Supplementary On-line Appendix for "Dynamic Collective Action and the Power of Large Numbers"

Marco Battaglini
Department of Economics
Cornell University
Ithaca, NY 14853

Thomas Palfrey
Division of the Humanities and Social Sciences
California Institute of Technology
Pasadena, CA 91125.

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1 Additional proof details

Details for the proof of Theorem 2

We proceed in three steps.

Step 1. For any history, h_t^k with k missing volunteers and lower bound $l_{h_t^k} = l$ define the set of possible equilibrium cutpoints as:

$$Z(l;w) = \left\{ c \geq l \;\middle|\; \begin{array}{c} c = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \\ \text{or } c = l \text{ if } e^{-\gamma\Delta} Q^{k-1}(l) \leq l, \\ \text{for some: } \{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1} \in \{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}, \\ \text{and } V^k(c,c) = w \end{array} \right\} \tag{1}$$

where w is the continuation value following the history if no member volunteers at stage t. The set of all possible future equilibrium value functions in the definition of $Z(l;w)$, $\{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}$, is defined by the induction hypothesis for all $j \in 1, 2, ..., k-1$, i.e. when at least one volunteer activates in stage t; and $\{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1}$ is a selection from $\{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}$: i.e., $\{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1}$ is the collection of future equilibrium value functions associated with one specific PBE. Note that we have proven in Section 3 that $V^1(c,c)$ is a continuous function of c, so a fortiori $\mathcal{V}^1(c)$ is an upper-hemicontinuous correspondence in c. We now assume as induction hypothesis that $\mathcal{V}^{k-j}(c)$ is an upper-hemicontinuous correspondence for all $j \in [1, k-1]$.

For any possible equilibrium cutoff, $c_w \in Z(l;w)$, and associated set of future equilibrium value functions, $\{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1} \in \{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}$ one obtains the corresponding value functions $Q_w^k(l), \left[V_w^k\right]^+(c,l)$, $\left[V_w^k\right]^-(c,l)$ and $V_w^k(c,l) = \max\{\left[V_w^k\right]^+(c,l), \left[V_w^k\right]^-(c,l)\}$. These functions directly depend on both c_w and w, since w is the expected continuation value in case of no volunteers. Note that $V_w^k(c,l) = \max\{\left[V_w^k\right]^+(c,l), \left[V_w^k\right]^-(c,l)\}$ is continuous in c since $\left[V_w^k\right]^+(c,l)$ and $\left[V_w^k\right]^-(c,l)$ are both continuous in c. Let $\mathcal{E}^k(l;w)$ be the set of possible equilibrium values for an uncommitted player of type l when the lower bound on types is l and k volunteers are missing;

and denote the convex hull of $\mathcal{E}^k(l;w)$ by $\Delta\mathcal{E}^k(l;w)$. Note that the set $\Delta\mathcal{E}^k(l;w)$ corresponds to the equilibrium values if the expected continuation values are PBE when at least one volunteer activates in period t, and equal to w is the value if there is no contribution. Values in the interior of the convex hull correspond to situations in which the public randomization device is used to mix between equilibria in the continuation game, for example mixing between the equilibria generating $V_w^k(l,l)$ and $\widetilde{V}_w^k(l,l)$, where both $V_w^k(c,l)$ and $\widetilde{V}_w^k(c,l)$ are equilibrium continuation values functions, constructed as described above.

Step 2. Define now as initial steps of a sequence, $Z_0(l) = Z(l;0)$, the set of equilibrium cutpoints if the expected continuation value in case of no contributions is 0 (i.e., the game were to be immediately terminated if there are no contributions), and $\Delta\mathcal{E}_0^k(l) = \Delta\mathcal{E}^k(l;0)$, the convex hull of the corresponding set of value functions. For each $T = 1, 2, ...$, given $\Delta\mathcal{E}_{T-1}^k(l)$, recursively define $Z_1(l), Z_2(l), ...$ similarly, that is:

$$Z_T(l) = \left\{ c \geq l \left| \begin{array}{c} c = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \\ \text{or } c = l \text{ if } e^{-\gamma\Delta} Q^{k-1}(l) \leq l \\ \text{for some: } Q^{k-j-1}(c), V^{k-j}(c,c) \in \mathcal{V}^{k-j}(c)\ \forall j\ j \in [1, k-1] \\ \text{and } V^k(c,c) \in \Delta\mathcal{E}_{T-1}^k(c) \end{array} \right. \right\} \tag{2}$$

In other words, the set $Z_T(l)$ is the set of cutpoints that can be an equilibrium at a history with k missing volunteers if the game were to be terminated after T periods of no additional volunteers. We call these the set of possible cutpoints for the T*-truncated game,* and it defines a sequence of sets of possible cutpoints, $\{Z_T(l)\}_{T=1}^{\infty}$. If $e^{-\gamma\Delta}Q^{k-1}(l) \leq l$, then $Z_T(l)$ is obviously not empty. If $e^{-\gamma\Delta}Q^{k-1}(l) > l$, note that the correspondence in c defined by

$$\varphi_l^T(c) = \left\{ \begin{array}{c} x \in [l,1] \left| x = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \right. \\ \text{for some } [Q^{k-j-1}(c), V^{k-j}(c,c)] \in \mathcal{V}^{k-j}(c)\ \forall j \in [1, k-1] \\ \text{and } V^k(c,c) \in \Delta\mathcal{E}_{T-1}^k(c) \end{array} \right\}$$

is non empty, convex- and closed-valued since $Q^{k-j-1}(c), V^{k-j}(c,c) \in \mathcal{V}^{k-j}(c)$ and $V^k(c,c) \in \Delta\mathcal{E}_{T-1}^k(c)$. Moreover, since $\mathcal{V}^{k-j}(c)$ and $\Delta\mathcal{E}_{t-1}^k(c)$ are upper-hemicontinuous in c, $\varphi_l^T(c)$ is upper-hemicontinuous in c as well. It follows that $\varphi_l^T(c)$ is closed valued and upper-hemicontinuous in c and hence has a closed graph. We conclude that $\varphi_l^T(c)$ is non-empty, convex-valued and has closed graph in c, so by the Kakutani fixed-point theorem implies it has a fixed point. This implies $Z_T(l)$ is non empty, since any fixed point of $\varphi_l^T(c)$ is an element of $Z_T(l)$. For each $\widetilde{c} \in Z_T(l)$, we can construct the corresponding value functions $Q_T^k(l)$ and $V_T^k(c,l)$, which are continuous in c. Define $\Delta\mathcal{E}_T^k(l)$ to be the convex hull of the set of continuation values for a type l when the lower bound is l. The set $\Delta\mathcal{E}_T^k(l)$ is non empty, convex and closed valued and upper-hemicontinuous in l. To verify the last property, note that for any sequence $\{l_\iota\} \to l$, we can select a corresponding sequence of $c_T^k(l_\iota) \in Z_T(l_\iota)$ and define the corresponding values for uncommitted players, $V_T^k(c, l_\iota)$.

Let $c_T^k(l) = \lim_{\iota\to\infty} c_T^k(l_\iota)$, since $c_T^k(l_\iota) \in Z_T(l_\iota)$ for all ι, then we must have at least a subsequence in which either the first or the second line of (2) is true for all ι: this implies that $c_T^k(l)$ satisfies the first or the second line of (2) as well, so $c_T^k(l) \in Z_T(l)$. Note that $\lim_{\iota\to\infty} V_T^k(c, l_\iota) = V_T^k(c, l)$ and $V_T^k(c, l)$ is an equilibrium value function since $c_T^k(l) \in Z_T(l)$. So, for any sequence $\{l_\iota\} \to l$, there is a selection $V_T^k(l_\iota, l_\iota) \in \Delta\mathcal{E}_T^k(l_\iota)$ with $V_T^k(l_\iota, l_\iota) \to V_T^k(l, l)$, such that $V_T^k(l, l) \in \Delta\mathcal{E}_T^k(l)$.

Step 3. For any $T > 0$, the procedure descibed above generates a sequence of cutpoints $\left(c_{T,\tau}^k(l)\right)_{\tau=1}^T$ where $c_{T,\tau}^k(l) = c_{T-\tau}^k(l)$ is the cutpoints when there are $T-\tau$ attempts, i.e. under the constraint that the game is terminated if there are $T-\tau$ periods without volunteers. For any τ, we can define $c_\tau^k(l) = \lim_{T\to\infty} c_{T,\tau}^k(l)$, and the value functions $Q^k(l)$ and $V^k(c, l)$ associated to the limit cutpoints $\left(c_\tau^k(l)\right)_{\tau=1}^\infty$. We claim that this is a PBE. Assume this is not true. Then there is a deviation for a player i that yields $\overline{V}_i^k(c, l)$ such that $\overline{V}_i^k(c, l) - V^k(c, l) > 2\varepsilon$ for some $\varepsilon > 0$. We now make two observations. First, let $\overline{V}_{i,T}^k(c, l)$ be the value of the strategies used in $\overline{V}_i^k(c, l)$ in the T-truncated game. Since utilities are bounded and $\Delta, \gamma > 0$, the truncated game is continuous at infinity (as defined in Fudenberg and Levine, 1983). We must therefore have $\overline{V}_{i,T}^k(c, l) \geq \overline{V}_i^k(c, l) - \varepsilon/2$ for T sufficiently large. Moreover, by construction, $V_T^k(c, l) \leq V^k(c, l) + \varepsilon/2$ for T sufficiently large. It follows that there exists a T^* such that for $T > T^*$:

$$\overline{V}_{i,T}^k(c, l) - V_T^k(c, l) \geq \varepsilon$$

But this is in contradiction with the fact that $V_T^k(c, l)$ is the equilibrium value function of a PBE in the T-truncated game.

Step 4. We conclude the induction step by proving that the set of equilibrium values $\mathcal{V}^k(l)$ is nonempty, closed, convex-valued and upper-hemicontinuous in l. We showed above that $Z(l)$ is non empty. We now prove that $Z(l)$ is upper hemicontinuous, which immediately implies the desired result. Consider a sequence $\{l_\iota\} \to l$ and the associated sequence $c^k(l_\iota) \in Z(l_\iota)$. We need to prove that if $c^k(l_\iota) \to \lim_{\iota\to\infty} c^k(l_\iota)$, then $\lim_{\iota\to\infty} c^k(l_\iota) \in Z(l)$. To show this, define $c^k(l_\iota) \in \arg\min_{c^k \in Z(l_\iota)} \left|c^k - \lim_{j\to\infty} c^k(l_j)\right|$ and assume by contradiction that $\left|c^k(l) - \lim_{j\to\infty} c^k(l_j)\right| > \varepsilon$ for some $\varepsilon > 0$. We can write:

$$\begin{aligned}\left|c^k(l) - \lim_{\iota\to\infty} c^k(l_\iota)\right| \leq{}& \left|c^k(l) - c_T^k(l)\right| + \left|c_T^k(l) - c_T^k(l_\iota)\right| \\ &+ \left|c_T^k(l_\iota) - c^k(l_\iota)\right| + \left|c^k(l_\iota) - \lim_{j\to\infty} c^k(l_j)\right|\end{aligned}$$

where $\left\{c_T^k(l)\right\}$ is a sequence of equilibrium cutpoints in the truncated game such that $c_T^k(l) \to c^k(l)$. Note that by definition of $c_T^k(l)$, there is a T^* such that for $T > T^*$, $\left|c^k(l) - c_T^k(l)\right| < \varepsilon/4$ and $\left|c_T^k(l_\iota) - c^k(l_\iota)\right| < \varepsilon/4$. Similarly, by definition of a limit, there is a ι^* such that for $\iota > \iota^*$, $\left|c^k(l_\iota) - \lim_{j\to\infty} c^k(l_j)\right| < \varepsilon/4$. Finally, note that for a given T, $c_T^k(l)$ is upper-hemicontinuous so it admits a selection such that $\lim_{j\to\infty} c_T^k(l_j) \in Z_T(l)$, implying that $\left|c_T^k(l) - c_T^k(l_\iota)\right| < \varepsilon/4$ for $\iota > \iota^*$ and some $c_T^k(l) \in Z_T(l)$ (if ι^* is chosen sufficiently large). We conclude that $\left|c^k(l) - \lim_{j\to\infty} c^k(l_j)\right| < \varepsilon$ for ι sufficiently large, a contradiction. ■

Details for the proof of Theorem 4

We proceed in two steps.

Step 1. We have proven in the main text that if $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, then $\lim_{n\to\infty} \frac{F(c_{n,1}^{m_n}(0))}{\alpha_n} \to L > 1$, where L is either bounded but strictly larger than 1 or infinite (and as, defined in the text, $\alpha_n = m_n/n$). We now prove that if $m_n = m$ for all n (or equivalently $m_n \to m$), then $\lim_{n\to\infty} \frac{F(c_{n,1}^{m}(0))}{m/n} = \infty$. Assume by contradiction that $\frac{nF(c_{n,1}^{m}(0))}{m} \to L < \infty$. In this case, it can be proven using standard methods that:

$$B(m-1, n-1, F(c_{n,1}^{m}(0))) \cong \binom{n-1}{m-1} \frac{\left[\left(\frac{m}{n}\right)^{\frac{m}{n}} (1-\frac{m}{n})^{1-\frac{m}{n}}\right]^n}{\frac{m}{n}} \cong \sqrt{\frac{1}{2\pi m(1-\frac{m}{n})}}$$

where the second step follows from the Strirling approximation formula, and "$\cong$" means that left hand side converges to zero or diverges to infinity at the same rate. Since $\frac{nF(c_{n,1}^{m}(0))}{m} \to L$ implies that $c_{n,1}^{m}(0) \to \frac{L}{f(0)}\frac{m}{n}$, so we must have that for sufficiently large n:

$$1 \geq \frac{f(0)B(m-1, n-1, F(c_{n,1}^{m}(0)))}{c_{n,1}^{m}(0)} \simeq \sqrt{\frac{1}{2\pi\left(\frac{m}{n}\right)^3(1-\frac{m}{n})n}} = \sqrt{\frac{n^2}{2\pi m^3(1-\frac{m}{n})}} \to \infty,$$

a contradiction. We therefore conclude that in equilibrium if m is constant, then: $\frac{nF(c_{n,1}^{m}(0))}{m} \to \infty$.

Step 2. We finally prove that if $m_n = m$, a constant, or if $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, then the probability of success in the first period converges to 1. Define for convenience here, $\zeta_n = \frac{F(c_{n,1}^{m_n}(0))}{\alpha_n}$. Note that the probability of failure in the first period is equal to the probability that the number of volunteers in period 1, j, is less than or equal to $\alpha_n n$ agents, which can be bounded above:

$$\begin{aligned}\Pr(j \leq \alpha_n n) &= \Pr(\frac{j}{n} \leq \alpha_n) = \Pr(\frac{j}{n} \leq F\left(c_{n,1}^{m_n}(0)\right) - (F\left(c_{n,1}^{m_n}(0)\right) - \alpha_n)) \quad (3)\\ &\leq \Pr\left[\left|\frac{j}{n} - F\left(c_{n,1}^{m_n}(0)\right)\right| \geq \alpha_n(\zeta_n - 1)\right]\\ &= \Pr\left[\left|\frac{j}{n} - F\left(c_{n,1}^{m_n}(0)\right)\right| \geq \sigma_{c_{n,1}^{m}(0)}(\frac{j}{n}) \cdot \frac{\sqrt{n\alpha_n}(\zeta_n - 1)}{\sqrt{\zeta_n(1 - F\left(c_{n,1}^{m}(0)\right))}}\right]\\ &\leq \left(\frac{\sqrt{\zeta_n(1 - F\left(c_{n,1}^{m_n}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)}\right)^2 \quad (4)\end{aligned}$$

where in the second line we used $F\left(c_{n,1}^{m_n}(0)\right) - \alpha_n = \alpha_n(\zeta_n - 1)$; in the third line we define $\sigma_{c_{n,1}^{m_n}(0)}(\frac{j}{n}) = \frac{\sqrt{F\left(c_{n,1}^{m_n}(0)\right)(1-F\left(c_{n,1}^{m_n}(0)\right))}}{\sqrt{n}}$ and used Chebyshev's inequality. We now have two cases to consider. If $m_n = m$, a constant, then by Step 1 we have $\zeta_n \to \infty$ and can rewrite (3) as:

$$\Pr(j \leq \alpha_n n) \leq \left(\frac{\sqrt{\zeta_n(1 - F\left(c^m_{n,1}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)} \right)^2 = \lim_{n\to\infty} \frac{1}{\zeta_n} \frac{1}{m(1-\frac{1}{\zeta_n})^2} = 0$$

If instead $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, we have by Step 1 that $\zeta_n \to L > 1$ and:

$$\Pr(j \leq \alpha_n n) \leq \left(\frac{\sqrt{\zeta_n(1 - F\left(c^{m_n}_{n,1}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)} \right)^2 = \lim_{n\to\infty} \frac{1}{m_n} \frac{L}{(L-1)^2} = 0$$

In both cases, we conclude that the probability of failure converges to zero. ■

Details for the proof of Theorem 5

Case 1.1. We prove here that reporting $c' > c$ and disobeying to a recomednation to volunteer is not a strictly optimal deviation for a player i. Assume the recommendation is to volunteer and the player strictly prefers to disobey by not volunteering. In this case, again, $\mathbf{c}_{-i}$ must be such that $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) \geq c'$ for some $t \leq S(\mathbf{c})$ and a sequence of cutoffs $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. As in Case 1 in the main proof of Theorem 5, let t' be the minimal period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Then $c_{t'',i}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and so $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'})$. If $t' > t$, then $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) < c$ for $t \leq S(\mathbf{c}, \theta)$, so if the agent reported truthfully, s/he would have received the recommendation to not volunteer. It follows that in this event reporting c' and disobeying induces the same action as reporting c and obeying: it cannot generate a strictly superior deviation in this event. If instead, $t' \leq t$, then $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Player i does not know $\mathbf{c}_{-i}$ and t, but s/he knows that conditioning on being asked to volunteer, $\mathbf{c}_{-i}$ is such that there is a $t \leq S(\mathbf{c}, \theta)$ in which $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c$. This implies the following. First that player i conditions on an event in which the set $I_{t-1}(c, \mathbf{c}_{-i}, \theta_{t-1})$ of players volunteers for sure (indeed i conditions on a family of events with this property). Second, i conditions on an event in which, for any $j \geq 0$, the cutoffs at $t + j$ are identical to the cutoffs in the PBE of the dynamic game (by construction) that follows the vector of cutoffs $c_t(c, \mathbf{c}_{-i}, \theta_t) = \{c_{t,1}(c, \mathbf{c}_{-i}, \theta_t), ..., c_{t,n}(c, \mathbf{c}_{-i}, \theta_t)\}$. It follows that i has the same expected values as in the PBE, and s/he weakly prefers to volunteer: s/he therefore cannot strictly prefer to disobey the mechanism and not volunteer. We conclude that if the player disobeys when asked to volunteer after reporting to be a type c', the deviation cannot be strictly superior that reporting honestly and then obeying the recommended action.

Case 2. In the case in which the recommendation is to abstain, we have two cases to consider:

Step 2.1. Consider first the case in which i reports c' and obeys to a recommendation to abstain. In this case, again, $\mathbf{c}_{-i}$ must be such that $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) \leq c'$ for some $t \leq S(\mathbf{c}, \theta)$ and a sequence of cutoffs $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the minimal period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Then $c_{t'',i}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and so by the same argument as in Step 1, $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'})$. If $t' > t$, then $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) < c$, so in this event reporting c' induces the same action as reporting c: it cannot generate a strictly superior deviation in this event. If instead, $t' \leq t$, then $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Player i does not know $\mathbf{c}_{-i}$ and t, but s/he knows that conditioning on being asked to volunteer, $\mathbf{c}_{-i}$ is such that it is as if s/he is a period $t \leq S(\mathbf{c}, \theta)$ with public signals θ_t in which $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c$. As in Step 1.2, this implies that i has the same expected values as in the PBE, and weakly prefers to volunteer. So reporting c' and obeying a recommendation to abstain cannot yield a higher expected utility than reporting truthfully and obeying the recommendation of the mechanism.

Step 2.2. Finally, consider the case in which i reports c' and disobeys to a recommendation to abstain. Again, let t' be the minimal period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. If $t' > t$, then $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) < c$ for all $t' \leq t$. So a type c would find it optimal to abstain. If instead, $t' \leq t$, then $c_{t,i}(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, and a type c would receive the same expected payoff from reporting truthfully and obeying than from reporting c' and disobeying a recommendation to abstain.

Since there is no scenario in which the player finds it strictly optimal to report to be a type $c' > c$, we conclude that the player is never strictly better off by reporting to be $c' > c$, no matter what obedience policy s/he follows afterwards. ■

Details for the proof of Theorem 3′

To simplify notation, we suppress the dependence of the lower bound of the posterior beliefs, l, on h_t^k. For any lower bound, l, define $\underline{c}^k(l)$ as the minimal x such that:

$$\begin{aligned} & v - x - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-2} \left(\frac{v}{e^{-\gamma\Delta}} - Q^{k-j-1}(x) \right) B\left(j, n-1-m+k, \widetilde{F}(x;l)\right) \\ & \geq v - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-1} \left(\frac{v}{e^{-\gamma\Delta}} - V^{k-j}(x,x) \right) B\left(j, n-1-m+k, \widetilde{F}(x;l)\right). \end{aligned} \tag{5}$$

The left hand side is the utility of a cutpoint type x who volunteers when the cutpoint used by the other agents is x. The right hand side is the utility of a type x who does not volunteer, when the others are using cutpoint x. Note that the left hand side may be strictly lower than the right hand side for any $x \in [l, v]$: in this case all types c strictly prefer not to volunteer and $c^k(l) = l$, in which case the cutpoint is not defined by an equality as in (5). It follows that there is no type $x < v$ that is willing to volunteer; when $\underline{c}^k(l) > l$, then any type $x \leq \underline{c}^k(l)$ is willing to volunteer. When $\underline{c}^k(l) = l$, then type l is willing to volunteer only (5) holds with equality.

We can write (5) as:

$$\underline{c}^k(l) = \min_{c \geq [l,1]} \left\{ c | c \leq e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\left(Q^{k-j-1}(c) - V^{k-j}(c,c) \right) B\left(j, n-1-m+k, \widetilde{F}(c;l)\right) \right] \right\} \tag{6}$$

where $Q^0(c) = v/e^{-\gamma\Delta}$. We now prove that there is a $v^*(n,m,\gamma,\Delta)$ such that for $v > v^*(n,m,\gamma,\Delta)$, $c^k(l) > l$ for any $k \leq m$ and $l < \min\{v,1\}$. We proceed in four steps.

Step 1. We have already proven in Lemma 2 that for any $l < v$, we have $\underline{c}^1(l) > l$. Moreover it is easy to verify that there must be a $v^*(n,1,\gamma,\Delta)$ such that for $v \geq v^*(n,1,\gamma,\Delta)$ we have $e^{-\gamma\Delta}Q^1(l) - l > 0$ for any $l \in [0,\min\{v,1\}]$: since as it can be verified using (4) in the paper, $c_t^1(l)$ is strictly increasing in v for all t; and $Q^1(l)$ is increasing in both v and $c_t^1(l)$ for all t.

Step 2. For the induction hypothesis, assume that for any $l \in [0,\min\{v,1\}]$ and for all $j = 1,...,m-1$ there is a $v^*(n,j,\gamma,\Delta) > 0$ such that for $v \geq v^*(n,j,\gamma,\Delta)$, we have: $c^j(l) > l$ and $e^{-\gamma\Delta}Q^j(l) - l > 0$. We prove that there is a $v^*(n,m,\gamma,\Delta)$ such that for $v \geq v^*(n,m,\gamma,\Delta)$, we have: $c^j(l) > l$ for all $l < v$ and $e^{-\gamma\Delta}Q^j(l) - l > 0$ for any $j \leq m$ and $l \in [0,1]$. There are two sub-cases to consider.

Step 2.1. Suppose by contradiction that for any v even arbitrarily large, $c^k(l) = l$ and we have that $\left[V^k\right]^-(c^k(l),l) > \left[V^k\right]^+(l,l)$ for some $l \leq \min\{1,v\}$. Hence, at l we therefore have a strict corner solution when there are k missing volunteers. In this case, the value function for a type l must be $V^k(l,l) = 0$, since the project will never be realized: all players expect that no other player of type $c \geq l$ is willing to contribute. Suppose that $v \geq v^*(n,k-1,\gamma,\Delta)$, as defined by the induction step. If a player of type l volunteers, s/he obtains: $\left[V^k\right]^+(l,l) \geq e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l > 0$, where the last inequality follows from the induction step: we thus have a contradiction.

Step 2.2. From the previous step, we conclude that, if Part 1 the theorem is not true, then for $v > v^*(n,k-1,\gamma,\Delta)$, if $c^k(l) = l$ then $V^k(c^k(l),l) = \left[V^k\right]^+(l,l)$. From $c^k(l) = l$, we have:

$$\begin{aligned} B\left(j, n-1-m+k, \widetilde{F}(c;l)\right) &= 0 \text{ for } j > 0 \\ B\left(0, n-1-m+k, \widetilde{F}(c;l)\right) &= 1 \end{aligned} \tag{7}$$

since $V^k(c^k(l),l) = \left[V^k\right]^+(l,l)$. Hence $\underline{c}^k(l)$ satisfies (6) at equality, which implies that (6) can be written as: $l = \underline{c}^k(l) = e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right)$. Note that when $\underline{c}^k(l) = l$, there are no other volunteers, so: $\left[V^k\right]^+(l,l) = e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l$. But then we have:

$$\begin{aligned} Q^{k-1}(l) - \left[V^k\right]^+(l,l) &= Q^{k-1}(l) - \left(e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l\right) \\ &= l + (1-e^{-\gamma\Delta})Q^{k-1}(l) \end{aligned}$$

This implies that we have $e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right) > l$ if $e^{-\gamma\Delta}l + e^{-\gamma\Delta}(1-e^{-\gamma\Delta})Q^{k-1}(l) > l$: or equivalently $e^{-\gamma\Delta}Q^{k-1}(l) > l$, an inequality that is always true if $v > v^*(n,k-1)$. But then we have $l = \underline{c}^k(l) = e^{-\gamma\Delta} \cdot \left(l + (1-e^{-\gamma\Delta})Q^{k-1}(l)\right) > l$, a contradiction. We conclude that for for any $k \leq m$ and $l \leq \min\{1,v\}$, $c^k(l) > l$ if $v > v^*(n,k,\gamma,\Delta)$.

Step 3. Finally, we conclude the inductive argument by proving that there is a $v^*(n,k,\gamma,\Delta) \geq v^*(n,k-1,\gamma,\Delta)$ such that for $v > v^*(n,k,\gamma,\Delta)$, then $e^{-\gamma\Delta}Q^k(l) > l$ for any $l \in [l_0, \min\{1,v\}]$. It is sufficient to prove $e^{-\gamma\Delta}Q^k(l) > 1$, for v sufficiently high. Assume not. Then it must be that

$c^k(l)$ converges to l as v increases, since if it converges to a constant $\widetilde{c} > l$. We must therefore have that $B\left(j, n-1-m+k, \frac{F(\widetilde{c})-F(l)}{1-F(l)}\right) > 0$ for all $j \geq k$ and $Q^k(l) \to \infty$ as $v \to \infty$, since it is strictly increasing in v (and diverging at infinity as v increases, given the lower bounds on the probabilities of $j \geq k$ volunteers): a contradiction. But if $c^k(l) \to l$, then we have: $\left[V^k\right]^+(l,l) \to e^{-\gamma\Delta}Q^{k-1}(l) - l$. Since, by the previous step, the equilibrium is interior in stage k for $v > v^*(n, k-1, \gamma, \Delta)$, we have: $c^k(l) = e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^+(l,l)\right)$. Note moreover that for $v > v^*(n, k-1, \gamma, \Delta)$, we have $e^{-\gamma\Delta}Q^{k-1}(l) > l$. It follows that as v increases, we have:

$$c^k(l) \to e^{-\gamma\Delta} \cdot \left(l + (1 - e^{-\gamma\Delta})Q^{k-1}(l)\right) > e^{-\gamma\Delta} \cdot \left(l + (1 - e^{-\gamma\Delta})\frac{l}{e^{-\gamma\Delta}}\right) = l$$

where the last inequality follows from $v > v^*(n, k-1, \gamma, \Delta)$. We thus have a contradiction. We conclude that there is a $v^*(n, k)$ such that for $v > v^*(n, k, \gamma, \Delta)$, $e^{-\gamma\Delta}Q^k(l) > l$. ■

Proof of Proposition 1

We show here that if $v > v^*(n, m, \gamma, \Delta)$, then the group achieves the objective if there are at least m players with type lower than v. Note that for $v > v^*(n, m, \gamma, \Delta)$, in the history h_t^k with k volunteers needed and a lower bound of types at $l_{h_t^k}$, then in period $t+1$, there will either be $j < k$ volunteers needed, with a lower bound of $c^k(l_{h_t^k})$; or there will still be k volunteers needed with a higher lower bound of types $c^k(l) > l$.

Now suppose, by way of contradiction, that there is some k for which $c_t^k \to c_\infty^k < v$ for some initial $l_{h_t^k}$, following a sequence of many periods where there are no additional volunteers beyond $m - k$. Note that as $c_t^k \to c_\infty^k$, we have:

$$\widetilde{F}(c_t^k; c_{t-1}^k) \to \frac{\lim_{t\to\infty}\left(F(c_t^k) - F(c_{t-1}^k)\right)}{1 - F(c_\infty^k)} = 0$$

Since (13) in the paper must hold, we therefore have:

$$c_\infty^k = \min_{c \in \left[c_\infty^k, 1\right]} \left\{c \geq e^{-\gamma\Delta} \cdot \left(Q^{k-1}(c) - \left[V^k\right]^+(c,c)\right)\right\},$$

But then the same argument as Step 2.2. above proves that for $v > v^*(n, m)$ we must have:

$$\min_{c \in \left[c_\infty^k, 1\right]} \left\{c \geq e^{-\gamma\Delta} \cdot \begin{bmatrix} Q^{k-1}(c) \\ -\left[V^k\right]^+(c,c) \end{bmatrix}\right\} > e^{-\gamma\Delta} \cdot \left(c_\infty^k + (1 - e^{-\gamma\Delta})\frac{c_\infty^k}{e^{-\gamma\Delta}}\right) > c_\infty^k,$$

a contradiction. We conclude that for all k, $c_t^k \to c_\infty^k = v$. ■

Proof of Proposition 2

Call E^{2+} the event comprising histories h_t in which at least two volunteers are missing and they both have a cost $c_i \in (v/2, 1)$. Clearly this event has positive probability for any $v \in (1, 2)$. We now prove that for any $v \in (1, 2)$, there is an equilibrium in which contributions are zero in E^{2+}, no matter what the level of γ, and Δ are, thus even in the limit as $\gamma, \Delta \to 0$. Consider an history

h_t^2 with the properties as above. Assume the lower-bound on types is $l > v/2$. An active player i who expects no other active player to contribute obtains from contributing at most a payoff

$$\begin{aligned} e^{-\gamma\Delta}Q^1(c^2(l)) - c_i &\leq e^{-\gamma\Delta}\left[v - c^2(l)\right] - c_i \leq e^{-\gamma\Delta}\left[v - l\right] - l \\ &\leq \left[e^{-\gamma\Delta} - \frac{1}{2}(1 + e^{-\gamma\Delta})\right] v < 0 \end{aligned} \tag{8}$$

where the first inequality follows from the fact that $Q^1(c^2(l))$ must be smaller than the utility of the lowest remaining type, i.e. $c^2(l)$, so it must be $Q^1(c^2(l)) \leq v - c^2(l) \leq v - l$; the second inequality follows from the fact that $c^2(l) \geq l$ and $c_i \geq l$. We conclude that no active player finds it optimal to contribute if s/he does not expect some other player to contribute with positive probability. ∎

Proof of Proposition 3

We prove here that for any $n > m_n$ and for any $\varepsilon > 0$, $\gamma \in (0,1)$ there exists a $\Delta_{n,\varepsilon,\gamma} > 0$ such that for $\Delta > \Delta_{n,\varepsilon,\gamma}$ the project is realized in an equilibrium with probability less than ε. This implies that for any $\widehat{v}$, there is a $\Delta_{\widehat{v}}$ such that for $\Delta > \Delta_{\widehat{v}}$ we have $v^*(n, m.\gamma, \Delta) > \widehat{v}$. For any F and for any $\varepsilon > 0$, there is a $c_\varepsilon > 0$ such that with probability $1-\varepsilon$, at least $n - m_n + 2$ players have cost strictly larger than c_ε (so that there are no more than $m_n - 2$ players with cost lower than c_ε). Consider a continuation game with $k \leq m_n$ and $l \geq c_\varepsilon$, where k is the number of missing volunteers, and l is the lower bound on types. For these continuation games, consider the path of future play along which no uncommitted member volunteers. To see that there is a $\Delta_{n,\varepsilon,\gamma}$ such that for $\Delta > \Delta_{n,\varepsilon,\gamma}$ this is an equilibrium, note that with these strategies the expected utility of a player who does not volunteer is zero; the expected benefit of volunteering player is not higher than $D_{n,\varepsilon,\gamma} \equiv -c_\varepsilon + e^{-\gamma\Delta}(v - c_\varepsilon)$: success can occur no sooner than a period after the deviator volunteers, and the expected payoff the period after a unilateral deviation cannot be larger than the utility of the lowest type, i.e. $v - c_\varepsilon$. For $\Delta > \frac{1}{\gamma}\log\left(\frac{v-c_\varepsilon}{c_\varepsilon}\right) = \Delta_{n,\varepsilon,\gamma}$, we have that $D_{n,\varepsilon,\gamma} < 0$, so the equilibrium strategies are optimal. Given these equilibrium strategies, assign to any other k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$, some corresponding equilibrium strategy for the continuation game. In the equilibrium of the overall game it must be that if $\Delta > \Delta_{n,\varepsilon,\gamma}$ then with probability $1-\varepsilon$ there are not enough members with cost $c < c_\varepsilon$ to complete the project, which then must fail: indeed, if $\Delta > \Delta_{n,\varepsilon,\gamma}$ then with probability at least $1-\varepsilon$ either we reach a continuation game corresponding to k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$ in which no player contributes; or we reach a state k, l with $k \leq m_n$ and $l \geq c_\varepsilon$, in which case again no player finds it optimal to contribute by construction. ∎

2 Formal statements and proofs of results in Section 3.3: The effects of n and Δ (or γ) on success and welfare

Since the distortion depends on a delay in realization, is natural to ask whether the distortion may be mitigated by an increase in n, or a decrease in the delay costs (i.e. a reduction in either Δ or γ). For any fixed sequence of cutpoints, $\{c_t\}_{t=1}^{\infty}$, and any initial lower bound on the cost types, l, an increase in n makes it easier to achieve the target m. On the other hand, an increase in

n has potentially negative equilibrium implications because the sequence of *equilibrium* cutpoints change with n. In fact, the following result shows that an increase in n leads to a uniform reduction in the equilibrium cutpoints, implying that players are individually more reluctant to contribute. Similar considerations are valid for γ or Δ: an increase would have a positive effect on welfare by reducing the distortions generated by delays; the increase, however, exacerbates the dynamic free rider problem, reducing the cutpoints and increasing equilibrium delays.

To study the overall effect of an increase in n, the following preliminary result will be instrumental. To make the dependence on n explicit, define $c_t(n)$ to be the cut-point with n players at period t (for some given lower bound on types, l, and discounting parameters, Δ, γ). Define as above $\Phi_t(n)$ as the cumulative probability of success up to and including the current period t, for an active player who chooses not to contribute. We say that an increase in n generates an *improvement in success probability* if $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. This implies that for a higher n the distribution is more skewed toward low values of t, which is good for the players (so an improvement). We have:

Lemma A1. *An increase in n shifts the cut-points downward so that $c_t(n) < c_t(n-1)$ for all t, but it generates an improvement in success probability.*

Proof: We proceed in two steps.
Step 1. We first prove that the cutpoints are declining in n so: $c_t(n) < c_t(n-1)$ for all $t > 0$. To see this note that $\left(1 - e^{-\gamma\Delta}\right) x / \left[1 - e^{-\gamma\Delta} \cdot x\right]$ is a strictly increasing function of x if, as always verified in our environment, $e^{-\gamma\Delta} \cdot x < 1$. Since $1 - F(c)/\left[1 - F(l)\right] < 1$ for $c > l$, it follows that the function:

$$\lambda_n(c; l, \gamma, \Delta) = \frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1 - e^{-\gamma\Delta} \cdot \left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}} \tag{9}$$

is strictly decreasing in n for any c, l, γ, Δ. It can also be verified that $\lambda_n(c; l, \gamma, \Delta)$ is strictly decreasing in c for all n, l, γ, Δ. From (23) in the paper, it follows that the fixed point $c^1_{n,1}$ is decreasing in n for any l, γ, Δ. To see this, note that:

$$0 = \lambda_n(c_1(n); l, \gamma, \Delta) - c_1(n) < \lambda_{n-1}(c_1(n); l, \gamma, \Delta) - c_1(n)$$

where the equality follows by the definition of $c_1(n)$, and the inequality follows by the monotonicity of $\lambda_n(c; l, \gamma, \Delta)$ in n. Since $\lambda_{n-1}(c; l, \gamma, \Delta) - c$ is strictly decreasing in c, it follows that we must have $c_1(n-1) > c_1(n)$, else it would be $\lambda_{n-1}(c_1(n-1); l, \gamma, \Delta) - c_1(n-1) > 0$, a contradiction. Assume the induction hypothesis that $c_j(n)$ is decreasing in n for all $j \leq t$. We prove the same is true for $j = t+1$. To see this note that an increase in n shifts the function $\left(\frac{1-F(c)}{1-F(c_t(n))}\right)^{n-1}$ downward for any c since it increases the exponent while reducing $F(c_t(n))$, thus increasing the denominator. It follows that $\lambda_n(c; c_t(n), \gamma, \Delta)$ shifts downward for any c after an increase in n, implying as above that:

$$0 = \lambda_n(c_{t+1}(n); c_t(n), \gamma, \Delta) - c_{t+1}(n) < \lambda_{n-1}(c_{t+1}(n); c_t(n), \gamma, \Delta) - c_{t+1}(n)$$

thus implying that $c_{t+1}(n-1) > c_{t+1}(n)$. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. The probability of success at or before period t, $\Phi_t(n)$, can be written as:

$$\Phi_t(n) = 1 - (1 - F(c_t(n)))^{n-1} = 1 - \prod_{j=1}^{t} \left(\frac{1 - F(c_j(n))}{1 - F(c_{j-1}(n))} \right)^{n-1}$$

From the cutpoint condition in equation (4) in the paper we have:

$$c_j(n) = \left[\frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}}{1 - e^{-\gamma\Delta} \cdot \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}} \right] v$$

Since the right hand side is increasing in $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}$, it follows that $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-1}$ is decreasing in n for any j, since $c_j(n)$ is decreasing in n. We conclude that an increase in n induces an increase in $1 - \prod_{j=1}^{t} \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-1}$, and thus in $\Phi_t(n)$. It follows that $\Phi_t(n) \geq \Phi_t(n-1)$ for all t and $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. ■

An increase in the size of the population induces players to be more reluctant to contribute in every period. Lemma A1 however shows that, from the point of view of any player, the increase in n more than compensates for this effect and generates an unambiguous improvement in the timing of the realization of the public good until a player decides to contribute.

The next result shows that this implies an unambiguous improvement in welfare for all players. Indeed, later where we generalize the analysis to allow for $m > 1$, we will show that the utility of all players converges to the first best v as $n \to \infty$ for any finite m (and thus for $m = 1$ as a special case as well). Let $EU_n(c)$ be the expected utility of a player of type c with n players.

Theorem A1. *An increase in n induces an increase in welfare for all types, strict for sufficiently high types: i.e., $EU_n(c) \geq EU_{n-1}(c)$ for all $c \in [0,1]$ and $EU_n(c) > EU_{n-1}(c)$ for $c_1(n)$.*

Proof. If $c \leq c_1(n)$, then a type c has a payoff of $v - c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v - c$ and with n players the payoff of a type c is not lower than $v - c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. We now prove the result by induction, using these findings as first step. Assume we have proven that for all types $c \leq c_t(n-1)$ for $t \leq j$ a player of type c with n players has utility $EU_n(c)$, weakly higher than the utility of a type c with $n-1$ players $EU_{n-1}(c)$. We have just proven this result for $j = 1$.

Consider first a type $c \in [c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$, if not empty. When there are $n-1$ players, the payoff of a type c is:

$$EU_{n-1}(c) = v\Phi_j(n-1)e^{-\gamma\Delta(t-1)} + [1 - \Phi_j(n-1)]\, e^{-\gamma\Delta j}\,(v - c)\,. \tag{10}$$

A type c with n players instead receives:

$$EU_n(c) = v\Phi_j(n)e^{-\gamma\Delta(t-1)} + [1 - \Phi_j(n)]\, e^{-\gamma\Delta j} \cdot (v - c) > V_{n-1}(c).$$

where the last inequality follows from the fact that $\Phi_t(n-1)$ strictly first order stochastically dominates $\Phi_t(\Delta, n)$ and $v > v - c$.

Alternatively, it could be that $[c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$ is empty. In that case, consider $c \in [\min\{c_{j+1}(n), c_{j+1}(n-1)\}, c_{j+1}(n-1)]$, which must be nonempty. In this case the payoff of a type c with $n-1$ players is again (10), which can be rewritten as:

$$EU_{n-1}(c) = v\Phi_j(n-1)e^{-\gamma\Delta(t-1)} + \sum_{t=j}^{\infty}\left[\Phi_{t+1}(n-1) - \Phi_t(n-1)\right]e^{-\gamma\Delta j}\cdot(v-c)$$

The payoff with n of a player with type c instead is:

$$EU_n(c) = v\Phi_j(n)e^{-\gamma\Delta(t-1)} + \sum_{t=j}^{\infty}\left[\Phi_{t+1}(n) - \Phi_t(n)\right]e^{-\gamma\Delta j}\cdot V_{n,j}(c)$$

where $V_{n,j}(c)$ is the expected continuation value function for a type c when there are n players and j contributors; and where we have $V_{n,j}(c) \geq (v-c)$ by revealed preferences, since $c > c_{j+1}(n)$. Once again we have that $V_{n-1}(c) < V_n(c)$. We have therefore proven that for all $c \leq c_{j+1}(n)$, we have $V_{n-1}(c) < V_n(c)$. It follows that for all types $c < \lim_j c_{j+1}(n) = v$, we have $EU_{n-1}(c) < EU_n(c)$, which proves the result. ■

The proof of this result follows from revealed preferences and a simple inductive argument on types. The core of the argument runs as follows. If $c \leq c_1(n)$, then a type c has a payoff of $v-c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v-c$ and with n players the payoff of a type c is not lower than $v-c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. In both cases the utility of a type $c \leq c_1(n-1)$ weakly increases with n. Assume we have proven this property for all types $c \leq c_t(n-1)$. Then this property together with Lemma A1 can be used to prove that a type c in $[c_t(n-1), c_{t+1}(n-1)]$ is strictly better off with n players than with $n-1$: even if type c does not change behavior the other players volunteer more with a higher n by Lemma A1; and if c behaves differently, then by revealed preferences this must induce an even higher utility. We can therefore extend the inductive assumption to types $c \leq c_{t+1}(n-1)$. Since $c_{t+1}(n-1) > c_t(n-1)$ and indeed we have proven above that $c_t(n-1) \to v$ as $t \to \infty$, this argument allows to show that all types $c \in [0,1]$ obtain a higher expected utility with n than with $n-1$.

Consider now the comparative statics with respect to γ and Δ. Similarly as for an increase of n, a reduction in Δ (or in γ) has an ex ante unambiguous effect on participation, leading to a downward shift in all cutpoints.[1] To make the dependence on Δ explicit, define $c_t(\Delta)$ and $\Phi_t(\Delta)$ similarly as above (for some given lower bound on types, l, and fixed values of n and γ), say that a decrease from Δ' to Δ generates an *improvement in success probability* if $\Phi_t(\Delta)$ first order stochastically dominates $\Phi_t(\Delta')$. Differently from n, now a reduction in Δ implies an deterioration in $\Phi_t(\Delta)$. This implies that although players are more patient, now success takes more time.

[1]We only consider a change in Δ, but the results also hold for changes in γ, since the equilibrium only depends on the product of the two parameters, $\gamma\Delta$.

Lemma A2. *A decrease in* Δ *shifts the cut-points downward so that* $c_t(\Delta) < c_t(\Delta')$ *for all* t *and* $\Delta < \Delta'$, *and a downward shift in* $\Phi_t(\Delta)$ *in the sense first order stochastic dominance, i.e.* $\Phi_t(\Delta) > \Phi_t(\Delta')$ *for all* t *and* $\Delta < \Delta'$.

Proof: We proceed again in two steps.
Step 1. We first prove that the cutpoints are declining in Δ so: $c_t(\Delta) < c_t(\Delta')$ for all t and $\Delta > \Delta'$. To see this note that $\frac{(1-e^{-\gamma\Delta})\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1-e^{-\gamma\Delta}\cdot\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}$ is a strictly increasing function of Δ. It follows that $\lambda_n(c; l, \gamma, \Delta)$, as defined in (9), is strictly increasing in Δ for any c, l, γ, n. By the same argument as in Lemma A1, it follows from (9) that the fixed point $c_1(\Delta)$ is increasing in Δ for any l, γ, n. Assume the induction hypothesis that $c_j(\Delta)$ is increasing in Δ for all $j \leq t$. We now prove the same is true for $j = t + 1$. To see this note that an increase in Δ certainly increases $\left(\frac{1-F(c)}{1-F(c_t(\Delta))}\right)^{n-1}$ for any given c, since it increases $F(c_t(\Delta))$. It follows that $\lambda_n(c; c_t(\Delta), \gamma, \Delta)$ shifts upward after an increase in Δ, implying that $c_{t+1}(\Delta)$ increases. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(\Delta)$ first order stochastically dominates $\Phi_t(\Delta')$ for $\Delta < \Delta'$. As in the proof of Lemma A1, we define:

$$\Phi_t(\Delta) = 1 - \left(\frac{1 - F(c_t(\Delta))}{1 - F(l)}\right)^{n-1}$$

Since $\left(\frac{1-F(c_j(\Delta))}{1-F(l)}\right)^{n-1}$ is decreasing in Δ for all j, then $\Phi_t(\Delta') \leq \Phi_t(\Delta)$ for all t and $\Delta < \Delta'$. ■

Because $\Phi_t(\Delta)$ deteriorates, the decrease in Δ has an ambiguous marginal effect on the welfare of an agent that cannot be signed: while success takes more time, the cost of delay has decreased, so these two effects go in opposite directions. The ambiguous overall effect, however, implies that contrary to what we will prove happens when $n \to \infty$, for a fixed n, efficiency is unattainable even in the limit as $\Delta \to 0$.[2]

Proposition A1: *For all* n, v *and* γ, *there exists* $\delta > 0$ *such that* $\lim_{\Delta\to 0} EU_n(c) < v - \delta$ *for all* $c \in [0, 1]$.

Proof. For any $\varepsilon > 0$, the probability of the event E in which no player has a type c lower than ε is $[1 - F(\varepsilon)]^n > 0$. In this event, no success can occur until we reach a period t in which the cutpoint is strictly larger than ε. Let t_ε be the minimal t such that $c_{t_\varepsilon} \geq \varepsilon$. For Δ small, we can assume without loss of generality that $c_{t_\varepsilon - 1} > \frac{\varepsilon}{2}$. To see this note that if $c_{t_\varepsilon - 1} \leq \frac{\varepsilon}{2}$, then (4) in the paper implies that for Δ sufficiently small, c_{t_ε} is arbitrarily close to $\frac{\varepsilon}{2}$ as well: so $c_{t_\varepsilon} < \varepsilon$, a contradiction. The utility of a player in event E is not larger than $(v - \frac{\varepsilon}{2})$ since in period $t_\varepsilon - 1$ no player can obtain a payoff larger than the lowest type. But then $\Delta \to 0$, the utility of a player is at most $(1 - [1 - F(\varepsilon)]^n)\, v + [1 - F(\varepsilon)]^n\, (v - \varepsilon/2) = v - [1 - F(\varepsilon)]^n\, \frac{\varepsilon}{2} < v$. The result is proven if we let $\delta = [1 - F(\varepsilon)]^n\, \frac{\varepsilon}{2}$. ■

[2] The fact that equilibrium converges to its efficient value as $n \to \infty$ when $m_n = 1$ will be proven as a special case of the case in which $m_n \geq 1$ and can potentially grow with n. See Theorem 7.

The intuition is as follows. For any $\varepsilon > 0$, there is a strictly positive probability that all types are strictly larger than ε. As $\Delta \to 0$, the cutpoints become smaller and smaller, so no player will volunteer until it is common knowledge that all types are strictly larger than, say, $\varepsilon/2 > 0$. In the equilibrium of this continuation game, no player can receive a payoff larger than the payoff of the lowest type, so no player receives more than $v - \varepsilon/2$. The result follows from the fact that this continuation game has a strictly positive probability of being reached.

3 Additional extensions and variations described in Section 6.3

Aggregate uncertainty and learning

In the previous analysis we assumed the players' types are i.i.d. In such environments, as time progresses players update their beliefs about the types of the other players because they know that the remaining active players must have a type higher than the previous equilibrium cutoffs. Yet, they do not learn anything new about the original environment, since the distribution of the other players' types is common knowledge. It is natural to consider scenarios in which players are also ex ante uncertain about the environment, for example about the shape of the distribution of types. In these environments, as time progresses, players also learn about the shape of the distribution of types and this learning also depends on equilibrium strategies.

To illustrate how the analysis can be generalized to this more complex case, we characterize here the equilibrium in the volunteer's dilemma with $m_n = 1$ as in Section 3. The analysis can extended in similar ways to the case with $m_n > 1$. We assume for simplicity that there are two states of nature $\vartheta = H, L$; and that the distribution of types is $F_\vartheta(c)$ in state ϑ, with density $f_\vartheta(c)$ and $F_H(c)$ first order stochastically dominating $F_L(c)$. In this environment, at $t = 1$, a player's belief that the state is H at the beginning of the first period is necessarily function of their type $\pi^1(c)$ since for any initial common prior π^0, they would update after observing their type. We assume here that $\pi^1(c)$ is increasing and continuous in c.[3]

Consider a threshold equilibrium with cutoffs $(\dot{c_t})_{t=0}^{\infty}$ and $\dot{c_t} > \dot{c}_{t-1}$ as in Section 3, such that at t all types $c \leq \dot{c_t}$ contribute, and types $c > \dot{c_t}$ wait. Given these cutoffs and belief $\pi^{t-1}(c)$ at $t-1$, the belief at the beginning of period $t > 1$ for a type c is given by:

$$\pi^t(c) = \left[1 + \frac{1 - \pi^{t-1}(c)}{\pi^{t-1}(c)} \frac{\left(\frac{1-F_H\left(\dot{c}_{t-1}\right)}{1-F_H\left(\dot{c}_{t-2}\right)}\right)}{\left(\frac{1-F_L\left(\dot{c}_{t-1}\right)}{1-F_L\left(\dot{c}_{t-2}\right)}\right)}\right]^{-1} \tag{11}$$

Note that, for any t, if $\pi^{t-1}(c)$ is continuous and increasing in c, then $\pi^t(c)$ is continuous and increasing in c as well since $\frac{1-\pi^{t-1}(c)}{\pi^{t-1}(c)}$ is continuous and decreasing in c.

[3]For an event $E = [c - \varepsilon, c + \varepsilon]$, we have $Pr(H; E) = \frac{\pi^0 \Delta F_H(E)}{\pi^0 \Delta F_H(E) + (1-\pi^0)\Delta F_L(E)}$, where we define $\Delta F_\vartheta(E) = [F_\vartheta(c+\varepsilon) - F_\vartheta(c-\varepsilon)]$. Taking the limit as $\varepsilon \to 0$, we have that $Pr(H; c) = \frac{\pi^0 f_H(c)}{\pi^0 f_H(c) + (1-\pi^0) f_L(c)}$ is increasing if $f_H(c)/f_L(c)$ is increasing in c.

Given the posterior at t, $\pi^t(c)$, and the equilibrium lower bound $c_{t-1}^{\cdot}$, an argument analogous to the argument leading to (3) in Section 3, gives us the cutoff at t as the fixed-point of:

$$v - c_t^{\cdot} = \sum\nolimits_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left\{ v \left[1 - \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} \right] + e^{-\gamma\Delta} \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} (v - c_t^{\cdot}) \right\} \tag{12}$$

note that now $c_t^{\cdot}$ does not only affect $\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)}$ in the right hand side of (12); but also the prior probabilities $\pi_\theta^t(c_t^{\cdot})$ at t with which the shape of the distribution is evaluated: this because all players of different types have different posteriors at each history of the game, since they start from heterogeneous priors $\pi^1(c)$. After some algebra, we have:

$$c_t^{\cdot} = \frac{\left(1 - e^{-\gamma\Delta}\right) \cdot \sum_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}}{1 - e^{-\gamma\Delta} \sum_{\theta=H,L} \pi_\theta^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}} v \equiv G(c_t^{\cdot}) \tag{13}$$

Condition (13) alone is no longer sufficient to characterize the equilibrium. The equilibrium now is determined by the system of difference equations (11) and (13): $\pi^1(c)$ and the initial lower bound $c_0^{\cdot}$ determine $c_1^{\cdot}$; $\pi^1(c)$, $c_0^{\cdot}$ and $c_1^{\cdot}$ determine $\pi^2(c)$; $c_1^{\cdot}$, $c_2^{\cdot}$ and $\pi^2(c)$ determine $c_3^{\cdot}$; and so on so forth for any $t > 0$.

Condition (13) can be used to show that a cutoff equilibrium has similar properties to the equilibria as in Section 3. To see this, consider the right hand side of (13), $G(c_t^{\cdot})$. This function is continuous in $c_t^{\cdot}$ and has the properties that $G(c_{t-1}^{\cdot}) = v$ and $G(1) = 0$; hence it has a fixed point $c_t^{\cdot} > c_{t-1}^{\cdot}$ for any $c_{t-1}^{\cdot}$ and $t-1$. We can moreover verify that $c_t^{\cdot} > c_{t-1}^{\cdot}$ and $c_t^{\cdot} \to v$. For this, assume by way of contradiction that $\lim_{t\to\infty} c_t^{\cdot} = c_\infty^{\cdot} < v$. Then we would still have $\lim_{t\to\infty} \pi^t(c_t^{\cdot}) = \pi^t(c_\infty^{\cdot})$ and $\sum_{\theta=H,L} \pi_\theta^t(c_\infty^{\cdot}) = 1$. It follows that:

$$\lim_{t\to\infty} c_t^{\cdot} = \frac{\left(1 - e^{-\gamma\Delta}\right) \cdot \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}}{1 - e^{-\gamma\Delta} \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(c_t^{\cdot}) \left(\frac{1-F_\theta(c_t^{\cdot})}{1-F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1}} v = v,$$

a contradiction.

When the players learn about the distribution of types, however, an additional complication may arise regarding whether an equilibrium is necessarily in cutoff strategies. To see this point, consider (12) which characterizes the indifferent type $c_t^{\cdot}$. For a type $c < c_t^{\cdot}$, the left hand side is $v - c$, so it decreases linearly with a slope of -1. The right hand side now is:

$$\sum\nolimits_{\theta=H,L} \pi_\theta^t(c) \left\{ v \left[1 - \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} \right] + e^{-\gamma\Delta} \left(\frac{1 - F_\theta\left(c_t^{\cdot}\right)}{1 - F_\theta\left(c_{t-1}^{\cdot}\right)} \right)^{n-1} (v - c) \right\}$$

where we note that c enters only in the posterior $\pi_\theta^t(c)$ and in the last term $v - c$, since $c_t^{\cdot}$ is the strategy used by the other players. When $f_H(c)/f_L(c)$ (and therefore a fortiori $\pi^1(c)$ and $\pi^t(c)$)

does not increase too sharply in c, then this term certainly declines in c at a rate slower than -1, so types $c > c_t^{\cdot}$ find it optimal to abstain and types $c < c_t^{\cdot}$ to contribute (just as in the case with no aggregate learning). But when $f_H(c)/f_L(c)$ can change sharply (as for example when the support of costs changes with the state, so that the posterior is discontinuous in c), then the equilibrium may not be in cutoff strategies. It is interesting that even in the simplest case with $m = 1$ we might have these complications.

Non-stationary environments

There are applications for collective action problems in which it seems natural to assume that the value of a group's success changes over time. In environmental problems, for example, the consequences of not solving the collective problem (i.e. failing to succeed in collective action) become more severe over time. In this section we show how non-stationarities can be easily incorporated in the analysis and lead to new insights.

Assume that if the group does not obtain the common goal at $t-1$, then at the beginning of t each player suffers a loss of Z^t for $Z > 1$. The loss from not obtaining the common goal (say closing the ozone hole) grows exponentially over time.[4] The equilibrium condition now becomes:

$$v - c_t^{\cdot\cdot} - Z^t = v\left[1 - \left(\frac{1-F\left(c_t^{\cdot\cdot}\right)}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}\right] + e^{-\gamma\Delta}\left(\frac{1-F\left(c_t^{\cdot\cdot}\right)}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}\left(v - c_t^{\cdot\cdot} - Z^{t+1}\right) - Z^t$$

At time t, the cost Z^t is sunk, and thus irrelevant for the decision. But the cost at $t+1$ still matters, since if the group is successful at t, it can avoid the cost Z^{t+1}. As in the previous sections, the left hand side is the utility from contributing, the right hand side is the utility for not contributing. In both cases, a players suffers a cost Z^t, which then can be simplified. In the utility for not contributing we now have Z^{t+1} times the probability that none of the other players contributes. The loss Z^{t+1} does not affect the decision at $t+1$, when it is a suck cost, but it affects the decision at t. This condition can be rewritten as:

$$c_t^{\cdot\cdot} = \left[\frac{\left(1 - e^{-\gamma\Delta}\left(1 - \frac{Z^{t+1}}{v}\right)\right)\left(\frac{1-F\left(c_t^{\cdot\cdot}\right)}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}}{1 - e^{-\gamma\Delta}\left(\frac{1-F\left(c_t^{\cdot\cdot}\right)}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}}\right] v \tag{14}$$

For a given $c_{t-1}^{\cdot\cdot}$ and $c_t^{\cdot\cdot}$, the right hand side increases in Z and t. The monotonic worsening of the environment reduces the utility of not contributing and leads to a lower utility of not contributing. It can be shown that the cutpoints $c_t^{\cdot\cdot}$ are strictly increasing in $\mathbf{t}$ and eventually success is achieved for all realizations of types.

[4] There are of course other ways to introduce non-stationary elements in the model (for example we could have assumed that the distribution of c or v changes over time). We chose to model non-stationarity as above because it seems it better captures the phenomenon described in the example of environmental protection described above.

On the time horizon and the effectiveness of imposed deadlines

Theorem 6 showed that, while the ex ante probability if success is strictly optimal for any n, the limit probability of success always converges to zero unless $\alpha_n \to 0$ sufficiently fast. This suggests the question of whether there are simple modifications in the strategic interaction of the players that may avert this "curse of large numbers" for the collective action problem. We leave the general problem of studying the optimal dynamic mechanism for future research, but focus here on the discussion of simple rules that the group could adopt hoping to improve the performance: *deadlines*, where the group commits to terminate the volunteering game if the goal is not reached by some specified finite period T (if such a commitment power is granted to the group). While terminating the game at T may be suboptimal once period T arrives, commitment to such a rule may be beneficial if it stimulates more volunteering in periods $\tau = 1, ...T$. Intuitively, in the dynamic collective action problem players have a free rider problem not only with respect to other participants, but to the future selves of all participants (including themselves); imposing a terminal period for contributions could, in principle, limit this problem.

The following result shows that in large groups, deadlines may have an especially undesirable side effect by generating a unique equilibrium in which participation is exactly zero, so the group does not even try to achieve the goal. For a sufficiently high n, therefore, a deadline of T periods is strictly suboptimal since it generates a payoff of exactly zero, while by Theorem 4 we know that in the unbounded game the probability of success is always strictly positive, thus the expected payoff is strictly positive.

Proposition A2. *Assume $m_n = \alpha n$ for some $\alpha < 1$. For any finite deadline T, participation is exactly zero if n is sufficiently large.*

Proof: We proceed in two steps.

Step 1. Consider period $T > 1$ and assume that the number of missing volunteers is larger or equal than $k_n = \frac{m_n}{T} > 0$; and that the lower bound of types is $l \geq 0$. We now prove that there is $n^{(T)}$ such that for $n > n^{(T)}$, the probability of contributing is zero for all players. It follows that at any history in which $k_n \geq \frac{m_n}{T}$ at stage T, then the continuation values are $Q_T^{k_n}(l) = V_T^{k_n}(c, l) = 0$ for any $c \geq l$ and $l \geq 0$.

At period T, let the equilibrium cutpoint be $c_n^{T,T}$ (we omit here for simplicity the dependence on h_t and l), which must satisfy:

$$c_n^{T,T} = vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(c_n^{T,T}; l)\right) = \Psi\left(c_n^{T,T}\right) \tag{15}$$

where we define the function $\Psi\left(c_n^{T,T}\right)$, and $z_n = (1-\alpha)\, n + k_n$ and $\beta_n = \frac{k_n}{z_n} \geq \frac{\alpha}{T} > 0$. We now prove that for n large enough, it must be $c_n^{T,T} = l$. If $l > 0$ and $c_n^{T,T} > l$, then the right hand side of (15) converges to zero, but the left hand side converges to a strictly positive value, a contradiction. Assume therefore that $l = 0$ and $c_n^{T,T} > 0$. Define $\widehat{\beta}_n$ to be the value such that $\widetilde{F}(\widehat{\beta}_n; l) = \frac{\beta_n}{1-1/z_n}$. This is the value that maximizes the right hand side of (15), i.e. $\Psi(\cdot)$. It is straightforward to verify that it must be that $\widehat{\beta}_n > 0$ for any n. Moreover, since $\beta_n \geq \frac{\alpha}{T}$, we have that for n large enough:

$$vB\left(\beta_n z_n - 1, z_n - 1, \frac{\beta_n}{1-1/z_n}\right) = vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(\widehat{\beta}_n; l)\right) < \widehat{\beta}_n \simeq \beta_n / f(0), \tag{16}$$

given that the first and second terms converge to zero, but $\widehat{\beta}_n$ converges to $\beta_n/f(0)$, which is strictly positive for all n. From the inequality in 16, we have that $\Psi'\left(c_n^{T,T}\right) < 1$, at any fixed point $c_n^{T,T}$ of Ψ. This follows from the fact that the right hand side of (15) has a maximum below the 45^o line, so if there is a strictly positive fixed point, there must be a fixed point at which $\Psi(c)$ intersects the 45^o line from above. But, as we now show, this is impossible. To see this note that:

$$B'\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right) = B\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right)\begin{bmatrix} \frac{\beta_n z_n - 1}{F\left(c_n^{T,T}\right)} \\ -\frac{z_n - \beta_n z_n}{1 - F\left(c_n^{T,T}\right)} \end{bmatrix} f\left(c_n^{T,T}\right) \tag{17}$$

$$\to \frac{f(0)\, c_n^{T,T}}{v}\left[\frac{\beta_n z_n - 1}{f(0)\, c_n^{T}} - z_n - \beta_n z_n\right] = \frac{1}{v}\left[1 - f(0)\, c_n^{T,T} - f(0)\frac{c_n^{T,T}}{\beta_n} - \frac{1}{\beta_n z_n}\right] \cdot \beta_n z_n \to \infty$$

So we have a contradiction, since the right hand side of (16) converges to a bounded value. We conclude that if $k_n \geq \frac{m_n}{T}$, then $c_n^{T,T} = l$ is the unique fixed point of (15) for any $l \geq 0$.

Step 2. Assume as an induction step that for some $t < T$ and all $\tau \geq t+1$ we have: there is a $n^{(\tau)}$ such that for $n > n^{(\tau)}$ we have $V_\tau^{k_n^\tau}(c, l_n) = Q_\tau^{k_n^\tau}(l_n) = 0$ for all $c \geq 0$ when $k_n^\tau \geq \overline{k}_n^\tau = (T-\tau+1)\frac{m_n}{T}$. This property is true for $t+1 = T$ by Step 1. We prove the result if we prove that:

$$c_n^{t,T} = e^{-\gamma\Delta}\sum\nolimits_{j=0}^{k_n^t - 1}\left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T})\right) B\left(j, n - 1 - m_n + k_n^t, \widetilde{F}(c_n^{t,T}; l_n)\right)\right]. \tag{18}$$

has a no strictly positive fixed point $c_n^{t,T}$ when $k_n^\tau \geq \overline{k}_n^\tau = (T-\tau+1)\frac{m_n}{T}$. To this goal define, similarly as in Step 1, $z_n^t = n - m_n + k_n^t$, $\beta_n^t = k_n^t / z_n^t$ and $\widetilde{F}_n^t = \widetilde{F}(c_n^{t,T}; l_n)$.

By the induction step we have

$$c_n^{t,T} = e^{-\gamma\Delta}\sum\nolimits_{j=\frac{m_n}{T}+1}^{k_n^t - 1}\left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T})\right) B\left(j, z_n^t - 1, \widetilde{F}_n^t\right)\right] \tag{19}$$

The right hand side of (19) can be bounded above by:

$$\begin{aligned} & e^{-\gamma\Delta} v \cdot \sum\nolimits_{j=\frac{m_n}{T}+1}^{k_n^t - 1}\left[B\left(j, z_n^t - 1, \widetilde{F}_n^t\right)\right] \\ \leq \; & \exp\left(-n\left(\frac{\alpha}{T}\log\frac{\alpha/T}{\widetilde{F}_n^t} + (1 - \frac{\alpha}{T})\log\frac{1-\alpha/T}{1-\widetilde{F}_n^t}\right)\right) = D_n(\widetilde{F}_n^t) \end{aligned}$$

where for the inequality we used the Chernoff bound of the upper tail of the Binomial distribution (see, for example, Ash [1990, 4.7.2)]. Without loss of generality we can assume that $\widetilde{F}_n^t < \frac{\alpha}{T}$ for n sufficiently large. Indeed, if this were not the case then we would have some $c^{\alpha/T} > 0$ such that $c_n^{t,T} > c^{\alpha/T}$, but this is impossible in equilibrium since the expected benefit of contributing for a single player converges to zero as $n \to \infty$. Note that for any $\widetilde{F}_n^t > \underline{F}$ for some $\underline{F} > 0$, we have

$D_n(\widetilde{F}_n^t) < \underline{F}$, since $D_n(\widetilde{F}_n^t) \to 0$. Moreover a Taylor approximation tells us that for $F < \underline{F}$ with $\underline{F}$ sufficiently small we have: $D_n(\widetilde{F}_n^t) = D_n(0) + D_n'(0)\widetilde{F}_n^t + o(\widetilde{F}_n^t)$, where $o(\widetilde{F}_n^t)/\widetilde{F}_n^t \to 0$ as $\widetilde{F}_n^t \to 0$. But then, if we have a positive fixed $c_n^{t,T}$ point, we have:

$$c_n^{t,T} \leq D_n(0) + D_n'(0)f(0)c_n^{t,T} + o(c_n^{t,T}) = c_n^{t,T}\left[D_n'(0)f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}}\right] < c_n^{t,T}$$

since $\left[D_n'(0)f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}}\right]$ can be chosen to be arbitrarily small, a contradiction. We can iterate the argument up to the first period. We must therefore have $m_n \geq \overline{k}_n^{1} = (T)\frac{m_n}{T} = m_n$, which implies that $c_n^1 = 0$ for $n > n^{(1)}$. ■

The threshold n_T^* on n such that participation is zero for $n > n_T^*$ may depend on the other parameters of the game. Proposition A2 says that as $n \to \infty$, the minimal deadline consistent with positive participation must also diverge at infinity; for any finite deadline T, positive participation is inconsistent with sufficiently large groups.

To see the intuition of this result, consider first the case in which $T = 1$. In this case the equilibrium cutpoint is determined by the equation:[5]

$$c_n^{1,1} = vB\left(m_n - 1, n - 1 - m_n, F\left(c_n^{1,1}\right)\right) \tag{20}$$

The left hand side is the cost of contributing for the marginal type; the right hand side is the expected benefit, that is v times the probability of being pivotal. An equilibrium cutoff is a fixed point of this equation.

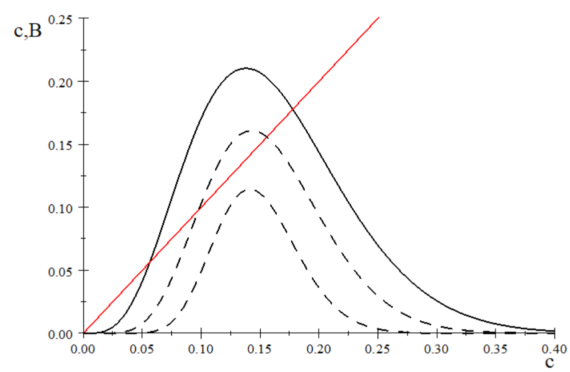

Figure 1: Illustration of fixed point equation (20) for $n = 30$ (solid top curve), $n = 50$ (dashed middle curve), and $n = 100$ (dashed bottom curve).

Figure 1 illustrates the result. The 45^o line is the left hand side of (20); the black curves are the right hand side of (20), for different values of n.[6] Equilibrium cutpoints correspond to intersections of the curve with the 45^o line. As n increases the right hand side shifts down; when n is 100 or

[5] We denote $c_n^{\tau,T}$ to be the cutoff at period τ in a model with a deadline with T periods.

[6] In the figure, F is Uniform, $v = 1$, and $n = 30$ (the solid line),50 (the intermediate dashed line) and 100 (the lower dashed line).

larger, the curve no longer intersects the 45^o degree line for any $p > 0$, implying that the only equilibrium cutpoint is $c = 0$. In Proposition A2 we indeed show that when the number of missing volunteers m_n grows at the speed of n (as when $m_n = \alpha n$ for some $\alpha < 1$) and $T = 1$ (or there only one period left before termination), then there $n^{(1)}$ such that for $n > n^{(1)}$ there is no equilibrium in which players with $c > 0$ contribute.

When $T > 1$, we can show that this phenomenon generalizes with an inductive argument, but there are some complications. Consider, for simplicity, $T = 2$. From the discussion above, we know that there is a $n^{(1)}$ such that for $n > n^{(1)}$ the probability of a contribution is zero when the remaining contributors are $k_n^2 \geq \alpha n/2$. It follows that at $T = 1$ a players knows that if volunteers at $T = 1$ are not at least $\alpha n/2$, then $k_n^2 \geq \alpha n/2$ and the project fails. The complication is that now a player can receive a positive payoff for any k_n^2 in which $k_n^2 < \alpha n/2$, not just when a specific threshold is reached. There are two possibilities. The first is when $F\left(c_n^{1,2}\right) < \alpha/2$, where $c_n^{1,2}$ is the cutoff at $T = 1$. In this case the probability of at least $k_n^1 \geq \alpha n/2$ contributors at $T = 1$ converges to zero fast, and indeed can be bounded above by

$$H(c_n^{1,2}) = \exp\left(-n \cdot D(\alpha \,\|F(c_n^{1,2})\,)\right).$$

where $D(\alpha \,\|F(c)\,) = \exp\left(-n\left(\frac{\alpha}{2}\log\frac{\alpha/2}{F(c)} + (1-\frac{\alpha}{2})\log\frac{1-\alpha/2}{1-F(c)}\right)\right)$ is the Kullback–Leibler divergence. This function of c lies below the 45^o line for n large, just as in Figure (1). The other possibility is that $F\left(c_n^{1,2}\right) \geq \alpha/2$. But this can be ruled out by the following argument. As $n \to \infty$, the expected benefit of contributing for an individual player always converges to zero; but then players with a strictly positive cost near $c = F^{-1}(\alpha/2)$ will not find it optimal to contribute in equilibrium. In any equilibrium sequence we must have $c_n^1 \to 0$, so we are always in the first case in which $F\left(c_n^{1,2}\right) < \alpha/2$ for n sufficiently large.

A notable implication of this result is that when m_n grows at the speed of n, the static one-shot game leads to *zero* probability of success for large enough n. Therefore, in such environments, the dynamic game leads to better outcomes in terms of welfare than the static game. I.e., the benefits of information transmission and coordination from the equilibrium dynamics outweighs the delay costs of dynamic free riding. This contrasts with the example in Section **??** of the paper, where welfare is higher in the static game than the dynamic game when m_n is a fixed constant that does not increase with n. An interesting conjecture is that dynamics produces welfare gains (losses) relative to the static game when the free riding problem is more (less) severe, where more severe corresponds to environments where m_n grows faster than speed of $n^{2/3}$.