we let $E^i(f) = \{e | f^i_e > 0\}$. $\mathcal{P}^i$ is the set of all directed s-t paths p on which for every $e \in p$, $f^i_e > 0$. We will use $e \in \mathcal{P}^i$ to mean that the edge e is in some path $p \in \mathcal{P}^i$; then $e \in \mathcal{P}^i \Leftrightarrow e \in E^i$. Let p be a directed simple s-t path. A *path flow* on path p is a directed flow on p of value f_p. A *cycle flow* along cycle C is a directed flow along C of value f_C. Any flow f can be decomposed into a set of directed path flows and directed cycle flows $\{f_p\}_{p \in \mathcal{P}} \cup \{f_c\}_{c \in C}$, [1]. This is a *flow decomposition* of f. Directed cycle flows cannot exist in atomic splittable or nonatomic games (this follows easily from Lemma 4). Thus, f^i in these games can be expressed as a set of path flows $\{f^i_p\}_{p \in \mathcal{P}^i}$ such that $f^i_e = \sum_{p \in \mathcal{P}^i : e \in p} f^i_p$. This is a *path flow decomposition* of the given flow. A *generalized path flow decomposition* is a flow decomposition along paths where we allow the path flows to be negative.

Series-Parallel Graphs. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and vertices $v_1 \in V_1$, $v_2 \in V_2$, the operation merge(v_1, v_2) creates a new graph $G' = (V' = V_1 \cup V_2, E' = E_1 \cup E_2)$, replaces v_1 and v_2 in V' with a single vertex v and replaces each edge $e = (u, w) \in E'$ incident to v_1 or v_2 by an edge incident to v, directed in the same way as the original edge.

Definition 5. *A tuple (G, s, t) is* series-parallel *if G is a single edge $e = (s, t)$, or is obtained by a series or parallel composition of two series-parallel graphs (G_1, s_1, t_1) and (G_2, s_2, t_2). Nodes s and t are* terminals *of G.*
(i) Parallel Composition*: $s = merge(s_1, s_2)$, $t = merge(t_1, t_2)$,*
(ii) Series Composition*: $s := s_1$, $t := t_2$, $v = merge(s_2, t_1)$.*

In directed series-parallel graphs, all edges are directed from the source to the destination and the graph is acyclic in the directed edges. This is without loss of generality, since any edge not on an s-t path is not used in an equilibrium flow, and no flow is sent along a directed cycle. The following lemma describes a basic property of flows in a directed series-parallel graph.

Lemma 6. *Let $G = (V, E)$ be a directed series-parallel graph with terminals s and t. Let h be an s-t flow of value $|h|$, and c is a function defined on the edges of the graph G. (i) If $\sum_{e \in p} c(e) \geq \kappa$ on every s-t path p, then $\sum_{e \in E} c(e) h_e \geq \kappa |h|$. (ii) If $\sum_{e \in p} c(e) = \kappa$ on every s-t paths p then $\sum_{e \in E} c(e) h_e = \kappa |h|$.*

Vectors and matrices in the paper, except for flow vectors, will be referred to using boldface. $\mathbf{1}$ and $\mathbf{0}$ refer to the column vectors consisting of all ones and all zeros respectively. When the size of the vector or matrix is not clear from context, we use a subscript to denote it, e.g. $\mathbf{1}_n$.

Uniqueness of Equilibrium Flow. The equilibria in atomic splittable and nonatomic routing games are known to be unique for affine delays, up to induced delays on the edges (this is true for a larger class of delays [4], [17], but here we only need affine delays). Although there may be multiple equilibrium flows, in each of these flows the delay on an edge remains unchanged. If the delay functions are strictly increasing, then the flow on each edge is uniquely determined. However with constant delays, for two parallel links between s and t with the same constant delay on each edge, any valid flow is an equilibrium flow. In this paper, we assume only that the delay functions are differentiable, nondecreasing and convex, hence we allow edges to have constant delays. We instead assume that in the graph, between any pair of vertices, there is at most one path on which all edges have constant delay. This does not affect the generality of our results. In graphs without this restriction there are Nash and Wardrop equilibrium flows in which for every pair of vertices, there is at most one constant delay path which has flow in either equilibrium. To see this, consider any equilibrium flow in a general graph. For every pair of vertices with more than one constant delay path between them, only the minimum delay path will be used at equilibrium. If there are multiple minimum constant delay paths, we can shift all the flow onto a single path; this does not affect the marginal delay of any player on any path, and hence the flow is still an equilibrium flow.

Lemma 7. *For atomic splittable and nonatomic routing games on series-parallel networks with affine delays and at most one path between any pair of vertices with constant delays on all edges, the equilibrium flow is unique.*

For technical reasons, for proving Theorem 1 we also require that every s-t path in the graph have at least one edge with strictly increasing delay. We modify the graph in the following way: we add a single edge e in series with graph G, with delay function $l_e(x) = x$. It is easy to see that for any flow, this increases the total delay by exactly v^2 where v is the value of the flow, and does not change the value of flow on any edge at equilibrium. In addition, if the price of collusion in the modified graph is less than one, then the price of collusion in the original graph is also less than one. The proof of Theorem 2 does not use this assumption.

3 Equilibria with Affine Delays

In this section we prove Theorem 1 where all delays are affine functions of the form $l_e(x) = a_e x + b_e$. Our main result in this section is:

Theorem 8. *In a series-parallel graph with affine delay functions, the total delay of a Nash equilibrium is bounded by that of a Wardrop equilibrium of the same total flow value.*

We first present the high-level ideas of our proof. Given a series-parallel graph G, terminals s and t, and edge delay functions $\boldsymbol{l}$, let $f(\cdot) : \mathbb{R}_+^k \to \mathbb{R}_+^{m \times k}$ denote the function mapping a vector of flow values to the equilibrium flow in the atomic splittable routing game. By Lemma 7, the equilibrium flow is unique and hence the function $f(\cdot)$ is well-defined. Let $(G, \boldsymbol{u}, \boldsymbol{l}, s, t)$ be an atomic splittable routing game. Our proof consists of the following three steps:

Step 1. Start with $v^i = \sum_{j=1}^{k} u^j / k$ for each player i, i.e. the players are symmetric.
Step 2. Gradually adjust the flow values $\boldsymbol{v}$ of the k players so that the total delay of the equilibrium flow $f(\boldsymbol{v})$ is monotonically nonincreasing.
Step 3. Stop the flow redistribution process when for each i, $v^i = u^i$.

In step 1, we make use of a result of Cominetti et al. [8].

Lemma 9. [8] *Let $(G, \boldsymbol{v}, \boldsymbol{l}, s, t)$ denote an atomic splittable routing game with k symmetric players. Let g be a Wardrop equilibrium of the same flow value $\sum_{i=1}^{k} v^i$. Then $C(f(\boldsymbol{v})) \le C(g)$.*

Step 2 is the heart of our proof. The flow redistribution works as follows. Let v^i denote the current flow value of player i. Initially, each player i has $v^i = \sum_{j=1}^{k} u^j/k$. Consider each player in turn from k to 1. We decrease the flow of the kth player and give it *evenly* to the first $k-1$ players until $v^k = u^k$. Similarly, when we consider the rth player, for any $r < k$, we decrease v^r and give the flow evenly to the first $r-1$ players until $v^r = u^r$. Throughout the following discussion and proofs, player r refers specifically to the player whose flow value is currently being decreased in our flow redistribution process.

Our flow redistribution strategy traces out a curve $\mathcal{S}$ in $\mathbb{R}_+^k$, where points in the curve correspond to flow value vectors $\boldsymbol{v}$.

Lemma 10. *For all $e \in E$, $i \in [k]$, the function $f(\boldsymbol{v})$ is continuous and piecewise linear along the curve $\mathcal{S}$, with breakpoints occurring where the set of edges used by any player changes.*

In what follows, we consider expressions of the form $\frac{\partial J(f(\boldsymbol{v}))}{\partial v^i}$, where J is some differentiable function defined on a flow (e.g., the total delay, or the marginal delay along a path). The expression $\frac{\partial J(f(\boldsymbol{v}))}{\partial v^i}$ considers the change in the function $J(\cdot)$ evaluated at the equilibrium flow, as the flow value of player i changes by an infinitesimal amount, keeping the flow values of the other players constant. Though $f(\boldsymbol{v})$ is not differentiable at all points in $\mathcal{S}$, $\mathcal{S}$ is continous. Therefore, it suffices to look at the intervals between these breakpoints of $\mathcal{S}$. In the rest of the paper, we confine our attention to these intervals.

We show that when the flow values are adjusted as described, the total delay is monotonically nonincreasing.

Lemma 11. *In a series-parallel graph, suppose that $v^1 = v^2 = \cdots = v^{r-1} \geq v^r \geq \cdots \geq v^k$. If $i < r$, then $\frac{\partial C(f(\boldsymbol{v}))}{\partial v^i} \leq \frac{\partial C(f(\boldsymbol{v}))}{\partial v^r}$.*

Proof of Theorem 8. By Lemma 9, the equilibrium flow in Step 1 has total delay at most the delay of the Wardrop equilibrium. We show below that during step 2, $C(f(\boldsymbol{v}))$ does not increase. Since the total volume of flow remains fixed, the Wardrop equilibrium is unchanged throughout. Thus, the price of collusion does not increase above 1, and hence the final equilibrium flow when $\boldsymbol{v} = \boldsymbol{u}$ also has this property.

Let $\boldsymbol{v}$ be the current flow values of the players. Since $C(f(\boldsymbol{v}))$ is a continuous function of $\boldsymbol{v}$ (Lemma 10), it is sufficient to show that the $C(f(\boldsymbol{v}))$ does not increase between breakpoints. Define $\boldsymbol{x}$ as follows: $\boldsymbol{x}^r = -1$; $\boldsymbol{x}^i = 0$, if $i > r$; and $\boldsymbol{x}^i = \frac{1}{r-1}$, if $1 \leq i < r$. The vector $\boldsymbol{x}$ is the rate of change of $\boldsymbol{v}$ when we decrease the flow of player r in Step 2. Thus, using Lemma 11, the change in total delay between two breakpoints in $\mathcal{S}$ satisfies

$$\lim_{\delta \to 0} \frac{C(f(\boldsymbol{v} + \delta \boldsymbol{x})) - C(f(\boldsymbol{v}))}{\delta} \;=\; -\frac{\partial C(f(\boldsymbol{v}))}{\partial v^r} + \sum_{i=1}^{r-1} \frac{\partial C(f(\boldsymbol{v}))}{\partial v^i} \frac{1}{r-1} \;\leq\; 0 \;.$$

□

The proof of Lemma 11 is described in Section 3.2. Here we highlight the main ideas. To simplify notation, when the vector of flow values is clear from the context, we use f instead of $f(\boldsymbol{v})$ to denote the equilibrium flow.

By chain rule, we have that $\frac{C(f)}{\partial v^i} = \sum_{e \in E} \frac{\partial L_e(f_e)}{\partial f_e} \frac{\partial f_e}{\partial v^i}$. The exact expressions of $\frac{\partial C(f)}{\partial v^i}$, for $1 \leq i \leq r$, are given in Lemmas 18 and 19 in Section 3.2. Our derivations use the fact that it is possible to simplify the expression $\frac{\partial f_e}{\partial v^i}$ using the following "nesting property" of a series-parallel graph.

Definition 12. *A graph G with delay functions $\boldsymbol{l}$, source s, and destination t satisfies the* nesting property *if all atomic splittable routing games on G satisfy the following condition: for any players i and j with flow values v^i and v^j, $v^i > v^j$ if and only if on every edge $e \in E$, for the equilibrium flow f, either $f_e^i = f_e^j = 0$ or $f_e^i > f_e^j$.*

Lemma 13 ([5]). *A series-parallel graph satisfies the nesting property for any choice of non-decreasing, convex delay functions.*

If a graph satisfies the nesting property, symmetric players have identical flows at equilibrium. When the flow value of player r is decreased in Step 2, the first $r-1$ players are symmetric. Thus, by Lemma 13, these players have identical flows at equilibrium. Hence, for any player $i < r$, $f_e^i = f_e^1$ and $L_e^i(f_e^i, f_e) = L_e^1(f_e^1, f_e)$ for any edge e. With affine delays, the nesting property has the following implication.

Lemma 14 (Frozen Lemma). *Let f be an equilibrium flow in an atomic splittable routing game $(G,\boldsymbol{v},\boldsymbol{l},s,t)$ with affine delays on the edges, and assume that the nesting property holds for $(G,\boldsymbol{l},s,t)$. Then for all players j, $j \neq i$ with $E^j(f) \subseteq E^i(f)$ and all edges e, $\dfrac{\partial f_e^j}{\partial v^i} = 0$.*

The frozen lemma has two important implications for our proof. Firstly, in Step 2, players $r+1, \cdots, k$ will not change their flow at equilibrium. Secondly, this implies a simple expression for $\frac{\partial f_e}{\partial v^i}$, $1 \leq i \leq r$,

$$\frac{\partial f_e}{\partial v^r} = \sum_{i=1}^{k} \frac{\partial f_e^i}{\partial v^r} = (r-1)\frac{\partial f_e^1}{\partial v^r} + \frac{\partial f_e^r}{\partial v^r} \ . \tag{3}$$

$$\frac{\partial f_e}{\partial v^i} = \sum_{i=1}^{k} \frac{\partial f_e^i}{\partial v^i} = \frac{\partial f_e^i}{\partial v^i}, \quad \forall i < r \ . \tag{4}$$

3.1 Proof of Lemma 14 (Frozen Lemma)

By Lemma 10, we can assume that f is between the breakpoints of $\mathcal{S}$ and is thus differentiable.

Lemma 15. *If player h has positive flow on every edge of two directed paths p and q between the same pair of vertices, then $\frac{\partial L_p^h(f)}{\partial v^i} = \frac{\partial L_q^h(f)}{\partial v^i}$.*

Proof. Since f is an equilibrium, Lemma 4 implies that $L^h_p(f) = L^h_q(f)$. Differentiation of the two quantities are the same since f is maintained as an equilibrium. □

Lemma 16. *Let G be a directed acyclic graph. For an atomic splittable routing game $(G, \boldsymbol{v}, \boldsymbol{l}, s, t)$ with equilibrium flow f, let c and κ be defined as in Lemma 6. Then $\sum_{e \in E} c(e) \frac{\partial f^i_e(\boldsymbol{v})}{\partial v^j} = \kappa$ if $i = j$, and is zero otherwise.*

Proof. Define $\boldsymbol{x}$ as follows: $\boldsymbol{x}^j = 1$ and $\boldsymbol{x}^i = 0$ for $j \neq i$. Then

$$\begin{aligned}\sum_{e \in E} c(e) \frac{\partial f^i_e(\boldsymbol{v})}{\partial v^j} &= \sum_{e \in E} c(e) \left(\lim_{\delta \to 0} \frac{f^i_e(\boldsymbol{v} + \delta \boldsymbol{x}) - f^i_e(\boldsymbol{v})}{\delta} \right) \\ &= \lim_{\delta \to 0} \frac{\sum_{e \in E} c(e)(f^i_e(\boldsymbol{v} + \delta \boldsymbol{x}) - f^i_e(\boldsymbol{v}))}{\delta} \quad ,\end{aligned}$$

where the second equality is due to the fact that $f^i_e(\cdot)$ is differentiable.

For any two s-t flows f^i, g^i, it follows from Lemma 6 that $\sum_{e \in E} c(e)(f^i_e - g^i_e) = \kappa(|f^i| - |g^i|)$. If $i \neq j$ then $|f^i(\boldsymbol{v} + \delta \boldsymbol{x})| = |f^i(\boldsymbol{v})|$, hence $\sum_{e \in E} c(e)(f^i_e(\boldsymbol{v} + \delta \boldsymbol{x}) - f^i_e(\boldsymbol{v})) = 0$. If $i = j$, then $|f^i(\boldsymbol{v} + \delta \boldsymbol{x})| - |f^i(\boldsymbol{v})| = \delta$, implying that $\sum_{e \in E} c(e)(f^i_e(\boldsymbol{v} + \delta \boldsymbol{x}) - f^i_e(\boldsymbol{v})) = \kappa \delta$. The proof follows. □

Proof of Lemma 14. We prove by induction on the decreasing order of the index of j. We make use of the folllowing claim.

Claim 17 *Let $S^j = \{h : E^h(f) \supseteq E^j(f)\}$. For player j and an s-t path p on which j has positive flow,*

$$\begin{aligned}|S^j| \frac{\partial L^j_p(f)}{\partial v^i} - \sum_{h \in S^j \setminus \{j\}} \frac{\partial L^h_p(f)}{\partial v^i} &= (|S^j| + 1) \textstyle\sum_{e \in p} a_e \frac{\partial f^j_e}{\partial v^i} \\ &+ \textstyle\sum_{e \in p} a_e \frac{\partial \sum_{h : E^h(f) \subset E^j(f)} f^h_e}{\partial v^i} \quad .\end{aligned}$$

Proof. Given players i and h,

$$\frac{\partial L^h_p(f)}{\partial v^i} = \sum_{e \in p} a_e \frac{\partial (f_e + f^h_e)}{\partial v^i} \quad . \tag{5}$$

Summing (5) over all players h in $S^j \setminus \{j\}$ and subtract it from $|S^j|$ times (5) for player j gives the proof. □

Let $G^j = (V, E^j)$. By definition, all players $h \in S^j$ have flow on every s-t path in this graph. Lemma 15 implies that for any s-t paths p, q in G^j and any player $h \in S^j$, $\frac{\partial L^h_p(f)}{\partial v^i} = \frac{\partial L^h_q(f)}{\partial v^i}$. The expression on the left hand side of Claim 17 is thus equal for any path $p \in \mathcal{P}^j$, and therefore so is the expression on the right.

For the base case $j = k$, the set $\{h : E^h(f) \subset E^j(f)\}$ is empty. Hence, the second term on the right of Claim 17 is zero, and by the previous discussion, the quantity $\sum_{e \in p} a_e \frac{\partial f_e^k}{\partial v^i}$ is equal for any path $p \in \mathcal{P}^k$. Define $c(e) = a_e \frac{\partial f_e^k}{\partial v^i}$ for each $e \in E^k$ and $\kappa = \sum_{e \in p} a_e \frac{\partial f_e^k}{\partial v^i}$ for any s-t path p in G^k. By Lemma 16, $\sum_{e \in E^j(f)} c(e) \frac{\partial f_e^k}{\partial v^i} = \sum_{e \in E^j(f)} a_e \left(\frac{\partial f_e^k}{\partial v^i}\right)^2 = 0$. Hence, $\frac{\partial f_e^k}{\partial v^i} = 0$, $\forall e \in E$.

For the induction step $j < k$, due to the inductive hypothesis, $\frac{\partial f_e^h}{\partial v^i} = 0$ for $h > j$. Since by the nesting property if $E^h(f) \subset E^j(f)$ then $h > j$, the second term on the right of Claim 17 is again zero. By the same argument as in the base case, $\frac{\partial f_e^j}{\partial v^i} = 0$, for each $e \in E$, proving the lemma. □

3.2 Proof of Lemma 11

An unstated assumption for all lemmas in this section is that the nesting property holds. For the proof of Lemma 11, our first step is to express the rate of change of total delay in terms of the rate of change of marginal delay of the players, as the flow value of player r is being decreased. The next lemma gives this expression for the first $r - 1$ players.

Lemma 18. *For $f = f(\boldsymbol{v})$, and for each $i < r$,* $\frac{\partial C(f)}{\partial v^i} = L^i(f) + \frac{\partial L^i(f)}{\partial v^i} \frac{\sum_{j=2}^k v^j}{2}$.

Proof. For any player j, the set of edges used by player j is a subset of the edges used by player $i < r$, since player i has the largest flow value and we assume that the nesting property holds. Hence, the total delay at equilibrium $C(f) = \sum_{e \in E^i(f)} C_e(f_e)$.

$$\begin{aligned} \frac{\partial C(f)}{\partial v^i} &= \sum_{e \in E^i(f)} \frac{\partial C_e(f_e)}{\partial f_e} \frac{\partial f_e}{\partial v^i} = \sum_{e \in E^i(f)} (2a_e f_e + b_e) \frac{\partial f_e}{\partial v^i} \\ &= \sum_{e \in E^i(f)} \frac{\partial f_e}{\partial v^i} \left(L_e^i(f_e^i, f_e) + a_e \sum_{j \neq i} f_e^j \right) . \end{aligned} \tag{6}$$

By Lemma 16 with $c(e) = L_e^i(f_e^i, f_e)$ and $\kappa = L^i(f)$, $\sum_{e \in E^i} L_e^i(f_e^i, f_e) \frac{\partial f_e}{\partial v^i} = L^i(f)$. Thus, $\dfrac{\partial C(f)}{\partial v^i} = L^i(f) + \sum_{j \neq i} \sum_{e \in E^i} a_e f_e^j \dfrac{\partial f_e}{\partial v^i}$.

By (4), we have that $a_e \frac{\partial f_e}{\partial v^i} = \frac{1}{2} a_e \frac{\partial (f_e + f_e^i)}{\partial v^i} = \frac{1}{2} \frac{\partial L_e^i(f_e^i, f_e)}{\partial v^i}$. It follows that

$$\begin{aligned} \frac{\partial C(f)}{\partial v^i} &= L^i(f) + \frac{1}{2} \sum_{j \neq i} \sum_{e \in E^i} f_e^j \frac{\partial L_e^i(f_e^i, f_e)}{\partial v^i} \\ &= L^i(f) + \frac{1}{2} \sum_{j \neq i} \sum_{e \in E^i} \sum_{q \in \mathcal{P}^i : e \in q} f_q^j \frac{\partial L_e^i(f_e^i, f_e)}{\partial v^i} \quad , \end{aligned}$$

where the last equality is because for any player j, $f_e^j = \sum_{q\in\mathcal{P}^j:e\in q} f_q^j = \sum_{q\in\mathcal{P}^i:e\in q} f_q^j$, and the nesting property. Reversing the order of summation and observing that $\sum_{e\in p:p\in\mathcal{P}^i} \frac{\partial L_e^i(f_e^i,f_e)}{\partial v^i} = \frac{\partial L^i(f)}{\partial v^i}$ and $v^i = v^1$, we have the required expression. □

We obtain a similar expression for $\frac{\partial C(f)}{\partial v^r}$.

Lemma 19. *Let $f = f(\boldsymbol{v})$. For player r whose flow value decreases in Step 2,*

$$\begin{aligned}\frac{\partial C(f)}{\partial v^r} &= L^1(f) + \frac{r-1}{r+1}\left(\frac{\partial L^1(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right) + \frac{1}{r+1}\left(\frac{\partial L^r(f)}{\partial v^r}\sum_{i=r}^{k} v^i\right)\\ &\quad + (r-2)\left(\sum_{e\in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r}\right) \quad . \end{aligned} \tag{7}$$

Let $\mathcal{P}$ denote the set of all s-t paths in G, and for equilibrium flow f, let $\{f_p^i\}_{p\in\mathcal{P},i\in[k]}$ denote a path flow decomposition of f. For players $i,j \in [r]$ with player r defined as in the flow redistribution, we will be interested in the rate of change of marginal delay of player i along an s-t path p as the value of flow controlled by player j changes. Given a decomposition $\{f_p^i\}_{p\in\mathcal{P},i\in[k]}$ along paths of the equilibrium flow, this rate of change can be expressed as

$$\begin{aligned}\frac{\partial L_p^i(f)}{\partial v^j} &= \sum_{e\in p} a_e \frac{\partial(f_e + f_e^i)}{\partial v^j} = \sum_{e\in p} a_e \sum_{q\in\mathcal{P}:e\in q} \frac{\partial(f_q + f_q^i)}{\partial v^j}\\ &= \sum_{q\in\mathcal{P}} \frac{\partial(f_q + f_q^i)}{\partial v^j} \sum_{e\in q\cap p} a_e \quad . \end{aligned} \tag{8}$$

Let $u_{pq} = \sum_{e\in p\cap q} a_e$ for any paths $p,q \in \mathcal{P}$ and the matrix $\mathbf{U}$ is defined as the matrix of size $|\mathcal{P}| \times |\mathcal{P}|$ with entries $[u_{pq}]_{p,q\in\mathcal{P}}$.

Lemma 20. *For an equilibrium flow f, there exists a generalized path flow decomposition $\{f_p^i\}_{p\in\overline{\mathcal{P}}^i,i\in[k]}$ so that $\overline{\mathcal{P}}^i \subseteq \mathcal{P}^i$ for all $i \in [k]$ and $\overline{\mathcal{P}}^1 \supseteq \overline{\mathcal{P}}^2 \supseteq \cdots \overline{\mathcal{P}}^k$. Moreover, each of the submatrices $\mathbf{U}^i = [u_{pq}]_{p,q\in\overline{\mathcal{P}}^i}$ of $\mathbf{U}$ is invertible, $\forall i \in [k]$.*

Since $\overline{\mathcal{P}}^i \subseteq \overline{\mathcal{P}}^{i-1}$, we can arrange the rows and columns of $\mathbf{U}$ so that $\mathbf{U}^i$ is a leading principal submatrix of $\mathbf{U}$ for every player i.

Since matrix $\mathbf{U}^i$ is invertible, we define $\mathbf{W}^i = \mathbf{U}^{-1}$. For a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we use $\mathbf{A}_p$ to refer to the pth row vector and a_{pq} to refer to the entry in the pth row and qth column. We define $\|\mathbf{A}\| = \sum_{i\in[m],j\in[n]} a_{ij}$.

Lemma 21. *For equilibrium flow f and sets $\overline{\mathcal{P}}^i \subseteq \mathcal{P}$ as described in Lemma 20, for all players $i \in [k]$, $\|\mathbf{W}^i\| \geq \|\mathbf{W}^{i+1}\|$ and $\|\mathbf{W}^k\| > 0$.*

The next lemma gives the rate of change of marginal delay at equilibrium.

Lemma 22. *For player r defined as in the flow redistribution process and any player $i < r$, for $f = f(\boldsymbol{v})$,*

(i) $\dfrac{\partial L^i(f(\boldsymbol{v}))}{\partial v^i} = \dfrac{2}{\|\mathbf{W}^i\|}$,

(ii) $\dfrac{\partial L^i(f)}{\partial v^r} = \dfrac{1}{\|\mathbf{W}^i\|}$,

(iii) $\dfrac{\partial L^r(f)}{\partial v^r} = \dfrac{r+1}{r}\dfrac{1}{\|\mathbf{W}^r\|} + \dfrac{r-1}{r}\dfrac{1}{\|\mathbf{W}^1\|}$.

If we have just two players, it follows by substituting $i = 1$ and $r = 2$ and the expressions from Lemma 22 into Lemma 18 and Lemma 19 that $\frac{\partial C(f)}{\partial v^2} - \frac{\partial C(f)}{\partial v^1} = \frac{1}{2}v^2\left(\frac{1}{\|\mathbf{W}^2\|} - \frac{1}{\|\mathbf{W}^1\|}\right)$. By Lemma 21, $\|\mathbf{W}^1\| \geq \|\mathbf{W}^2\|$, and hence $\frac{\partial C(f)}{\partial v^2} - \frac{\partial C(f)}{\partial v^1} \geq 0$, proving Lemma 11 for the case of two players. However, if we have more than two players, when $r \neq 2$ the fourth term on the right hand side of (7) has nonzero contribution. Calculating this term is complicated. However, we show the following inequality for this expression.

Lemma 23. *For $f = f(v)$ and the player r as defined in the flow redistribution process,* $\displaystyle\sum_{e \in E^1} a_e f_e^1 \frac{\partial f_e}{\partial v^r} \geq \frac{v^1}{\|\mathbf{W}^1\|} - \frac{v^r}{r}\left(\frac{1}{\|\mathbf{W}^1\|} - \frac{1}{\|\mathbf{W}^r\|}\right)$.

Proof of Lemma 11. For any player $i < r$, substituting the expression for $\frac{\partial L^i(f)}{\partial v^i}$ from Lemma 22 into Lemma 18, and observing that $L^i(f) = L^1(f)$ and $\|\mathbf{W}^i\| = \|\mathbf{W}^1\|$ since the flow of the first $r-1$ players is identical,

$$\frac{\partial C(f)}{\partial v^i} = L^1(f) + \frac{\sum_{j=2}^k v^j}{\|\mathbf{W}^1\|} . \tag{9}$$

Similarly, substituting from Lemmas 22 and 23 into Lemma 19 and simplifying,

$$\frac{\partial C(f)}{\partial v^r} \geq L^1(f) + \frac{\sum_{i=2}^k v^i}{\|\mathbf{W}^1\|} + \frac{1}{r}\left(\frac{1}{\|\mathbf{W}^r\|} - \frac{1}{\|\mathbf{W}^1\|}\right)\left(\sum_{i=2}^k v^i - (r-2)(v^1 - v^r)\right) . \tag{10}$$

We subtract (9) from (10) to obtain, for any player $i < r$,

$$\frac{\partial C(f)}{\partial v^r} - \frac{\partial C(f)}{\partial v^i} \geq \frac{1}{r}\left(\frac{1}{\|\mathbf{W}^r\|} - \frac{1}{\|\mathbf{W}^1\|}\right)\left(\sum_{i=2}^k v^i - (r-2)(v^1 - v^r)\right) . \tag{11}$$

From Lemma 21 we know that $\|\mathbf{W}^1\| \geq \|\mathbf{W}^r\|$. Also, $\sum_{i=2}^k v^i = (r-2)v^1 + \sum_{i=r}^k v^i \geq (r-2)(v^1 - v^r)$. Hence, the expression on the right of (11) is nonnegative, completing the proof. □

4 Convex delays on series-parallel graphs

Let $\mathcal{C}$ denote the class of continuous, differentiable, nondecreasing and convex functions. In this section we prove the following result.

Theorem 24. *The price of collusion on a series-parallel graph with delay functions taken from the set $\mathcal{C}$ is at most the price of collusion with linear delay functions.*

This theorem combined with Theorem 8, suffices to prove Theorem 1. The following lemma is proved by Milchtaich.[3]

Lemma 25 ([14]). *Let $(G,v,\boldsymbol{l},s,t)$ and $(G,\tilde{v},\tilde{\boldsymbol{l}},s,t)$ be nonatomic routing games on a directed series-parallel graph with terminals s and t, where $v \geq \tilde{v}$, and $\forall x \in \mathbb{R}^+$ and $e \in E$, $l_e(x) \geq \tilde{l}_e(x)$. Let f and $\tilde{f}$ be equilibrium flows for the games with delays l and $\tilde{l}$ respectively. Then $C(f) \geq \tilde{C}(\tilde{f})$.*

We now use Lemma 25 to prove Theorem 24.

Proof of Theorem 24. Given a series-parallel graph G with delay functions $\boldsymbol{l}$ taken from $\mathcal{C}$, let g denote the atomic equilibrium flow and f denote the nonatomic equilibrium. We define a set of linear delay functions $\tilde{\boldsymbol{l}}$ as follows. For an edge, $\tilde{l}_e(x) = a_e x + b_e$, where $a_e = \frac{\partial l_e(f_e)}{\partial f_e}\Big|_{f_e=g_e}$ and $b_e = l_e(g_e) - a_e g_e$. Hence, the delay function $\tilde{l}_e$ is the tangent to the original delay function at the atomic equilibrium flow. Note that a convex continuous differentiable function lies above all of its tangents.

Let $\tilde{g}$ and $\tilde{f}$ denote the atomic and nonatomic equilibrium flows respectively with delay functions $\tilde{\boldsymbol{l}}$. Then by the definition of $\tilde{\boldsymbol{l}}$, $\tilde{g} = g$ and $\tilde{\boldsymbol{l}}(\tilde{g}) = \boldsymbol{l}(g)$. Hence, $\tilde{C}(\tilde{g}) = C(g)$. Further, by Lemma 25, $C(f) \geq \tilde{C}(\tilde{f})$. Since $\frac{C(g)}{C(f)} \leq \frac{\tilde{C}(\tilde{g})}{\tilde{C}(\tilde{f})}$, the proof follows.

5 Total Delay without the Nesting Property

If the nesting property does not hold, the total delay can increase as we decrease the flow of a smaller player and increase the flow of a larger player, thus causing our flow redistribution strategy presented in Section 3.2 to break down. See the full version for an example.

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[3] Milchtaich in fact shows the same result for *undirected* series-parallel graphs. In our context, every simple s-t path in the underlying undirected graph is also an s-t path in the directed graph G.

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