Dynamic Collective Action and the Power of Large Numbers*

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Abstract

Collective action problems arise when a group's common goal can only be achieved if enough members engage in a costly action. We study the equilibrium properties of such problems when decisions are made dynamically over time, delay is costly, and members have heterogeneous and privately known preferences. In these environments, time acts as both a curse and a blessing: individuals have incentives to delay their actions to observe the commitments of others, yet time can also serve as a coordination device. This tension generates an equilibrium dynamic that combines learning about the eventual probability of success with free-riding incentives and coordination challenges. For finite n, dynamic collective decisions are inherently probabilistic: the probability of success is always positive, but there remains a strictly positive probability that the process halts inefficiently—stopping "cold turkey" before success is achieved. As n becomes large, however, the outcome becomes essentially deterministic: success is achieved either almost instantly at minimum cost or not at all, depending on how quickly the threshold number of participants grows with n. We establish this result by uncovering a novel connection between the equilibria of dynamic Bayesian games and the class of optimal static direct mechanisms requiring Honest and Obedient behavior—an insight that may have broader applicability to mechanism design, beyond collective action environments.

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1 Introduction

Collective action problems unfold dynamically, and require time to achieve success. Public protests often start small and, if they do not die out, reach a critical mass only gradually.[1] International agreements are initiated by small group of countries, but then require years to rally further support and collect ratifications from enough participants.[2] It is customary to leave public good and charitable-giving fund drives open for weeks or months. In these problems a common goal is achieved if and only if individual participation is sufficiently high: time allows participants to better coordinate, getting a better sense of whether the goal is achievable, which avoids wasting resources if a cause does not have enough support from early contributors.

In these situations, time is both a curse and a blessing. It is a curse because it creates incentives for individuals to delay their participation, waiting to see what others do; and this moral hazard problem is not just with respect to other players, but also against an individual's future selves. It is however also a blessing because it enables coordination and information transmission.

Some progress has been made in the study of dynamic moral hazard problems with free riding, identifying some inefficiencies that arise, and even characterizing conditions under which efficient allocations are possible in special cases (Schelling [1960], Fershtman and Nitzan [1991], Admati and Perry [1991], Marx and Matthews [2000], Gale [1995, 2001], Matthews [2013], Lockwood and Thomas (2002), Battaglini et al. [2014] among others). A key takeaway from those studies is that the cost of moral hazard is not so much that projects are not completed or that there are excess contributions, but that they are completed with inefficient delays. Under special conditions, for example when the players are very patient and have long horizons, the delay inefficiencies and the costs of overcontribution can disappear entirely.

There is however an important factor affecting the ability of social groups to dynamically achieve common goals that has not been adequately studied, which paints a less optimistic picture - *private information.* A limitation of the existing studies of dynamic moral hazard is that they focus almost exclusively on environments with complete information, where pref-

[1] For example, a classic historical account of the dynamic evolution of the Great English Agricultural Uprising of 1839 is presented by Hobsbaum and Rude [1968]. More recently, Lohmann [1994] discusses the dynamic evolution of demonstrations leading to the fall of the Berlin wall in 1989.

[2] The average time of ratification of the 1992 *United Nations Framework Convention on Climate Change* was 810 days. Indeed more than a year passed by the time a quarter of the countries had ratified it, and more than two years before half of the countries had ratified it (Fredrickson and Gaston [2000]).

erences are common knowledge, so the uncertainty faced by individuals is entirely strategic: How many others will do their part? Will additional participants engage? But in environments with private information about preferences, players may become more (or less) hesitant to volunteer over time because of what they learn about the prospects for success. The relevant question then becomes: Are there a sufficient number of committed citizens for whom it is worthwhile to participate in the collective action? Because of the strategic effects of information transmission and learning, there is an additional curse from time: as time progresses, players are uncertain whether delays are simply due to procrastination (i.e., moral hazard), or because other players lack a willingness to participate due to high private costs (i.e., adverse selection). Players gradually accumulate information on the others' preferences and willingness to contribute. Because delays from procrastination are unavoidable, this process generates a *systematic bias* toward failure. It is indeed possible that any further participation stops "cold turkey" after some histories because uncommitted individuals become too pessimistic about the prospects for eventual success and simply give up. How and to what extent can the passage of time still help solve the collective action problem? These questions have not been addressed and this paper provides some answers.

We study these and related questions in a simple but natural dynamic collective action model with private information about preferences. The collective action problem is modeled as a dynamic participation game with n individuals, or group members. The game takes place over a possibly infinite sequence of discrete periods. In period one, each of the n members independently and simultaneously decides whether or not to join in the collective action. Participation decisions are binary and the associated sunk cost of participation for member i, c_i, is private information and is borne immediately. If the threshold number of required participants for success, m, is met in period one, the game ends; each non-participant member receives the benefit v and each participant receives $v - c_i$. If the threshold is not met, the game continues to the second period, and all non-participants again must decide whether or not to join the action. The game continues indefinitely like this, with discounting, until the threshold is met at which point the game ends and each member receives the success payoff of v, discounted by the number of periods it took to reach the threshold; in addition each participant loses $-c_i$, discounted by the period when they participated. If the threshold is never met, then the game continues forever, with final payoffs equal to 0 for each non-participant, and $-c_i$ for each participant, discounted by the period when they participated.

When $m = 1$ the game can be interpreted as a classic war of attrition. The prize is achieving the public good without paying for it (i.e. v); at any time, a player can terminate the game by participating and secure a lower payoff $v - c_i$. When $m > 1$, however, the game is fundamentally different from a war of attrition. If a player quits, the payoff now is endogenous, because it depends on the *externality* generated by the *other* players completing the game. Differently from the classic war of attrition, moreover, here there is no exogenous flow cost to be paid for each period in which a player stays in the game, and it would not be natural to assume any: the cost of procrastination is already captured by the delay in receiving the public good.[3]

Two basic lessons emerge from the analysis. The first is that when $m > 1$ collective decisions are probabilistic in all equilibria: the process starts for sure in the sense that some group members volunteer with positive probability in early periods; the final outcome, however, depends on the trajectory of participation decisions, which in turn depends on the exact realization of types. With positive probability the required threshold m is reached and the public good is obtained. On the other hand, the process will also get "stuck" with strictly positive probability, where additional participation ceases forever, resulting in failure, even if there is a sufficiently large number of players whose cost of contributing is strictly smaller than the private benefit of success. This fundamentally differs from the case with $m = 1$, in which if there is at least one player i with type $c_i < v$, the collective goal is achieved for sure in finite time.

The second lesson is that outcomes become essentially deterministic with large populations. This however does not imply that success is guaranteed (or even possible in some equilibrium) nor that other sources of inefficiencies disappear. We indeed show that the group's success depends on the speed with which the threshold fraction of players required for success $\alpha_n = m_n/n$ converges to zero as $n \to \infty$: if α_n converges faster than the *cube root of* $1/n$, then there is a sequence of equilibria converging to the efficient collective decision with no delay; if instead it converges slower than the cube root of $1/n$, then the expected utility in the game converges to zero for all types of all players in *all* equilibria. We call this phenomenon the *Curse of Large Numbers* for collective action because the requirement for efficiency is very strong: even if an arbitrarily small share ε of population is required to contribute, the project is doomed for failure for large n.[4]

[3]While this is true when $m = 1$ as well, in that case it does not have qualitative implications since the game ends as soon as one player contributes. As we will explain, however, it is essential to the strategic analysis of the game when $m > 1$.

[4]To prove the limit efficiency result, we rely on the characterization of symmetric PBE, thus making the result stronger since we show there is no loss in assuming symmetric equilibria. The finding that the expected utility in the game converges to zero when α_n converges slower than the cube root of $1/n$, however, is not restricted to symmetric equilibria, and actually holds for all equilibrium thus giving us a fully general impossibility result for dynamic collective action games.

The result that success is achieved if and only if α_n decreases at a rate of order of $n^{-1/3}$ or less offers two important qualitative insights: one regarding the limits of dynamic collective action, and the other concerning a deep but yet unexplored connection between dynamic games with incomplete information and optimal mechanisms without transfers.

Regarding the first point, a key result in several landmark papers in the literature is that the inefficiency in the war of attrition and related games disappears as $n \to \infty$. But these results are based on the restrictive assumption that $m_n = 1$ for all n (Bliss and Nalebuff, 1984). Our results significantly strengthen these findings by showing that full efficiency in the limit can still be achieved even when m_n increases with n, as long as the number of required contributors, m_n, grows at a rate on the order of $n^{2/3}$ or less. If, however, the prevailing lesson in the literature was that large numbers solve the free rider problem in the war of attrition, our result significantly qualifies this conclusion: while our condition is substantially less restrictive than previous ones, it is still very demanding. Our threshold implies that efficiency fails for sure in the limit if success always requires even an arbitrarily small positive fraction of the population.

As for the connection with mechanism design, we show that the condition for asymptotic efficiency in the dynamic war of attrition is exactly the same as the condition for efficiency in the optimal Honest and Obedient (static) direct mechanism without transfers. This finding is important not only because it shows that the expected outcome generated by spontaneous contributions over time cannot be improved upon in limit economies through more sophisticated forms of communication or coordination, but also because it highlights a deep connection between dynamic games with incomplete information and optimal Honest and Obedient (HO) mechanisms, which we are able to exploit in the impossibility result described above — a point we now explain.

In seminal contributions, Ausubel and Deneckere [1989a, 1989b] characterized the equilibria of a dynamic two-person bargaining game with one-sided private information in terms of the equilibria of a suitable static incentive-compatible (IC) and individually rational (IR) mechanism. This characterization allowed them to study the welfare properties of the equilibrium payoffs of the dynamic game using the static IC and IR mechanism.

As we will explain in greater detail, the equivalence highlighted by Ausubel and Deneckere does not hold in games with multilateral private information—such as the one we study—in which the equilibria cannot be characterized in terms of simple IC and IR constraints of a direct mechanism. In our work, we instead generalize the approach by demonstrating that for any perfect Bayesian equilibrium (PBE) of the dynamic contribution game, we can construct a corresponding static *Honest and Obedient (HO) mechanism* that achieves exactly the same expected payoffs for each player type.[5] Given this, the payoffs attainable in any PBE can be bounded above by the payoff attainable in the best HO mechanism. This argument allows us to apply Theorem 4 in Battaglini and Palfrey [2024], which shows that if α_n converges to zero slower than the cube root of $1/n$, then the payoff in the best Honest and Obedient mechanism converges to zero, thereby proving the impossibility part of the result.

The connection between the PBE of dynamic games with multilateral private information and static HO mechanisms that we exploit in our analysis is, to our knowledge, new and potentially applicable for characterizing properties of equilibria in dynamic games with incomplete information in other environments as well.

The connection between HO mechanisms and the PBE of the dynamic game, however, does not allow us to prove that if α_n converges to zero faster than the cube root of $1/n$, an efficient limit equilibrium exists, since the set of payoffs obtainable in HO mechanisms is larger than the set of payoffs supported in PBE. We therefore prove the possibility result directly using the characterization of the game. The reason the cube root of $1/n$ plays a key role for efficiency stems from fundamental features of the strategic interaction and will be intuitively explained in Section 5.

The reminder of the paper is organized as follows. In the next subsection we discuss the related literature. We presents the model in Section 2. In Section 3, we study the dynamic volunteer's dilemma in which $m_n = 1$, a case of independent interest that we refer to as the *dynamic volunteer's dilemma*.[6] Building on this analysis, in Section 4, we study the *dynamic collective action problem* in which $m_n > 1$, showing that it is qualitatively and substantively different. Section 5 is dedicated to the study of the properties of equilibria in large economies as $n \to \infty$. In Section 6 we present extensions and variations of the model.

1.1 Related literature

Our paper is most closely related to three lines of research. The first line, briefly mentioned above, studies dynamic contributions to public goods. This research has focused on settings with perfect information in which there is no uncertainty regarding the environment (say, for example, regarding the players' evaluations of the public good or their cost of contributing).

[5] HO mechanisms are a more demanding class of mechanisms, first introduced by Myerson (1982).

[6] The classic volunteer's dilemma was introduced by Diekmann [1985] and refers to a static situation with complete information about preferences in which a group can achieve a collective goal if at least one member volunteers to pay a fixed cost c. It has been adopted as a paradigm of cooperation in economics (Bergstrom [2018], Battaglini and Palfrey [2025]), biology (Patel et al. [2018]), neuroscience (Park et al. 2019), and other fields.

The key issue in these works is the moral hazard problem faced by participants who would like the public good, but prefer for others to contribute and thus may postpone their contributions.[7] Our paper extends this analysis by considering environments where players' costs of contribution are private and heterogeneous. The key new feature of this environment is that, as time unfolds, players learn about the distribution of types and re-evaluate whether it is optimal to contribute.

The second line of research to which our work is connected is the war of attrition. Bliss and Nalebuff [1984] present a continuous time model in which a public good is achieved if at least one player volunteers for it. Players have private and heterogeneous costs of volunteering and may choose to wait hoping that other players will do it for them. In this problem, there is the usual moral hazard problem with public goods, but in addition there is uncertainty regarding the conditions under which other players will be willing to contribute. As time progresses, players update their beliefs about the cost of contributing of the remaining players. Our work extends this volunteer's dilemma framework by addressing the collective action problem with *multiple volunteers* ($m_n > 1$), and even allowing m_n to grow with n without bounds. These differences are essential to model collective action in realistic environments, since it seems natural to require multiple contributors for success in common projects with even moderately sized groups. As highlighted above, the analysis of the general collective action problem with $m_n > 1$ is qualitatively different from the volunteers dilemma. While the war of attrition has been used as leading framework in numerous important economic problems (see Alesina and Drazen [1991] for a prominent example), applications restrict the analysis to setting in which only one player needs to concede to terminate the game, as in the volunteers dilemma.[8]

[7] Seminal contributions are Admati and Perry [1991] and Marx and Matthews [2000]. The first paper characterizes the unique equilibrium in a game in which two players alternate contributing until the sum of contributions pass a threshold. They show that equilibrium generally implies delay and characterized conditions for efficiency, showing that they are demanding. Marx and Matthews [2000] extend the analysis to environments in which players can contribute simultaneously in each period, showing that although equilibria involve delay, the conditions for the existence of equilibria that eventually reach efficient outcomes are generally satisfied if there is a positive jump when a threshold is reached, as in our environment (or if the utility for contributions is linear). Battaglini et al. [2014] show that efficient outcomes are attainable even in environments with continuous, non-linear utilities and no threshold.

[8] The only other papers we are aware of that consider extensions of the war of attrition with m multiple volunteers are Haigh and Cannings [1989] and Bulow and Klemperer [1999], but these works study different economic environments, restrict attention to fixed m, and lead to very different results. The first restricts the analysis to environments with complete information. Bulow and Klemperer [1999] study an all pay auction with $m + n$ players and m prizes in which players pay a strictly positive exogenous cost κ per period to stay in the game. In the all pay auction the payoff of quitting is exogenous even with $m > 1$ since the auction allocates private goods among the other players. In this setting, they show there is a unique equilibrium in which the allocation is ex post efficient and the ex ante inefficiency converges to zero as $n \to \infty$.

A key feature of the war of attrition, especially in our version requiring multiple volunteers, is that players learn over time the distribution of other players' types. Recent works have studied this dynamic signaling problem in contribution games that are related but different from ours. Mutluer [2024] studies a dynamic volunteer's dilemma with $m = 1$ under the assumption that players decide sequentially, one at a time. In this model, with an exogenous order of play, players cannot self-select as in our model. However, observing previous decisions is informative in that game because players have common values and receive correlated signals. The author shows that, in this environment, there is a unique finite population size above which increasing the group size is associated with a reduction in the probability of providing the public good. Deb et al. [2024] study a model of reward-based crowdfunding, in which contributions to a business project can be made by randomly arriving private investors who make one-shot decisions in the period of arrival, and by a long-lived donor who has an interest in the success of the project. They study the extent to which the donor can alleviate the coordination problem of the private investors by signaling his valuation of the project with donations.

The third related line of research studies the optimal design of static public good mechanisms (d'Aspremont et al. [1990], Mailath and Postlewaite [1990], Hellwig [2003], Battaglini and Palfrey [2024], among others). As in our work, this literature focuses on environments with private information; but, in addition to restricting attention to static settings, it has a normative flavor, allowing for potentially complex communication mechanisms that require commitment power. Our analysis is positive and explicitly dynamic: we are not interested in characterizing the optimal mechanism, but in studying collective action in a natural dynamic environment in which society cannot commit to complex cooperation mechanisms. As discussed above, however, there is a deep connection between our work and this literature, especially insofar as it studies HO mechanisms (Battaglini and Palfrey [2024]).

Two other recent contributions are related to our work. Madsen and Schmaya [2025] study optimal mechanisms for public good provision in dynamic environments and characterize the optimal mechanism for public goods requiring recurrent non-monetary contributions for maintenance. Besides taking a normative approach, a key difference between Madsen and Schmaya's environment and ours is that they focus on settings with excludable public goods, allowing the mechanism to restrict access in order to elicit preferences. Jehiel and Leduc [2025] present a dynamic model of collective action in which a principal can intervene to punish volunteers. In the model, both the agents and the principal receive opportunities at random times to volunteer or punish a volunteer, respectively. The main result is that deterrence of collective action is possible only if the principal can intervene sufficiently quickly after an agent's arrival.

2 The Dynamic Collective Action Model

A *dynamic collective action problem* is a public good game played over a possibly infinite sequence of periods, $t = 1, 2, ..., \infty$. There is a group with n members, and each member i has a privately known participation cost c_i that is an independent draw from a commonly known cost distribution $F(c)$ with a continuous density function, $f(c)$ that is strictly positive on the interval $[0, 1]$. In each period before the game ends, each member simultaneously and independently decides whether to participate or not, a binary choice. A decision by any member i to contribute in period t is irreversible and incurs the cost c_i in the period at which i contributes. If at least m members of the group have chosen to participate up to and including in period t the game ends, and we say the group *succeeds in period* t. If fewer than m members have contributed by period t, the game continues to period $t + 1$. Participation decisions are publicly observed.

Group success yields a common benefit of $v \in (0, 1)$ to all group members. Payoffs are discounted, and the discount factor is $e^{-\gamma\Delta}$, where Δ denotes the time delay between periods and $\gamma > 0$ denotes the discount rate. Hence, if the group succeeds in period t the payoff to inactive members is $ve^{-\gamma\Delta(t-1)}$ and the payoff to each member who contributed in period $\tau \leq t$ is $ve^{-\gamma\Delta(t-1)} - c_i e^{-\gamma\Delta(\tau-1)}$. If the game continues indefinitely and success is never achieved, then all members who never chose to participate receive a payoff of 0 and each member who participated in period τ receives a payoff $-c_i e^{-\gamma\Delta(\tau-1)}$.[9]

We study the set of Perfect Bayesian equilibria of this dynamic game. A strategy is a function that assigns, for each (public) history of play at period t and for each type, c, a (possibly mixed) current action to either participate or not.[10] A history at t, h_t, has two components. The first component is the sequence of participation decisions by all members in the previous periods, $t = 1, ..., t - 1$. The sequence of the sets of members choosing to participate in each period before t is $I_t = (I^1, ..., I^{t-1})$, where I^τ for $\tau \leq t - 1$ is the set of players who volunteer at τ. We define $\kappa_t = (\kappa^1, ..., \kappa^{t-1})$ as the sequence of the numbers of members choosing to participate in each period before t, where $\kappa^\tau = |I^\tau|$ for $\tau \leq t-1$ is the set of players who volunteer at t. The second component of the public history is a public

[9] The standard collective action problem is a one shot game and corresponds to an extreme case in our model where $\gamma\Delta = \infty$.

[10] Because participation decisions are irreversible, any individual who chooses to participate in some period τ is inactive in all future periods $t > \tau$.

signal that is observed at the beginning of each period t, θ^t, which is the outcome of a single independent draw from the uniform distribution on $[0,1]$.[11] Thus, a history at period t is denoted by $h_t = (I_t, \theta_t)$, where $\theta_t = (\theta^1, ..., \theta^{t-1})$. Given a history h_t we denote by k_t the minimum number of remaining members at period t who must contribute in order for the group to achieve success. That is, $k_t = m - \sum_{\tau=1}^{t-1} \kappa^\tau$.

In Sections 2 and 3, we characterize the set of symmetric Perfect Bayesian Equilibria (PBE) in the game. Focusing on symmetric equilibria is both natural in this setting and standard in the literature on the war of attrition (e.g., Bliss and Nalebuff [1984], Bulow and Klemperer [1999]). However, in our analysis, this focus is also without loss of generality and strengthens our results. Indeed, one of our key results is that limit efficiency can be achieved in equilibrium when α_n converges to zero faster than the cube root of $1/n$ as $n \to \infty$. The focus on symmetric PBE allows us to conclude that efficiency is achievable with symmetric equilibria. Another key result is that if α_n converges to zero slower than the cube root of $1/n$, then in all sequences of PBE we have $\lim_{n\to\infty} EU_n(c) = 0$. Notably, for this impossibility result, we do not need the restriction to symmetric equilibria.

As we will prove, in a symmetric PBE, the strategies of the game are characterized as a sequence of history-dependent cutpoints, $\{c(h_t)\}_{t=1}^{\infty}$, whereby, in period t, following history h_t, any player with a type $c \leq c(h_t)$ who has not yet participated chooses to participate. Hence, in an equilibrium, the game ends in period 1 if there are at least m members with $c \in [0, c(h_1)]$, where $h_1 = (\varnothing, \theta_1)$; the game ends in period 2 if, for some $j < m$, there are exactly j members with $c \in [0, c(h_1)]$, and at least $m - j$ members with $c \in [c(h_1), c(h_2)]$, where $h_2 = (j, \theta_2)$; and so forth. In a symmetric equilibrium, moreover, players are treated anonymously by the other players: two histories h_t and h'_t, that differ only in the identity of the volunteers (so $\kappa_t = \kappa'_t$) are associated with the same cutpoint for the active players: $c(h_t) = c(h'_t)$.

In an equilibrium, as the game progresses each remaining member's belief about the distribution of the other remaining members' types are updated simply by increasing the lower bound of the distribution of types, which we denote by $l_{h_t} = c(h_{t-1})$, with $l_{h_1} = 0$. An equilibrium consists of a history-dependent *cutoff strategy,* $c(h_t)$, and conditional beliefs about the distribution of remaining members, derived by Bayes rule: $\widetilde{F}(c; l_{h_t}) = \max\left\{0, \frac{F(c)-F(l_{h_t})}{1-F(l_{h_t})}\right\}$. Associated with an equilibrium are two *value functions.* For any history, h_t, $Q(h_t)$ is the continuation value for a member who has previously participated (i.e., any member with $c \leq l_{h_t}$); and $V(c, h_t)$ is the continuation value for a member with cost c who

[11] The public signal does not affect the characterization of equilibrium, but simplifies the existence proof.

has not yet contributed (i.e., any member with $c > l_{h_t}$).

To solve this class of games, a key observation is that, for any given cutoff strategy each public history of play results in a new continuation game $\Gamma(h_t)$ defined by the *lower bound on the cost distribution*, l_{h_t}, and the *minimum number of contributors that are still needed for success*, k_t. This is just another collective action problem with a new group size, $n' = n - m + k_t$, a new threshold, $m' = k_t$, and a new distribution F' which is F truncated below at l_{h_t}. Hence, a solution to the original collective action involves solving for all games in this general class of collective action problems.

To characterize the equilibria of this more general class of games, we initially begin by solving the case where the continuation game is a *dynamic volunteers dilemma*, i.e., the special case of $k = 1$, and *any lower bound* l, and proceed inductively. That is, given the solutions for $k = 1$ for all $l \in [0, 1]$, and assuming we have a characterization for each $k' = 2, ..., k - 1$ for all $l_{h_t} \in [0, 1]$ we characterize the PBE for k.

3 The Dynamic Volunteer's Dilemma

3.1 Equilibrium

In this case, we solve the continuation game for a group of n members following a history h_t, at which point exactly $m - 1$ members have already contributed, so $k_t = 1$ and there are $(n - 1) - (m - 1) = n - m$ remaining uncommitted members, and the lower bound on the distribution of types is l_{h_t}. We call this continuation game, where the group is missing exactly one contributor to succeed, *the dynamic volunteers dilemma*. We start with a preliminary result. A PBE is in *cutoff strategies* if there is a cutoff $c(h_t)$ such that any type $c \leq c(h_t)$ find it optimal to contribute, and any type $c > c(h_t)$ find it strictly optimal to wait. It is straightforward to show:

Lemma 1. *All PBE of a continuation game starting from any history h_t in which only one contributor is missing for success are in cutoff strategies.*

Proof: See the appendix. ■

We next show that the symmetric equilibrium of the continuation game, i.e., the equilibrium cutoff strategy function, $c(\cdot)$, is *uniquely determined.* The argument is as follows.

Suppose that at some history h_t we have reached a point at which the lower bound of the support is l_{h_t} and exactly $m - 1$ members have already contributed, so $k_t = 1$. Denote by $V^-(c, h_t)$ the expected value for a type c who does not contribute in the current period, and $V^+(c, h_t)$ the expected value for a type c who chooses to contribute in the current period.

Since $k_t = 1$, success is automatically achieved if the member contributes, so $V^+(c, h_t) = v - c$. The expression for $V^-(c, h_t)$ is slightly more complicated and depends on the continuation value if no other member contributes, which in turn depends on the current cutpoint, $c(h_t)$ (to be solved for) and the continuation value of the game if no other member contributes in the current period, $V(c, h_{t+1})$, in which case the lower bound of the distribution of types will change in the next period to $l_{h_{t+1}} = c(h_t)$:

$$V^-(c, h_t) = v\left[1 - \left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m} V(c, h_{t+1}) \qquad (1)$$

We have the following observation:

Lemma 2. $l_{h_t} < v \Rightarrow c(h_t) > l_{h_t}$. *Furthermore,* $\lim_{t\to\infty} c(h_t) = v$.

Proof: See the appendix $\blacksquare$.

In equilibrium, it must be that $V^+(c(h_t), h_t) = V^-(c(h_t), h_t)$. Furthermore, from Lemma 2, we know that $c(h_{t+1}) > c(h_t)$, so, the $c(h_t)$ type will contribute for sure in period $t+1$ if the game continues. Hence:

$$V^-(c(h_t), h_t) = v\left[1 - \left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m} (v - c(h_t)).$$

We conclude that the indifference condition that characterizes the equilibrium cutpoint is:

$$v - c(h_t) = v\left[1 - \left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F(c(h_t))}{1 - F(l_{h_t})}\right)^{n-m} (v - c(h_t)) \qquad (2)$$

More generally, since the game can continue for many periods without anyone contributing, (2) can be rewritten as a difference equation for all $\tau \geq t$:

$$c(h_\tau) = \left[\frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}{1 - e^{-\gamma\Delta}\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}\right] v \qquad (3)$$

with $c(h_{t-1}) = l_{h_t}$. Condition (3) is illustrated in Figure 1.

The left hand side is the opportunity cost of contributing in the current period for the cutoff type, $c(h_\tau)$; the right hand side is the discounted net expected benefit of waiting and contributing next period, for the cutoff type. The right hand side is a function of $c(h_\tau)$ itself, since it depends on the strategy followed by the other players, which is itself determined by the cutoff $c(h_\tau)$. The equilibrium cutoff is a fixed-point of (3). As illustrated by Figure 1, the right hand side of (3) is always decreasing, higher than c at $c = c(h_{\tau-1})$, and lower than

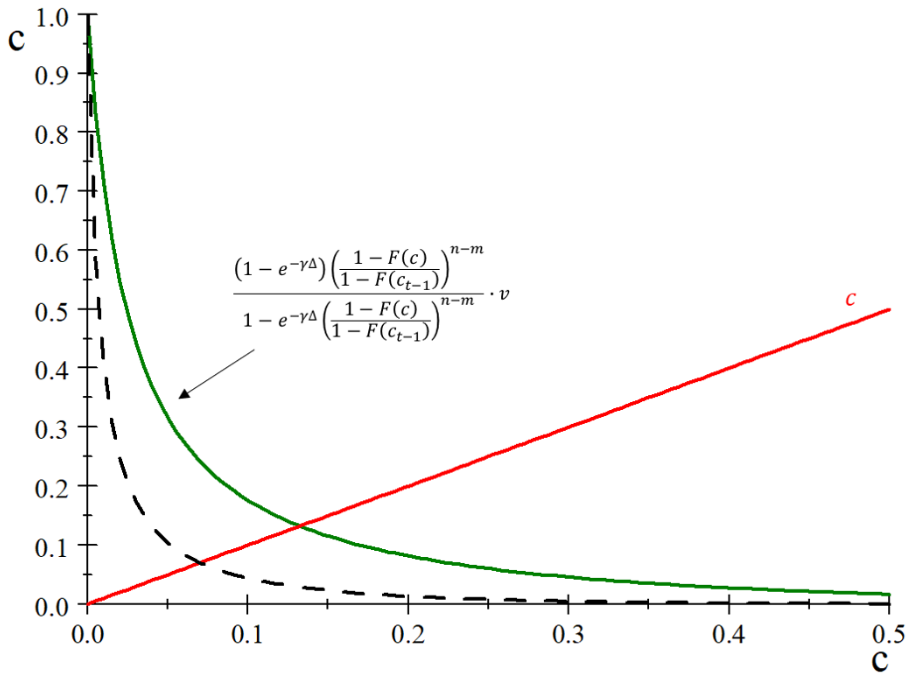

Figure 1: The equilibrium for $m = 1$. The solid and dashed curves are, respectively, the right hand side of (3) when $n = 5$ and $n = 10$, under the assumption that F is uniform, γ and Δ are such that $e^{-\gamma\Delta} = 0.95$ and $c_{t-1} = 0$.

c at $c = 1$: so there is a unique interior fixed-point $c(h_\tau) \in (c(h_{\tau-1}), 1)$. Difference equation (3) can therefore be used to mechanically construct all the equilibrium cutpoints for all h_t and the associated value functions, and thus fully characterize the unique symmetric PBE for the case of $k = 1$.

3.2 Value functions

The unique characterization of equilibrium cutpoints in the system of equations given by (3) imply two relevant value functions: (1) the equilibrium continuation value for an agent who has committed before t, given that the lower-bound of types is l_{h_t}, which we denote as $Q(l_{h_t})$; and (2) the equilibrium continuation value of a player of type c when the lower-bound on types is l who is still uncommitted, that we denote as $V(c, l_{h_t})$. These will be useful for the full characterization when $k > 1$.

Consider first the value in period t of a member who has already committed in some previous period before t, when the lower-bound on types is $l_{h_t} = c(h_{t-1})$ and the group is still missing *exactly* $k = 1$ contributors for success. Note there are $n-1-(m-2) = n-m+1$ other uncommitted players. The value $Q(l_{h_t})$ for such a committed player (net of the sunk cost of contributing) when the lower-bound on the types is l_{h_t} can be written as:[12]

$$Q\left(l_{h_t}\right) = v\left(1 - B(0, n-m+1, \widetilde{F}(c(h_t)))\right) + e^{-\gamma\Delta} Q\left(c(h_t)\right) B(0, n-m+1, \widetilde{F}(c(h_t); l_{h_t})) \quad (4)$$

where $B(0, n-m+1, x)$ is the binomial probability of 0 successes out of $n-m+1$ trials when the probability of success equals x. Using the notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ we can use recursion to solve for $Q\left(l_{h_t}\right)$ and write (4) as:

$$Q\left(l_{h_t}\right) = \sum\nolimits_{\tau=1}^{\infty} e^{-\gamma\Delta(\tau-1)} \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1-F\left(c_j\right)}{1-F\left(c_{j-1}\right)}\right)^{n-m+1}\right] \left[1 - \left(\frac{1-F\left(c_{\tau+1}\right)}{1-F\left(c_\tau\right)}\right)^{n-m+1}\right] v \quad (5)$$

where, by convention, $c_0 \equiv c_1 = l_{h_t}$, so $\frac{1-F(c_1)}{1-F(c_0)} = 1$. Note that (5) is defined only as a function of the primitives and the current and future cutpoints $(c_\tau)_{\tau=1}^{\infty}$, defined by (1).

We can also define the value of being uncommitted for a type c at $k=1$ when the lower-bound on types is l_{h_t} and the group is missing $k=1$ contributors for success as follows. Note that in this case there are $n-1-(m-1) = n-m$ other uncommitted players. If $c \in (l_{h_t}, c(h_t)]$, we have $V(c, l_{h_t}) = v - c$ since the game ends when they contribute. Hence, $V(c(h_t), l_{h_t}) = v - c(h_t)$. If $c > c(h_t)$, we can define $V(c, l_{h_t})$ when $c > c(h_t)$ as follows:

$$V(c, l_{h_t}) = \left[1 - \left(\frac{1-F\left(c(h_t)\right)}{1-F\left(l_{h_t}\right)}\right)^{n-m}\right] v + e^{-\gamma\Delta} \left(\frac{1-F\left(c(h_t)\right)}{1-F\left(l_{h_t}\right)}\right)^{n-m} V(c, c(h_t)) \quad (6)$$

Using the same notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ as in the derivation of $Q\left(l_{h_t}\right)$, define $T(c)$ as the largest τ such that $c_\tau \leq c$. Solving recursively, as with $Q\left(l_{h_t}\right)$, gives:

$$\begin{aligned} V(c, l_{h_t}) \;=\; & \sum\nolimits_{\tau=1}^{T(c)-1} e^{-\gamma\Delta(\tau-1)} \cdot \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1-F\left(c_j\right)}{1-F\left(c_{j-1}\right)}\right)^{n-m}\right] \left[1 - \left(\frac{1-F\left(c_{\tau+1}\right)}{1-F\left(c_\tau\right)}\right)^{n-m}\right] v \\ & + e^{-\gamma\Delta\cdot(T(c)-1)} \cdot \left[\prod\nolimits_{j=1}^{T(c)} \left(\frac{1-F\left(c_j\right)}{1-F\left(c_{j-1}\right)}\right)^{n-m}\right] \cdot (v-c) \end{aligned} \quad (7)$$

Note again that $V(c, l_{h_t})$ is fully determined by the cutpoints $(c_\tau)_{\tau=1}^{\infty}$.

The characterization of the equilibrium for the dynamic volunteers dilemma has two immediate implications. First, even when only one contributor is needed, the equilibrium is inefficient since the probability of immediate realization of the public good, $\Phi_t = 1 - [1 - F(c_1)]^{n-1}$, is strictly less than 1. Second, the distortion is not due to the fact that the

[12] In writing the continuation value for the committed players as $Q(l_{h_t})$ we are slightly abusing notation, since this value is both a direct function of the lower bound of types l_{h_t}, and the history h_t, that may directly affect future cutpoints if there are multiple equilibria. We avoid writing it as $Q(l_{h_t}; h_t)$ for simplicity when it does not generate confusion. When there is only one remaining missing volunteer, there is no loss of generality since the equilibrium is unique.

project is not realized when it should be realized, but because it is realized with a delay. In equilibrium the threshold for participation is always lower than v, but it gradually approaches this bound: so if there is at least one player with cost lower than v, then the project would be eventually realized.

3.3 The effects of n and Δ (or γ) on success and welfare

Since the distortion depends on a delay in realization, is natural to ask whether the distortion may be mitigated by an increase in n, or a decrease in the delay costs (i.e. a reduction in either Δ or γ). In the online appendix we present supplementary material studying the effect of n and Δ on equilibrium behavior when $k = 1$. With respect to n, we prove that an increase in n always leads to a uniform reduction in the equilibrium cutoff points, implying that players are individually more reluctant to contribute. This free-rider effect, however, is mitigated by the addition of more potential volunteers. A simple revealed preference argument allows us to prove that this second effect always dominates: no type c is worse off after an increase in n, and some types c are strictly better off; and the probability of success increases. The comparative statics with respect to Δ is less clear-cut. A decrease in Δ, also leads to a uniform reduction in the equilibrium cutoff points, thus making players individually more reluctant to contribute. The number of players, however, remains constant, so a decrease in Δ always reduces the probability of success in each period. However, since the costs of delays are lower when Δ is smaller, a decrease in Δ has a welfare effect that cannot be unambiguously signed. In Section 5, we characterize the condition under which the expected utility converges to the efficient level, which includes $m_n = 1$ as a special case. For any given n, however, full efficiency is unattainable even in the limit as Δ and/or γ converges to zero.

4 The General Dynamic Collective Action Problem

If $k > 1$ contributors are required, the analysis is more complicated and we modify the notation accordingly. We denote a history by h_t^k, to indicate a period t history at which there are exactly k missing contributors.[13] Let $Q^k(l_{h_t^k})$ denote the expected utility of a committed player (net of the sunk contributing cost) at history h_t^k when the lower bound on types is $l_{h_t^k}$; and define $V^k(c, l_{h_t^k})$ to be the expected utility of an active (uncommitted) player

[13] For the case of $k = 1$, we will henceforth use the notation h_t^1.

of type c at history h_t^k.[14] As in the previous section, we say a PBE is in cutoff strategies if, for every history h_t^k, there is a critical cost $c(h_t^k)$ with the property that every type $c < c(h_t^k)$ finds it strictly optimal to contribute at h_t^k, every type $c > c(h_t^k)$ finds it strictly optimal to wait at h_t^k, and a player with type $c(h_t^k)$ is indifferent between contributing and waiting at h_t^k. We have:

Lemma 3. *All PBE of a continuation game starting from any history h_t^k in which k contributors are missing for success are in cutoff strategies.*

Proof: See the appendix. ■

In the following, we will also use the notation $c^k(l_{h_t^k})$ to denote the cutpoint $c(h_t^k)$ in order to highlight that for a given equilibrium it directly depends on the missing contributors k and the lower-bound $l_{h_t^k}$.[15] For a given history h_t^k, when it does not generate confusion, we also define as before c_t^k recursively to be $c_t^k = c^k(c_{t-1}^k)$ with an initial condition $c_0^k = l_{h_{t-j}^k}$ in case there are no contributors in the periods between from $t-j$ to $t-1$.

The previous section uniquely characterized the equilibrium for any $l \in [0,1]$ when $k=1$, denoted here as $Q^1\left(l_{h_t^1}\right)$, $V^1\left(c, l_{h_t^1}\right)$, $c^1\left(h_t^1\right)$ for all $l_{h_t^1}$ and h_t^1, with the superscript indicating $k=1$. We now proceed inductively on k in the following way. Assume that, for all $j = 1, ..., k-1$, the functions $Q^j\left(l_{h_t^j}\right)$, $V^j\left(c, l_{h_t^j}\right)$, $c^j\left(h_t^j\right)$ are fully defined for all $l_{h_t^j}$ and h_t^j. In the next subsection, we first show that this information enables us to characterize the cutpoints $c_t^k(l_{h_t^k})$ for all h_t^k; then, using the cutpoints $\left(c_t^j\left(l_{h_t^k}\right)\right)_{j \leq k, l_{h_t^k} \in [0,1]}$ the value functions $Q^k\left(l_{h_t^k}\right)$, $V^k\left(c, l_{h_t^k}\right)$ can be derived.[16] In this way we can characterize all continuation games in all histories, for all $k \leq m$ and $l \in [0,1]$. In Section 4.2 we complete the analysis of the equilibria for finite n by studying when they lead to success.

4.1 Characterization and existence

Consider first the value of an uncommitted player who contributes at h_t^k. Note that when we are missing $k > 1$ contributors, for an uncommitted player, there are $n - 1 - (m - k) =$

[14] As for the case with $k=1$, we are slightly abusing notation here to keep it simple. For a given lower bound l and number of missing volunteers k, both $Q^k(l)$ and $V^k(c,l)$ may directly depend on h_t as well, since, with the possibility of multiple equilibria, the equilibrium that is played in a continuation game can depend on payoff irrelevant elements of the history. Omitting this information is without loss of generality when, as we will do here, we characterize the properties of an equilibrium in a continuation game for given continuation values.

[15] In general, for a given lower bound l and number of missing volunteers k, $c^k(l_{h_t})$ also depends on h_t since h_t may determine how the others play in the presence of multiple equilibria. We omit the dependence on h_t for simplicity when it does not generate confusion.

[16] Note that for any $t > 0$, given the threshold at $t-1$, i.e. $l_{h_t^k} = c(h_t^k)$, the posterior at h_t^k is uniquely defined as the truncated distribution $\widetilde{F}(c; l_{h_t})$, as defined in Section 2.

$n-1-m+k$ other uncommitted players in the game. Hence, the value for an uncommitted player with cost c who contributes at history h_t^k is:

$$\left[V^k\right]^+(c,l_{h_t^k}) = v\sum_{j=k-1}^{n-1-m+k} B\left(j,n-1-m+k,\widetilde{F}(c^k(l_{h_t^k});l_{h_t})\right) \tag{8}$$
$$+e^{-\gamma\Delta}\sum_{j=0}^{k-2} B\left(j,n-1-m+k,\widetilde{F}(c^k(l_{h_t^k});l_{h_t})\right)Q^{k-j-1}(c^k(l_{h_t^k}))-c.$$

The first term on the right hand side of (8) is the probability the group reaches success at t times the prize v; the second term collects the probabilities that an insufficient number $j<k$ of players volunteers times the discounted expected continuation value for a contributor, $e^{-\gamma\Delta}Q^{k-j-1}(c^k(l_{h_t^k}))$; the third term is the cost of contributing.

The function $Q^{k-j-1}\left(c^k(l_{h_t^k})\right)$ used in (8) does not depend on the type of the agent, but it depends on $c^k(l_{h_t^k})$ because $c^k(l_{h_t^k})$ becomes the minimal type at $t+1$. One can simplify $\left[V^k\right]^+(c,l_{h_t^k})$ by adding and subtracting v times the probability the game does not end at t to obtain:

$$\left[V^k\right]^+(c,l_{h_t^k}) = v-c-e^{-\gamma\Delta}\sum_{j=0}^{k-2}\left[\begin{array}{c}\left(\frac{v}{e^{-\gamma\Delta}}-Q^{k-j-1}(c^k(l_{h_t^k}))\right)\\ \cdot B\left(j,n-1-m+k,\widetilde{F}(c^k(l_{h_t^k});l_{h_t})\right)\end{array}\right] \tag{9}$$

Consider next the expected continuation payoff of an uncommitted agent of type c does not contribute at history h_t^k. Similarly as in (9), we can derive it as:

$$\left[V^k\right]^-(c,l_{h_t^k}) = v-e^{-\gamma\Delta}\sum_{j=0}^{k-1}\left[\begin{array}{c}\left(\frac{v}{e^{-\gamma\Delta}}-V^{k-j}(c,c^k(l_{h_t^k}))\right)\\ \cdot B\left(j,n-1-m+k,\widetilde{F}(c^k(l_{h_t^k});l_{h_t})\right)\end{array}\right] \tag{10}$$

There are now two possibilities. The first case arises if the equilibrium cutoff is a corner solution in which $c^k(l_{h_t^k})=l_{h_t^k}$ and all types $c\geq l_{h_t^k}$ choose not to contribute. This is possible only if $\left[V^k\right]^+(l_{h_t^k},l_{h_t^k})\leq\left[V^k\right]^-(l_{h_t^k},l_{h_t^k})$, which (as it can be easily verified from (9) and (10)) is equivalent to $e^{-\gamma\Delta}Q^{k-1}(l_{h_t^k})\leq l_{h_t^k}$. When this condition is satisfied, even a player of type $l_{h_t^k}$ (the lowest possible cost at h_t^k) is unwilling to commit if no other player with a higher cost will ever contribute.[17] Indeed, if this player contributes alone, then the discounted benefit is $e^{-\gamma\Delta}Q^{k-1}(l_{h_t^k})$ and the cost is $l_{h_t^k}$. On the other hand, if this condition is not satisfied, then any player of type close to $l_{h_t^k}$ is willing to contribute even if s/he expects no participation from the other players at t, in the hope that other players will continue contributing in history h_{t+1}^{k-1} in which the missing contributors are $k-1$.

[17]Strictly speaking, we have a corner solution if $\left[V^k\right]^+(l_{h_t^k},l_{h_t^k})<\left[V^k\right]^-(l_{h_t^k},l_{h_t^k})$. The properties of a PBE when type $l_{h_t^k}$ is indifferent between contributing or not are the same as when the inequality is strict.

The other possibility is that we have an interior solution with $c^k(l_{h_t^k}) > l_{h_t^k}$. In this case the threshold $c^k(l_{h_t^k})$ is given by an indifference equation similar to the indifference condition for the volunteer's dilemma in equation (2):

$$\left[V_t^k\right]^+ (c^k(l_{h_t^k}), l_{h_t^k}) = \left[V^k\right]^- (c^k(l_{h_t^k}), l_{h_t^k}) \tag{11}$$

For any h_t^k, the right and left hand side of (11) are defined as functions of exclusively the cutpoints $c^k(l_{h_t^k})$ for histories in which k contributors are missing. These cutpoints can be found solving (11) for any h_t^k. After some algebra, these conditions can be written as:

$$c^k(l_{h_t^k}) = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right) - V^{k-j}(c^k(l_{h_t^k}), c^k(l_{h_t^k}))\right) \\ \cdot B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right) \end{array} \right]. \tag{12}$$

where, by convention, we define $Q^0(\cdot) = v/e^{-\gamma\Delta}$. The system of equations defined in (12) characterizes $c^k(l_{h_t^k})$ in the same way as condition (3) defined the cutpoints in the case with $k = 1$.[18] It also has a similar interpretation. The left hand side is the cost of contributing by the cutpoint type $c^k(l_{h_t^k})$; the right hand side is the net expected discounted utility of contributing: the difference between the utility after contributing minus the continuation in the absence of a contribution (i.e., $V^{k-j}(c^k(l_{h_t^k}), c^k(l_{h_t^k}))$). In order to prove existence of a PBE, in Theorem 2 below we prove that a fixed-point of (12) exists for any h_t^k, a step that will be discussed below.

Once we have the cutpoints for all continuation games when k contributors are missing, we can close the circle and define an equilibrium in all continuation games up to $k = m_n$. Given the cutpoints $c^k(l_{h_t^k})$ defined above we can indeed define the expected continuation payoffs at history h_t^k to all the players as a function of $l_{h_t^k}$. The payoff to a player who has contributed in previous periods does not depend on the player's cost, c, and is given by:[19]

$$Q^k(l_{h_t^k}) = v - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-1} \left[\left(\frac{v}{e^{-\gamma\Delta}} - Q^{k-j}(c^k(l_{h_t^k}))\right) B\left(j, n-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right)\right]. \tag{13}$$

Using (9),(10) and (13), we can finally obtain $V^k(c, l_{h_t^k})$, which is equal to $\left[V^k\right]^+(c, l_{h_t^k})$ for $c \le c^k(l_{h_t^k})$, and equal to $\left[V^k\right]^-(c, l_{h_t^k})$ otherwise. With this, we have all the ingredients to complete the induction argument. Using $Q^j(l_{h_t^j})$ and $V^j(c, l_{h_t^j})$ for $j \le k$ we can now obtain $c^{k+1}(l_{h_t^{k+1}})$, and then $Q^j(l_{h_t^j})$ and $V^j(c, l_{h_t^j})$ for $j \le k+1$. We can therefore obtain $c^j(l)$, $Q^j(l)$, and $V^j(c, l)$ for $j \le m_n$, which fully characterizes the equilibrium. At each generic history, h_t^k, and lower bound on types $l_{h_t^k}$, the cutpoints in state k evolve according

[18] One can verify that, for $k = 1$ (the dynamic volunteers dilemma), equation (12) reduces to equation (3).

[19] To find (13) we first write $Q^k(l_{h_t^k})$ in terms of expected payoffs then, as we did for (9) and (10), we rewrite it by adding and subtracting v times the probability the game does not ends at t.

to $c_t^k = c^k(l_{h_t^k})$ where $l_{h_t^k} = c_{t-1}^k$, where $c_{t-1}^k = c^j(l_{h_{t-1}^j})$ for some $j \geq k$ and history h_{t-1}^j. The game is initialized at the history, h_1^m, where $l_{h_1^m} = 0$.

We now have a full characterization of the equilibria.

Theorem 1. *A symmetric PBE is characterized by a monotonically increasing sequence of cutpoints $c_t = c^k(l_{h_t^k})$ for $k \leq m$ and $t = 0, ..., \infty$, where $c^k(l_{h_t^k})$ is inductively defined by (9), (10), (12) and (13) as described above. For each k we have $c_{t-1} \leq c_t < v$ for each $t = 1, ..., \infty$. Furthermore if $c_t = c_{t-1}$ for any t, then $c_\tau = c_{t-1}$ for all $\tau > t$.*

Proof. Given the analysis above, we only need to prove that $c_t^k < v$. To see this, note that the right hand side of (12) is always strictly less than v since

$$e^{-\gamma\Delta}\left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right) - \left[V^{k-j}\right]^+ (c^k(l_{h_t^k}), c^k(l_{h_t^k}))\right) \leq e^{-\gamma\Delta}\left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right)\right) < v.$$

So we have that $c^k(l_{h_t^k}) < v$ for any $l_{h_t^k} < v$. It follows that $c_t = c^k(c_{t-1}) < v$. ■

We complete the analysis proving the existence of a PBE.

Theorem 2. *A symmetric Perfect Bayesian Equilibrium exists.*

Proof: See the discussion below. Details are provided in the online appendix. ■

To understand this result, consider (12). As discussed above, all continuation functions defining the right hand side are defined by the induction step. It is however the case that the value of the right hand side of (12) does not only depend on the cutpoint at t, i.e. $c^k(l_{h_t^k})$. The reason is that at $t+1$ the lower bound has moved to $c_1^k = c^k(l_{h_t^k})$, so the other players use the strategy $c_2^k = c^k(c^k(l_{h_t^k})) = c^k(c_1^k)$. If at $t+1$ we have an interior solution, $\left[V^k\right]^+ (c_1^k, c_1^k)$ can be written as:

$$\begin{aligned}\left[V^k\right]^+ (c_1^k, c_1^k) &= v \cdot \sum\nolimits_{j=k-1}^{n-1-m+k} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k\right) \\ &+ e^{-\gamma\Delta} \cdot \sum\nolimits_{j=0}^{k-2} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k)\right) Q^{k-j-1}(c_2^k) - c,\end{aligned}$$

so c_1^k depends on c_2^k. Analogously, c_2^k is itself a function of c_3^k, and so on. This implies that (12) defines c_1^k as a function of all equilibrium cutoffs that follows it along the "worst history" in which there are no additional contributors. When condition (12) is required for all histories h_t^k, it defines the equilibrium cutoffs $\left(c_j^k\right)_{j=1}^\infty$ as a fixed-point of a correspondence that maps the sequence of cutoffs to itself. To prove existence, we proceed as follows. We first define an auxiliary truncated game in which the players give up and stop contributing if there are T attempts at which no player contributes (for some finite $T > 0$). We then show that, in this game, equilibrium cutpoints $\left(c_j^{T,k}\right)_{j=1}^T$ exist and are defined as fixed points of a condition similar to (12). Finally, we prove that as $T \to \infty$, these cutpoints converge to a limit $\left(c_j^k\right)_{j=1}^{\infty}$ that is an equilibrium of the original game. A key step to prove that the truncated game has a fixed-point is to show that the set of continuation values, and thus the right hand side of (12), is a non-empty, convex-, closed- valued and upper-hemicontinuous correspondence in $c^k(l_{h_t^k})$. We can prove this with an inductive argument over k. To this goal, note that Section 3.1 established, by construction, the existence of a unique PBE for any lower bound on types l when only one contributor is needed. Moreover, we showed that the associated value functions $Q^1(l)$ and $V^1(c,l)$ are continuous both in l and c for any $c \geq l$. For the induction hypothesis, assume that, for all $j = 1, ..., k-1$, the set of continuation value functions $Q^j(l)$ and $V^j(c;l)$ corresponding to a PBE in the continuation game is non-empty, convex-, closed- valued and upper-hemicontinuous in $l_{h_t^j}$. Using this property, we can prove that the the set of continuation value functions at k, $Q^k(l)$ and $V^k(c;l)$ are non empty, convex valued and upper-hemicontinuous in l.

4.2 Participation and comparative statics

In the previous section we mentioned that it is possible the equilibrium cutoff gets stuck, implying contributions stop after some history reached with positive probability in equilibrium (and this possibility has to be contemplated in the characterization). We however did not prove such an event occurs in equilibrium. Indeed, the following result shows that, while *the probability of success is always strictly positive* in every equilibrium, *the probability of prematurely getting stuck is also strictly positive* in equilibrium:

Theorem 3. *For any $n > 2$, for any $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0,1)$, in every symmetric equilibrium:*

1. *The probability of success is strictly positive.*
2. *There are no interior equilibria: in every equilibrium, there is a positive probability of reaching an effectively terminal history h_t^k with $c^k(l_{h_\tau^k}) = l_{h_t^k} < v$* for all $\tau \geq t$.

Proof: Part (1): A sufficient condition for the probability of success to be strictly positive is that $c^m(0) > 0$, since the probability of success would then be bounded below by $1 - (1 - F(c^m(0)))^n > 0$. First, notice that we already proved that $c^1(0) > 0$. We now proceed by induction on the number of volunteers that are needed. Assume that for all $j = 2, ..., m-1$, $c^j(0) > 0$ in every equilibrium and therefore $Q^j(0) > 0$ for $j = 2, ..., m-1$. Now suppose that there is some equilibrium of the $(n, m, \gamma, \Delta, v)$ game in which the probability of success is 0.

This implies that $c^m(0) = 0$ and hence $[V^m]^+(0,0) = 0$. But $[V^m]^+(0,0) \geq e^{-\gamma\Delta}Q^{m-1}(0)v > 0$, a contradiction. Hence, $c^m(0) > 0$ in every equilibrium, so the probability of success is strictly positive in all equilibria.

Part (2) Suppose by way of contradiction that there exists an equilibrium that contains a sequence $\{c_\tau^m\}_{\tau=1}^\infty$ with $\lim_{\tau\to\infty} c_\tau^m = c_\infty^m = v$. (If not, then the result is proved.) This is the sequence of equilibrium cutpoints along those histories where no player has contributed up to period τ. Along such a sequence, it must be that $c_\tau^m - c_{\tau-1}^m \to 0$. Hence:

$$\widetilde{F}(c_\tau^m; c_{\tau-1}^m) \to \frac{F(c_\tau^m) - F\left(c_{\tau-1}^m\right)}{1 - F(v)} \to 0$$

It follows that:

$$\begin{aligned}
\lim_{\tau\to\infty} c_\tau^m &= e^{-\gamma\Delta} \lim_{\tau\to\infty} \sum_{j=0}^{m-1} \left[\left(Q^{m-j-1}(c_\tau^m) - V^{m-j}(c; c_\tau^m) \right) B\left(j, n-1, \widetilde{F}(c_\tau^m; c_{\tau-1}^m)\right) \right] \\
&\to e^{-\gamma\Delta} \left(Q^{m-1}(v) - V^m(c; v) \right) = 0 < v, \text{ a contradiction.}
\end{aligned}$$

The last step follows from the fact that if the lower bound on types is v, then the expected probability that an active player contributes is zero, so $Q^{m-1}(v) = V^m(c; v) = 0$. ■

Since no player with a cost $c > v$ would find it optimal to contribute, the highest possible probability of success achievable is $\overline{p}_n = 1 - [1 - F(v)]^n$. We use this benchmark to evaluate the performance of an equilibrium in a dynamic collective action game. We define a group to be *constrained-successful* if, in all equilibria, there will be at least m contributors by some finite date t whenever there are at least m players with cost $c < v$. That is, $\lim_{t\to\infty} c_t^k = v$, so a group is constrained-successful if the probability of success in all equilibria is $\overline{p}_n$.[20] The following corollary follows immediately from Theorem 3:

Corollary 1. *If $m > 1$ then the group is not constrained-successful, and the probability of success is strictly less than $\overline{p}_n$ in all symmetric equilibria: types with sufficiently low c contribute in early periods, but there is a positive probability of reaching an effectively terminal history h_t^k with $k < m$ and $l_{h_t^k} < v$.*

The main implication of Corollary 1 is that the dynamic process of contributing is probabilistic: the process starts for sure, and players contribute with positive probability in early periods; the final outcome, however, depends on the dynamics of participation which in turn depends on the realization of types. With positive probability the required threshold m is

[20] Being constrained-successful does not imply that the equilibrium is efficient. An equilibrium is efficient if the sum of the costs is lower than nv. When m is finite or when we have a threshold m_n that depends on n but such that $m_n/n \to \alpha < 1$, this condition is always satisfied for n sufficiently large. But these efficient equilibria are unachievable in a honest and obedient mechanism with no transfers.

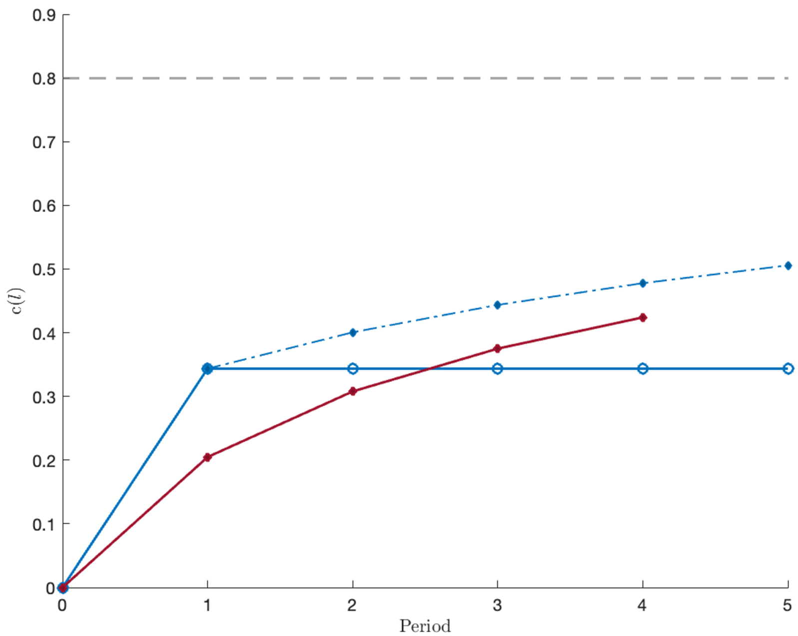

Figure 2: Equilibrium dynamics of cutpoints for $m = 1$ (red) and $m = 2$ (blue). $n = 3$, $v = 0.8$, $n = 3$, $e^{-\gamma\Delta} = 0.8$, F Uniform.

reached and the public good is obtained; but with strictly positive probability the process gets "stuck". This finding shows that the general dynamic collective action problem with $m > 1$ is fundamentally different than the dynamic Volunteer's dilemma with $m = 1$ (in discrete or continuous time as in Bliss and Nalebuff [1984]). It also illustrates a sharp contrast with the multi-person war of attrition studied by Bulow and Klemperer [1999]. In all these cases, the probability the efficient allocation is reached equals 1.

Figure 2 displays the equilibrium dynamics for a very simple example ($n = 3$, $v = 0.8$, $n = 3$, $e^{-\gamma\Delta} = 0.8$, F Uniform) to illustrate some of the key differences in the evolution of equilibrium cutpoints between $m = 1$ and $m > 1$. The horizontal axis is the period number, with cutpoints on the vertical axis.

The solid red upward sloping curve marks the sequence of cutpoints for each period for the war of attrition model ($m = 1$), along the path where there has not yet been a contribution. The initial cutpoint is approximately 0.2 and increases each period after that, until a contribution is made.

The solid and dashed blue curves mark the history-dependent sequence of cutpoints for $m = 2$. The key difference to note is that the initial cutpoint of 0.34 is much higher (by more than 60%) than the initial cutpoint for $m = 1$. If there are zero contributions in the first period, the game is effectively over, since the equilibrium cutpoints never increase again, as indicated by the solid blue line. The higher period 1 cutpoint is needed to support an equilibrium for two reasons, first to reduce the probability of immediate and complete failure (which doesn't happen if $m = 1$), and second to significantly increase the probability of success. These two reasons are complementary, since the second is needed to induce more members to be willing to take the risk of making an initial contribution. The dashed blue line shows the evolution of cutpoints following a single contribution in period 1, and corresponds to the the $m = 1$ equilibrium with an initial lower bound is 0.34. Another reason that the initial cutpoint for $m = 1$ is lower than the initial cutpoint for $m = 2$ is that the dynamic free riding problem is lessened with $m > 1$, due to the threat of immediate failure. One can interpret this as a *bandwagon effect*, since an early contribution increases the likelihood of future contributions.

Equation (12) and, more generally, the characterization of Theorem 1 allow us to compute the equilibria and evaluate the ex ante expected utility to the players as we vary the parameters. The equilibrium ex ante expected value to a member of the group of size, n, for given parameters γ, Δ, v, m, F is defined as:

$$E_n(\gamma, \Delta, v, m, F) = \int_0^1 V^m(c, 0|\gamma, \Delta, v, n) dF(c).$$

Figures 3 and 4 illustrate how E_n varies with n, fixing $m = 2$ and F Uniform. Group size is on the horizontal axis with E_n on the vertical axis. Figure 3 illustrates how E_n varies with v with a given discount factor $e^{-\gamma\Delta} = 0.8$. Figure 4 illustrates how E_n varies with the discount factor, $e^{-\gamma\Delta}$ for $v = 0.65$. In both cases, when there are multiple equilibria, we select the equilibrium with the highest cut-offs.

An increase in n leads to a higher (and earlier) probability of success and hence a higher ex ante per capita value of the game. Moreover, in the examples of Figures 3 and 4, E_n converges rapidly to the fully efficient ex ante value for any v and $e^{-\gamma\Delta}$, so in the limit, $E_\infty = v$. We will characterize the exact conditions for this property in the next section, which is independent of v or $e^{-\gamma\Delta}$, but instead exclusively depends on the speed with which the share of required volunteers m/n converges to zero.

The comparative statics of changing v are perhaps unsurprising: an increase in v is associated with higher utility. The comparative statics of changing the discount factor are more complicated than the effect of changing v. Increasing the discount factor leads to greater cost efficiency, but exacerbates the adverse incentive to delay contributions. So, an increase in patience, $e^{-\gamma\Delta}$, results in more delay, a lower probability of success, and lower ex

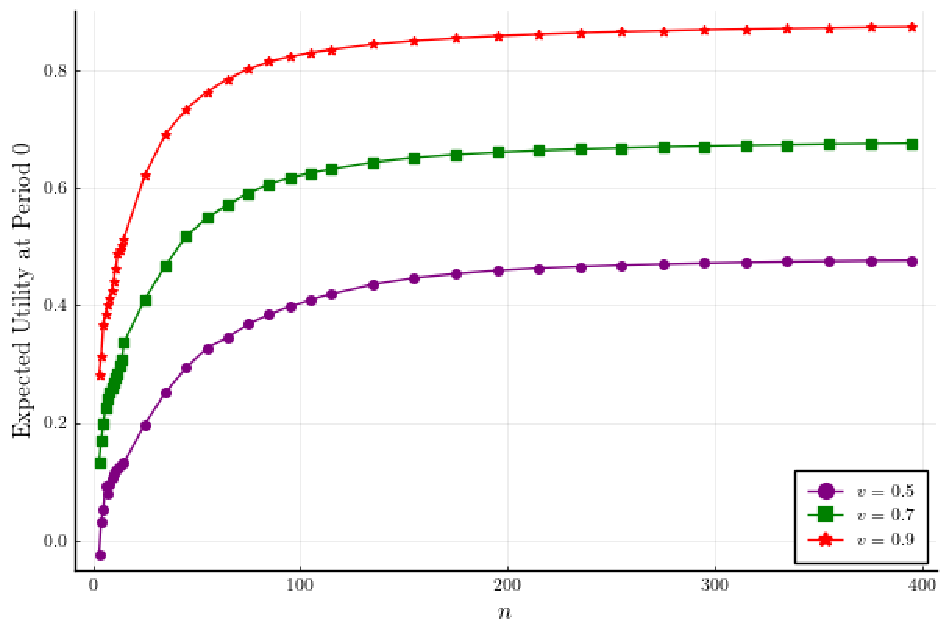

Figure 3: Ex ante discounted value, E_n.
$v = 0.5, 0.7, 0.9$, $m = 2$, $n \leq 400$, and $e^{-\gamma\Delta} = 0.8$, F Uniform..

ante group value. The examples of Figures 3 and 4 suggest that for any positive discount factor and any n, the players would be better off in a static game where they could commit to immediate termination after a single period (corresponding to $e^{-\gamma\Delta} = 0$). While this is true in the example, it is not generally the case that the delay costs of dynamic free riding outweigh the coordination benefits from coordination benefits of information transmission. If m_n grows at the same rate as n (as opposed to this example, where $m_n = 2$ for all n), then for sufficiently large n the probability of success in the static game is exactly 0, while the probability of success in the dynamic game is strictly positive. We discuss this result in Section 6.3.

We conclude this section with an additional comment on the comparative statics properties of the equilibria. In Section 3, we showed that when only one volunteer is needed (i.e., $m = 1$), a unique symmetric PBE exists. This allowed for clean comparative statics results concerning the equilibrium cut points (see Section 3.3). However, this is no longer the case when more than one volunteer is needed. For example, consider the case where F is uniformly distributed in $[l, 1]$, $v = 0.3$, $n = 3$, $m = 2$, $e^{-\gamma\Delta} = 0.8$ and $l = 0.05$. In this case, we have a PBE in which (12) admits a unique solution: the initial equilibrium cutpoint is $c_0(n) = 0.095$. If we increase the number of players from $n = 3$ to $n' = 5$, we have two interior equilibria, $c_0(n') = 0.053$ and $c'_0(n') = 0.109$; and a corner equilibrium in which no player volunteers, $c''_0(n') = l$. It is therefore possible that an increase of n from 3 to 5 is

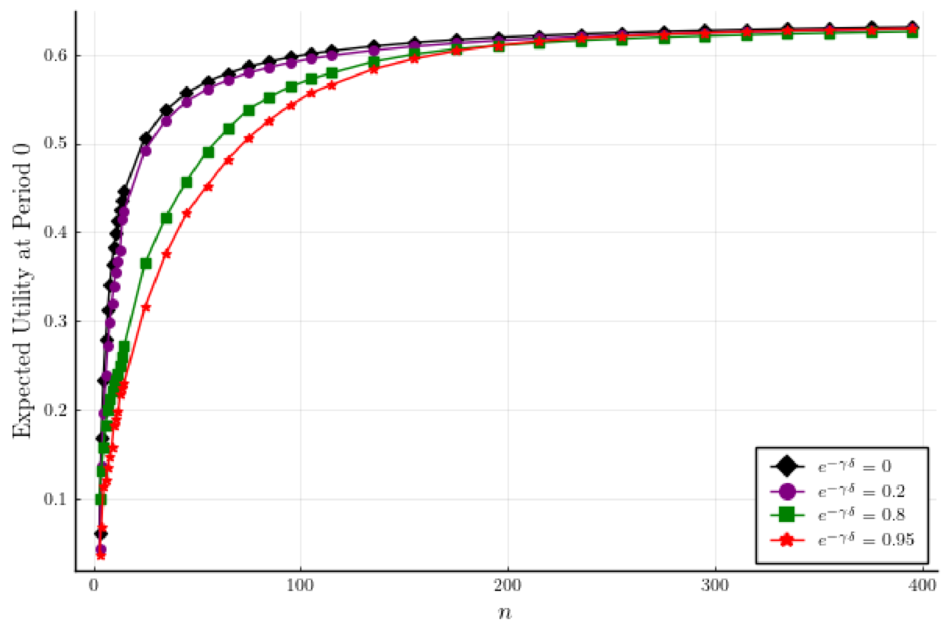

Figure 4: Ex ante discounted value, E_n.
$v = 0.65$, $m = 2$, $n \leq 400$, $e^{-\gamma\Delta} = 0, 0.2, 0.8, 0.95$, F Uniform.

associated to a decrease of the cutpoint at $t = 0$ from $c_0(n) = 0.095$ to $c_0(n') = 0.053$ or $c_0''(n') = 0.05$; or that it is associated to an increase from $c_0(v) = 0.095$ to $c_0'(v') = 0.109$.

To understand why the cutpoints for volunteering may not be monotonic in n when more than one volunteer is needed, compare the cases of $m = 1$ and $m = 2$. In the $m = 1$ case, an increase in n does not affect the expected utility of volunteering: it remains v, regardless of the number or behavior of other players. The only effect of increasing n, ceteris paribus, is a higher probability that someone other than player i volunteers, thereby reducing the probability that i is pivotal. This unambiguously lowers the equilibrium cutpoint for volunteering.

Now, consider the $m = 2$ case. If player i volunteers while no other player does, the value of volunteering for i, net of the realized cost c_i, is equal to the value of a passive player in the continuation game with $n - 1$ active players where only one volunteer is needed. It is easy to prove that an increase in n increases i's expected value in this continuation game, since when n is larger, the expected wait in the continuation game is smaller. This generates two opposing effects. On the one hand, a larger n increases the probability that i's action is superfluous (i.e., more than two other players volunteer), which, as in the $m = 1$ case, pushes the cutpoint downward. On the other hand, it also raises the expected value of volunteering in the event in which no other player volunteers, which pushes the cutpoint upward. These conflicting forces explain why cutpoints may not be monotonic in n when $m > 1$.

5 Large groups

We next turn to an analysis of the properties of the PBE and welfare as $n \to \infty$. In the continuous time model of the war of attrition with $m_n = 1$ by Bliss and Nalebuff [1984], the equilibrium is asymptotically efficient as $n \to \infty$. Large numbers, therefore, eliminate the free rider problem in a collective action.[21] In this section we study the welfare properties of our more general environment in which m_n contributions are needed. In collective action problems with large groups is natural to assume that m_n is larger than one, and indeed that it grows with n. In these cases, we will show that the existence of an efficient equilibrium critically depends on the relative speed with which m_n grows with n.

As previously defined, the share of required contributors is $\alpha_n = m_n/n$. For any sequence $\{\alpha'_n\}_{n=1}^{\infty}$, we say that α_n converges to zero slower (resp., faster or at the same rate) than α'_n if $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = \infty$ (resp., $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = 0$, or $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = l$ for some finite l). For future reference, for two sequences a_n, b_n with $a_n \to 0$, $b_n \to 0$, we write $a_n \succ b_n$ if $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, and $a_n \prec b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

In the following, we maintain the assumption that $\lim_{n\to\infty} \alpha_n < F(v)$, so achieving group success with the minimum threshold is always ex ante efficient as $n \to \infty$. In the first best, the expected per capita utility, as $n \to \infty$, is $W_n^* = v - E\left[\sum_{j=1}^{m_n} c^{[j]}(n)/n\right]$, where $c^{[j]}(n)$ is the jth lowest cost with n samples from $F(c)$. Note that when $\alpha_n \to 0$, then $W_n^* \to v$ as $n \to \infty$; when $\alpha_n \to \alpha < F(v)$, then we still have $W_n^* \to W^* \geq v - F^{-1}(\alpha) > 0$ as $n \to \infty$. It is also immediate to see that if $m_n > 1$, then efficiency is not guaranteed by large numbers, since equilibria with arbitrarily low probability of success are possible for any n if Δ is sufficiently large, even if m_n is constant.

As anticipated in the introduction, the key threshold for efficiency is the speed at which $n^{-1/3}$ converges to zero: if $\alpha_n \prec n^{-1/3}$, then we there is a sequence of equilibria that achieves asymptotic efficiency; if $\alpha_n \succ n^{-1/3}$, then the expected utility converges to zero in all equilibria. Before we formally prove the results, it is useful to discuss intuitively the reason the cube root of $1/n$ plays such a key role in the characterization and why this threshold stems from fundamental features of the strategic interaction.

Suppose for simplicity that types are uniformly distributed in $[0, 1]$, and consider a sequence of equilibria converging to a limit in which success is achieved with no delay. A necessary condition for this to happen is that for large n the first period cutoff for par-

[21] The same is true in the all pay auction model by Bulow and Klemperer [1999] in which $m > 1$, but fixed. The existence of an asymptotically efficient equilibrium is also the typical result in the literature with perfect information (see Marx and Matthews [2000] and Battaglini et al. [2014] for instance).

ticipation, $c^*_{1,n}$, is sufficiently high that the at least $\alpha_n n$ members have a cost less than or equal to $c^*_{1,n}$ with probability close to 1. This, in turn, requires that $c^*_{1,n}$ converges to zero slower or at the same rate as α_n as $n \to \infty$ (hence, $c^*_{1,n} \gtrsim \alpha_n$), otherwise, with probability close to 1, the share of volunteers would fall short of the requirement. The cutoff type $c^*_{1,n}$, moreover, must be on the order of the expected benefit of contributing, i.e. the probability for an individual player to affect the decision in the first period, which can be shown to be proportional to $B(\alpha_n n, n, c^*_{1,n})$, the binomial probability of $\alpha_n n$ contribution out of n trials when the probability of an individual contribution is $c^*_{1,n}$.[22] If $c^*_{1,n}$ does not converge to zero at the same rate as $B(\alpha_n n, n, c^*_{1,n})$, then an agent with cost $c^*_{1,n}$ would not be indifferent, thereby contradicting the assumption that $c^*_{1,n}$ represents a suitable cutoff value. Putting this all together gives:

$$\alpha_n \lesssim c^*_{1,n} \simeq B(\alpha_n n, n, c^*_{1,n}) \lesssim B(\alpha_n n, n, \alpha_n) \simeq \frac{1}{\sqrt{\alpha_n n}}, \tag{14}$$

where in the last step we use the well known facts that $B(\alpha_n n, n, c^*_{1,n})$ is maximized at $c^*_{1,n} = \alpha_n$, and that $B(\alpha_n n, n, \alpha_n)$ is on the order of $1/\sqrt{\alpha_n n}$ for large n. But condition (14) cannot hold if $\alpha_n / (1/n)^{1/3} \to \infty$, since in this case α_n converges to zero slower than $1/\sqrt{\alpha_n n}$ as $n \to \infty$.

Note that this argument only establishes this rate of convergence as a necessary condition for instant success: not that it is a necessary and sufficient condition for efficiency. To prove the result, in Theorem 4 of Section 5.1 we show that there is guaranteed to exist a sequence of PBEs for which the probability of being pivotal converges to zero at the rate of $B(\alpha_n n, n, \alpha_n)$ whenever $\alpha_n \prec n^{-1/3}$, which in turn implies instant success with probability 1 in the limit. In Theorem 5 of Section 5.2, we prove that the expected utility converges to zero in all equilibria if $\alpha_n \succ n^{-1/3}$.

5.1 A possibility result

The following theorem establishes an asymptotic efficiency result for large populations for the case in which α_n converges to zero sufficiently fast, showing that there is always a sequence of equilibria such that the benchmark W^*_n is asymptotically achieved as $n \to \infty$. To prove this result we rely on the characterization of symmetric equilibria of the previous section.

[22] Indeed, a contribution may be beneficial even if the agent is not pivotal at $t = 1$, since it changes the state at which the game is played in the following periods. A key step in our analysis is to show that, in any sequence of equilibria converging to a limit efficient equilibrium, these expected benefits have only a second order effect, and can thus be ignored in this discussion.

We can, therefore, not only prove that asymptotic efficiency is achievable in equilibrium, but also that it can be achieved with symmetric equilibria.

We start from a preliminary lemma that is useful to prove the result, but has independent interest. Define $c_{n,1}^{m}(0)$ to be the initial cutpoint in a symmetric equilibrium at the start of the game with n members, where $t=1$, $k=m$ and $l=0$. The lemma shows that, for any sequence of symmetric equilibria, the share of players who volunteer in the first period, i.e. $F\left(c_{n,1}^{m}(0)\right)$, converges to zero at the same speed or slower than $\alpha_n = m_n/n$ if m_n diverges at infinity slower than $n^{2/3}$. We have:

Lemma 4. *If* $m_n \prec n^{2/3}$*, then for any sequence of symmetric equilibria, we have* $\lim_{n\to\infty} F(c_{n,1}^{m_n}(0))/\left(\frac{m_n}{n}\right) > 1.$.

Proof. A proof outline is as follows (see the appendix for the complete argument).

Step 1. We first find that if $m_n \prec n^{2/3}$ then $\frac{F[vB(m_n-1,n-1,\alpha_n)]}{\alpha_n} \to \infty$. To establish this property, we use Stirling's approximation of $B\left(m_n-1, n-1, \alpha_n\right)$ and the fact that, in the neighborhood of 0, $F(c)$ is an approximately a linear function of c.

Step 2. From Theorem 3, any period 1 equilibrium cutpoint must be positive, i.e., $c_{n,1}^{m_n}(0) > 0$, and hence is given by (12) evaluated at $k=m_n$ and $l=0$:

$$c_{1,n}^{m_n}(0) = e^{-\gamma\Delta}\sum_{j=0}^{m_n-1}\left[\begin{array}{c} B\left(j, n-1, F\left(c_n^{m_n}(0)\right)\right) \\ \cdot\left(Q^{m_n-j-1}\left(c_n^{m_n}(0)\right) - \left[V^{m_n-j}\right]^+\left(c_n^{m_n}(0), c_n^{m_n}(0)\right)\right)\end{array}\right]. \tag{15}$$

The maximal fixed-point consistent with equation (15) can be bounded below by $\overline{c}_n^{m_n}(0)$ defined as follows:

$$\overline{c}_n^{m_n}(0) = \max_{c\in[0,1]}\left[c | c \le e^{-\gamma\Delta}\sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c\right)\right)\left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+\left(c,c\right)\right]\right]. \tag{16}$$

Let $Z_n^{m_n}(c)$ be the expression in the right hand side of the inequality in (16):

$$Z_n^{m_n}(c) = e^{-\gamma\Delta}\sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c\right)\right)\left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+\left(c,c\right)\right] \tag{17}$$

In the appendix we show that, for any period 1 cutpoint $c \ge L\cdot\alpha_n$ with $L>1$, $Z_n^{m_n}(c)$ is bounded below by a function $z_n^{m_n}(c) = \xi\cdot B\left(m_n-1, n-1, F(c)\right)$, where ξ is a positive constant. A key step in finding the lower bound is the observation that if $c \ge L\cdot\alpha_n$ we can ignore the payoffs obtained in all histories in which $j < m_n - 1$ players contribute: this implies that we can ignore all the terms in the summation in (17), except for the term with $j = m_n - 1$. This step would be obvious if $Q^{m_n-j-1}\left(c\right) \ge \left[V^{m_n-j}\right]^+\left(c,c\right)$ for all j, implying that all the terms are non-negative. While this property may seem intuitive, it is not automatically satisfied.[23] To bypass this complication, in the proof, we show that the expected utility of a player conditioning on fewer than $m_n - 1$ volunteers out of the remaining players is negligible relative to $B\left(m_n - 1, n-1, F(c)\right)$ if $c \geq L \cdot \alpha_n$, so that (17) can be bounded below by $\xi \cdot B\left(m_n - 1, n-1, F(c)\right)$ for a given positive constant $\xi \in (0,1)$.

Step 3. We conclude the proof by showing that Steps 1 and 2 imply that, for all sufficiently large n, $F(c_{1,n}^{m_n}(0)) \geq L\frac{m_n}{n}$ for some factor $L > 1$, where L may depend on $\gamma\Delta$. To this goal we use the lower bound (17) to show that (16) has a fixed point in the set $(L \cdot \alpha_n, 1)$. The logic of this step will be illustrated below using Figure 5. ■

We now use the above lemma to show that the equilibrium is successful in the first period with probability approaching 1 as $n \to \infty$ if $m_n \prec n^{2/3}$. Define P_n to be the group's ex ante probability of success, and EU_n to be the ex ante utility of a player. We have:

Theorem 4. *If* α_n *converges to zero faster than the cube root of* $1/n$*, then for all* $\gamma, \Delta > 0$ *there is a sequence of symmetric equilibria in which* $\lim_{n\to\infty} EU_n(c) = v$ *for all* $c \in (0,1)$.

Proof. See appendix.

The key passage in proving Theorem 4 is Step 3 of Lemma 4, where we show that:

$$\lim_{n\to\infty} F(c_{n,1}^{m_n}(0))/\left(\frac{m_n}{n}\right) \geq L > 1. \tag{18}$$

This proves that as n grows, the expected fraction of members who activate in the first period becomes strictly larger than the required threshold for success, i.e. α_n. Theorem 4 follows using this property and Chebyshev's inequality: it shows that, as $n \to \infty$, the share of players willing to contribute is larger than α_n with probability converging to 1 (and it is comprised only of types with cost arbitrarily close to zero).

The logic of Step 3 of Lemma 4 can be explained using Figure 5, where, for the purpose of this discussion we assume that the right hand side of (16) and (17) are continuous functions

[23]The first term, $Q^{m_n-j-1}(c)$, is the expected utility of a passive player when the other players contribute knowing that the lower bound on types is c and $m_n - j - 1$ volunteers are missing. The second term is the expected utility of an active player, say player i, when the lower bound on types is c and $m_n - j$ are missing, but player i plans to contribute for sure. The second term therefore includes the cost of contribution $-c_i$, which the first term does not include. Moreover, since player i contributes for sure, in the second term, the missing volunteers for success are effectively only $m_n - j - 1$. These observations therefore would suggest that the second term should be smaller. However, in the second term, the other players volunteer as if there are $m_n - j$ missing volunteers because they do not know i's intention to contribute for sure. If the fact that they are marginally more distant from success ($m_n - j$ instead of $m_n - j - 1$ missing volunteers) induces players $-i$ to volunteer with higher probability, then the higher probability of success may compensate for the fact that player i has to contribute c_i.

of c.[24] The solid curve represents $Z_n^{m_n}(c)$. Condition (16) admits a fixed point larger than $F^{-1}(L \cdot \alpha_n)$ if the solid curve evaluated at $F^{-1}(L \cdot \alpha_n)$ is above the 45^o line: in this case it admits an intersection on the right of $F^{-1}(L \cdot \alpha_n)$, since this curve converges to zero as $c \to 1$.[25] By Step 2 of Lemma 4, we can bound the solid curve below by $z_n^{m_n}(c)$, which is illustrated by the dashed curve. Hence, a sufficient condition for (18) is that $z_n^{m_n}(c)$ evaluated at $F^{-1}(L \cdot \alpha_n)$ is above the 45^o line: in this case the dashed curve intersects the 45^o line on the right of $F^{-1}(L \cdot \alpha_n)$ since this curve converges to zero as $c \to 1$, and a fortiori so does the solid line (thus implying the existence of a fixed-point $c_{n,1}^{m_n}(0) > F^{-1}(L \cdot \alpha_n)$). The key condition therefore is that $z_n^{m_n}(F^{-1}(L \cdot \alpha_n)) > F^{-1}(L \cdot \alpha_n)$, or:

$$\frac{\xi B(m_n - 1, n-1, L \cdot \alpha_n)}{F^{-1}(L \cdot \alpha_n)} > 1 \tag{19}$$

To see that this is the case, note that using Stirling's approximation, we have:

$$\frac{\xi B(m_n - 1, n-1, L \cdot \alpha_n)}{F^{-1}(L \cdot \alpha_n)} \simeq \frac{\xi f(0)}{L} \cdot \sqrt{\frac{1}{2\pi \cdot \alpha_n^3 (1-\alpha_n) n}}$$

When $\alpha_n \to 0$ faster than the cube root of $\frac{1}{n}$, the denominator of the right hand side converges to zero, so (19) is guaranteed for sufficiently large n.

5.2 An impossibility result

We have proven above that if α_n converges to zero faster than the cube root of n, then the equilibrium payoff in the most efficient PBE converges to the efficient allocation: immediate success with probability 1. The next result shows that the cube root of n is the critical dividing threshold between complete efficiency and complete failure in large groups: if α_n converges to zero slower than the cube root of n, then in all equilibria—both symmetric and asymmetric—the expected utility of every type of every player converges to zero. In an asymmetric equilibrium, players may adopt different cutpoints, $(c_1^*(h_t), ..., c_n^*(h_t))$, where player $i's$ cutpoint $c_n^*(h_t)$ may depend on the identity of the past contributors, not just their numbers. We have:

[24] In general, $Z_n^{m_n}(c)$ and $z_n^{m_n}(c)$ are correspondences in c, since we might have multiple equilibria, and thus multiple continuation value functions for each c. In Theorem 3 we however show that the set of continuation values defines a non empty, convex valued, upper hemicontinuous correspondence in c. These properties are sufficient for the argument outlined here to go through.

[25] Note that in Figure 5, $Z_n^{m_n}(c)$ has a positive intersection at $c = 0$. This reflects the fact that the expected benefit of contributing for a $c = 0$ type is positive even if no other player contributes (recall that, in $Z_n^{m_n}(c)$, c is the cutoff adopted by the other players). In this case when $m_n > 1$, although success would be impossible in the current period, it may move the game to a state in which success will be more likely in the future.

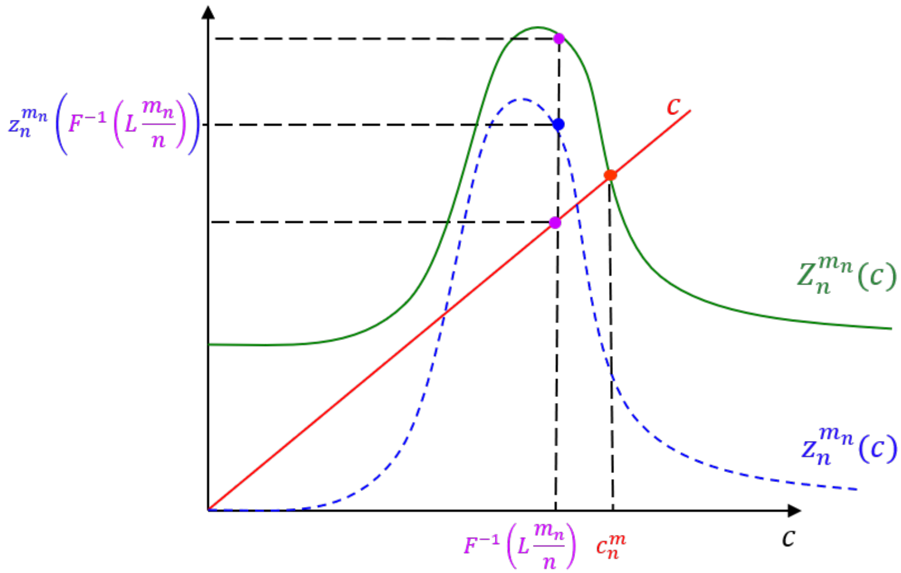

Figure 5: Condition (15) and the existence of an asymptotically efficient equilibrium.

Theorem 5. *If α_n converges to zero slower than the cube root of $1/n$, then, for all $\gamma, \Delta > 0$, in every sequence of PBE,* both symmetric and asymmetric, we have $\lim_{n\to\infty} EU_n(c) = 0$ *for all $c \in (0, 1)$.*

Proof. See the appendix.

The idea behind the proof of Theorem 5 is to prove that for any PBE of the *dynamic* collective action game we can define a Honest and Obedient mechanism (Myerson 1982) of a related *static* contribution game that generates the same expected payoffs.[26] This implies that the supremum of the payoffs achievable in a PBE can be bounded above by the maximal payoff achievable in the best static Honest and Obedient mechanism. As proven in Battaglini and Palfrey (2024), the per capita payoff in the best Honest and Obedient mechanism of the static game converges to zero as $n \to \infty$ if $\alpha_n/\sqrt[3]{1/n} \to \infty$. Since the payoff of a player in any PBE is non-negative, this implies that the per capita payoff in a PBE converges to zero as well if $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$.

This idea can be interpreted in terms of the revelation principle. In the revelation principle it is shown that for any mechanism, there is a direct mechanism with the same payoffs for the players: the game forms associated to the mechanisms differ in terms of the players' action space; but they share the same outcome space and utility functions. Here, the dynamic game has a very different outcome space and thus different preferences over it than

[26]The related static contribution game is the same contribution game as the game described in Section 2.1, but there is only one period, after which if success was not reached no player can no longer contribute.

the corresponding static game: in the latter, the outcome is just a vector of activated players (and the associated success/failure of the common project); in the dynamic game, the outcome is a distribution over time of activated players evaluated over time.

As observed in the introduction, Ausubel and Deneckere [1989a, 1989b] previously characterized the equilibria of a two-person, dynamic bargaining game with one-sided private information in terms of the allocation achievable through a direct IC, IR mechanism, under the assumption that agents are sufficiently patient. This important result, however, does not extend to more general games with multilateral private information. The key distinction is that with one-sided private information, only the agent possesses private information. When the agent receives a recommendation from the mechanism, the agent's information set remains unchanged. In contrast, in a model with multilateral incomplete information, the agent must update beliefs about the types of other agents after any recommendation, as the recommendation depends on the information collected by the mechanism. This belief updating is precisely what is incorporated into the Obedience constraint. The Obedience constraint is key for the result, since it is the reason why the expected utility converges to zero in the best direct HO-mechanism when $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$. Efficient outcomes are indeed possible in the best IC and IR (but not Obedient) mechanism even when $\alpha_n/\sqrt[3]{1/n} \to \infty$ as $n \to \infty$.

To see why any PBE of the dynamic game defines an equivalent static HO mechanism, consider here, for the sake of the argument, an equilibrium in which the cutoffs $(c_\tau(\mathbf{c}))_{\tau=1}^{\infty}$ depend only on the realized types $\mathbf{c}$ (so there is no other public signal observed by the players).[27] Note that, because we are allowing for asymmetric equilibria, here $c_\tau(\mathbf{c}) = \{c_{\tau,1}(\mathbf{c}), ..., c_{\tau,n}(\mathbf{c})\}$, where $c_{\tau,i}(\mathbf{c})$ is player i's cutpoint at τ when the type profile is $\mathbf{c}$. Even if we fix the PBE, this sequence is stochastic since it depends on the realized profile of types $\mathbf{c}$. But, for a given PBE and a given $\mathbf{c}$, it is deterministic.[28] Given a realized profile of individual costs, $\mathbf{c}$, these cutoffs define $S(\mathbf{c})$, i.e. the first period in which there are m volunteers (which may be never); $T_i(\mathbf{c})$, the first period in which player i volunteers (i.e. the first t in which $c_{t,i}(\mathbf{c}) \geq c_i$); $I_t(\mathbf{c})$, the set of volunteers up to and including period t; and $k_t(\mathbf{c})$ is the number of missing volunteers for success at the end of t.

A static mechanism can be defined as a function $\mu : [0,1]^n \to \Delta 2^I$, mapping the set of

[27] Given a type profile $\mathbf{c}$, and ignoring here for simplicity the public signals, the equilibrium cutoff strategy $c(h_t)$ defines a unique history $h_t(\mathbf{c})$, describing the sets of volunteers in all periods $\tau \leq t$. The cutpoint at t can therefore be written as $c_t(\mathbf{c}) = c\left(h_t(\mathbf{c})\right)$.

[28] In the presence of public signals, the sequence may also depend on the realization of the signals, which determines the continuation equilibrium that is chosen. See the proof of Theorem 6 in the appendix for details.

profiles of types to a distribution over the set of players who are asked to contribute. To see that a vector of cutoffs $(c_\tau(\mathbf{c}))_{\tau=1}^{\infty}$ define such a mechanism, consider the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals i with a type below $c_{1,i}(\mathbf{c})$ are asked to volunteer (i.e., the set $I_1(\mathbf{c})$). If there are at least m such individuals, i.e., $k_1(\mathbf{c}) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c}) = 1$ and $\mu_{I_1(\mathbf{c})}^{DYN}(\mathbf{c}) = 1$ (we denote by $\mu_g^{DYN}(\mathbf{c})$ the probability a group g is asked to volunteer when the profile is $\mathbf{c}$). If $k_1(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > 1$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c}) > 1$ and $\mu_{I_1(\mathbf{c})}^{DYN}(\mathbf{c}) = 1 - e^{-\gamma\Delta}$. With probability $e^{-\gamma\Delta}$, instead, the algorithm proceeds to Step 2. In Step 2, all individuals i with a type in the interval $(c_{1,i}(\mathbf{c}), c_{2,i}(\mathbf{c})]$ are also asked to volunteer and the process continues. In general, at any Step t at which the algorithm has not yet stopped, individuals i with a type in the interval $(c_{t-1,i}(\mathbf{c}), c_{t,i}(\mathbf{c})]$ are asked to volunteer. If there are at least $k_{t-1}(\mathbf{c})$ such individuals, i.e., $k_t(\mathbf{c}) = 0$ and $S(\mathbf{c}) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c})$ without proceeding to step $t+1$. If $k_t(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > t$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability $e^{-\gamma\Delta}$ the algorithm proceeds to step $t+1$. The algorithm described above defines the following static mechanism:

$$\mu_g^{DYN}(\mathbf{c}) = \begin{cases} \sum_{\{\tau | I_\tau(\mathbf{c}) = g\}} \left(1 - e^{-\gamma\Delta}\right) e^{-\gamma\Delta(\tau-1)} & |g| < m \\ e^{-\gamma\Delta(S(\mathbf{c})-1)} & \text{if } |g| \geq m \text{ and } g = I_{S(\mathbf{c})}(\mathbf{c}) \\ 0 & \text{else} \end{cases} \tag{20}$$

which mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1 - e^{-\gamma\Delta}$ after any step at which the threshold m has not yet been achieved.

The proof is completed by showing that $\mu_g^{DYN}(\mathbf{c})$ is Honest and Obedient mechanism. This fact is intuitive. By construction, the static mechanism asks a player to contribute if and only if the player is in an event that mimics an history in which the player finds it optimal to contribute in the dynamic game. The event "mimics" such an history in the sense that conditioning on such an event, the player has a the same posterior on the other players types as after such an history. It follows therefore that if the static mechanism is not honest and obedient, then we would have a deviation in the PBE, a contradiction.

6 Extensions and variations

6.1 Dynamic collective action with high value ($v \geq 1$)

Up to this point we assumed $v < 1$. If $v \geq 1$, then it is common knowledge that all players would willingly participate if they are pivotal. This has a number of implications, which we explain here. First, there exist asymmetric equilibria that achieve success instantly: in equilibrium, at $t = 1$ a subset of exactly m members participates, regardless of their private cost, and the remaining $n - m$ members free ride. Of course, this requires some coordination device among the players. If such coordination devices are not readily available, then we are back to characterizing the symmetric PBE of the game. In this high value case, results are not entirely negative. For symmetric PBE, we have the following results. We extend Theorem 3 to the high-value case as follows:

Theorem 3′. *If $v \geq 1$ then for all $n > 2$, for all $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0, 1)$, in every symmetric equilibrium:*

1. *The probability of success is strictly positive.*

2. *There exists a minimum value threshold, $1 < v^*(n, m, \gamma, \Delta) < \infty$, such that all equilibria are interior iff $v > v^*(n, m, \gamma, \Delta)$, and success is achieved with probability* 1.

3. *For all $v \in [1, v^*(n, m, \gamma, \Delta)]$, there is at least one equilibrium in which there is a positive probability of reaching an effectively terminal history h^k_t with $c^k(l_{h^k_\tau}) = l_{h^k_t} < v$* for all $\tau \geq t$.

Proof: See the online appendix. ■

The proof of part (1) is the same as in Theorem 3. Part (2) is proved by induction. A detailed proof is in the online appendix, which we sketch here. When $m = 1$, all equilibria are interior by Lemma 2 for all v, (2) therefore holds for $m = 1$. The properties of the interior equilibrium when $k = 1$ moreover guarantee that we have $e^{-\gamma\Delta}Q^1(l) - l > 0$ for any $l \in [0, \min\{v, 1\}]$. For the induction hypothesis, assume that for all $j = 1, ..., k - 1$ there exists a $v^*_{k-1}(n, j, \gamma, \Delta) < \infty$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^j(l) > l$ and $e^{-\gamma\Delta}Q^j(l) - l > 0$, if and only if $v > v^*_{k-1}(n, j, \gamma, \Delta)$. The next step of the proof is to show that the induction hypothesis implies the existence of a $v^*_k(n, k, \gamma, \Delta) > 1$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^k(l) > l$ and $e^{-\gamma\Delta}Q^k(l) - l > 0$ if and only if $v > v^*_k(n, k, \gamma, \Delta)$. This requires some care since $Q^k(l)$ is typically a complicated function of l for $k > 1$. This argument allows us to conclude that for any $k \leq m$, a player with type close to l finds it optimal to contribute, even if s/he expects all other active players not to contribute.

Part (3) of Theorem 3′ can be seen as the residual case and follows as a corollary to Part (2). If $v \leq v^*(n, m, \gamma, \Delta)$, then there is a history with positive probability at which if a player expects no other player to contribute, then s/he does not find it optimal to contribute as well: thus we have an equilibrium in which players stop contributing. This in itself, however, does not imply that there is no interior equilibrium in which success is eventually achieved since the payers' decisions to contribute may be strategic complements, implying that we could have additional equilibria in which participation is stimulated by the expectation that other players contribute with positive probability. As shown in Theorem 3, however, this is impossible if $v < 1$.

Theorems 3 and 3′ highlight the fact that, except when v is very high, there is uncertainty regarding whether the group can get stuck at an effectively terminal history where no player is willing to contribute anymore, or, alternatively, the cutpoints continually increase in all periods. However it leaves open two issues. First, it is possible that for all $k < m$ we have $c_t^k > c_{t-1}^k$, but $c_t^k \to c_\infty^k < 1$. In this case, the equilibrium is interior but the project may still remain unrealized even if all types are below v. The second question concerns the size of $v^*(n, m, \gamma, \Delta)$. Should we expect $v^*(n, m, \gamma, \Delta)$ to be close to 1, at least when players are patient (i.e., $\gamma, \Delta \to 0$)? The following proposition addresses the first issue.

Proposition 1. *If $v > v^*(n, m, \gamma, \Delta)$, then the group is constrained-successful, i.e., $\overline{p}_n = 1 - [1 - F(v)]^n = 1$ in all symmetric equilibria. If $v \in [1, v^*(n, m, \gamma, \Delta)]$, then the group is constrained successful in any interior equilibrium.*

Proof: The proof shows that $c_t^k \to c_\infty^k = v$ in any interior equilibrium. See the online appendix for details. ■

The next result addresses the second issue, the size of $v^*(n, m, \gamma, \Delta)$. We have:

Proposition 2. *For any $n > 2$, for any $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0, 1)$: $v^*(n, m, \gamma, \Delta) \geq 2$ for any n, m, γ and Δ.*

Proof: See the online appendix for details. ■

This result tells us that even if $v \in [1, 2]$, where it is common knowledge that all players desire the public good even if their participation is required to get it, the dynamic process of participation is probabilistic, and a strictly positive probability of failure is unavoidable. This is true even if the group is large, the threshold is small, and even if the players are arbitrarily patient. The inability of a group to reach success does not depend on the frequencies of interaction Δ, so it remains true even in the limit as $\Delta \to 0$.

For large groups $v^*(n, m, \gamma, \Delta)$ grows without bound. Specifically, we have:

Proposition 3. *If* $m_n > 1$, *then* $\lim_{\Delta\to\infty} v^*(n, m_n, \gamma, \Delta) = \infty$

Proof: See the online appendix for details. ■

It is important to note that Proposition 3 does not preclude the possibility that even if Δ is large, there can be efficient or approximately efficient equilibria in the limit. In particular, it is easy to see that Theorem 4 holds for all $v > 0$, including for the high value case.

6.2 Random thresholds

In the preceding analysis we have assumed the agents play a simple threshold public good game in which the public good is obtained by the group if and only if there are at least m_n contributors. It is easy to consider extensions in which success is determined under alternative, less stylized rules. A natural case to consider is when the realization of the public good depends on the number of contributors in a probabilistic way. As an illustration, consider here the case in which at t the public good is realized with probability $\xi(k_t)$ if contributors up to t are k_t, where $\xi(\cdot)$ is a non-decreasing function with $\xi(k_t) = 0$ if $k_t < \lceil \underline{\theta} m_n \rceil$, $\xi(k_t) > 0$ for $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$, and $\xi(k_t) = 1$ if $k_t \geq \lceil \overline{\theta} m_n \rceil$ for $\underline{\theta} \in (0,1)$ and $\overline{\theta}_n \in \left(1, \frac{n}{m_n}\right)$ (here $\lceil x \rceil$ is the smallest integer that is not smaller than x). This implies that the players are uncertain about the minimal number of volunteers that are required for success which may take values from $\lceil \underline{\theta} m_n \rceil$ to $\lceil \overline{\theta}_n m_n \rceil$.

It is not difficult to see that the characterization of the equilibrium, the existence proof and the limit results of Section 4 all generalize to this more complex environment. The equilibrium conditions are qualitatively similar to (8), (10) and (12), although now they are complicated by the fact that for any $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$ success is probabilistic. Consider now the limit results presented in Section 5. A simple argument shows that, in this case too, a sufficient condition for the existence of a limit efficient equilibrium is that $\alpha_n = m_n/n$ converges to zero sufficiently fast, faster than the cube root of $1/n$; the fact that the probability of success is lower than one for $k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)$ is completely irrelevant. To see this, let $\xi_* = \min_{k_t \in \left[\lceil \underline{\theta} m_n \rceil, \lceil \overline{\theta}_n m_n \rceil\right)} \xi(k_t)$, which is strictly positive by assumption. Consider a modified game in which the value of the public good is $v' = v\xi_* > 0$, and the probability of success is 1 for $k_t \geq \lceil \overline{\theta}_n m_n \rceil$. This gives a game similar to the games studied in the previous sections, in which an asymptotically efficient equilibrium exists if $\alpha_n / \sqrt[3]{1/n} \to 0$. But the incentives to contribute in the original game with probabilistic success are even higher than in this modified game, since in the modified game we assume the value of the public good v' and the probability of success are always lower than in the original game: thus a limit efficient equilibrium must exists in the game with probabilistic success. To show the impossibility of an equilibrium with positive payoff in the limit when $\alpha_n/\sqrt[3]{1/n} \to \infty$, the analysis is similar. We can now define a modified game in which the threshold for success is at $m_n = \lceil \underline{\theta} m_n \rceil$ and $\xi(k_t) = 1$ if $k_t \geq \lceil \underline{\theta} m_n \rceil$. The preceding analysis shows that all PBE generate zero utility in the limit in this game; thus it can be shown that the same must be true in the game with probabilistic public good, since in this game the expected utility of contributing is always strictly lower.

6.3 Other extensions

In the online supplementary appendix of this paper we present three additional extensions that further explore the robustness of the model and the possibility of alternative dynamic mechanisms to improve group success. First we show that the basic model can be extended to allow for environments in which there is aggregate uncertainty about the distribution of types. Second, we show how the analysis can be extended to incorporate non stationary environments, in which the cost of inaction for a group increases over time, as it might occur in environmental problems.

Third, we explore whether it might be beneficial for the group to self-impose a deadline, so that the group only has T periods to achieve success. We show that committing to such deadlines can be counter-productive: for any deadline T there exists a group size $n(T)$ such that all groups larger than $n(T)$ *completely fail*, in the sense that the only equilibrium involves no member ever participating, even those with arbitrarily low costs. A notable implication of this result is that when m_n grows at the speed of n, the static one-shot game leads to zero probability of success for large enough n. Therefore, in such environments, the dynamic game leads to better outcomes in terms of welfare than the static game. I.e., the benefits of information transmission and coordination from the equilibrium dynamics outweighs the delay costs of dynamic free riding. This contrasts with the example in Section 4.2 of the paper, where welfare is higher in the static game than the dynamic game when m_n is a fixed constant that does not increase with n. An interesting conjecture is that dynamics produce welfare gains (losses) relative to the static game when the free riding problem is more (less) severe, where more severe corresponds to environments where m_n grows faster than speed of $n^{2/3}$.

7 Conclusions

Collective action problems arise when a group's collective goal can only be achieved if at least some fraction of its members engage in a costly action to help the group succeed.

We study the equilibrium properties of collective action problems when decisions are taken dynamically over a potentially long horizon, delay is costly, and members have heterogeneous and privately known preferences.

We present two categories of characterization results. The first half of the paper characterizes the properties of dynamic equilibrium, as well as providing efficiency results, for any fixed group size and any fixed participation threshold for group success. The simplest such case is known as the volunteers dilemma, or bystander intervention problem, where exactly one member must undertake the action - to rescue a drowning swimmer, or call 911 to report an accident or ongoing violent crime. This also happens to be the only case that had been analyzed as a dynamic stochastic game with incomplete information (Bliss and Nalebuff, 1984). Our first finding establishes that the dynamic volunteers dilemma is a very special case: except in the limit with arbitrarily large groups, the equilibrium properties of the dynamic version of this game do not extend to the more general, and arguably more realistic, case where group success requires the action of more than one member.

In the volunteers dilemma case where only one member is needed, the group always succeeds as long as at least one member of the group has an action cost that is less than the benefit of success. But if group success requires the coordinated action of multiple members, this is no longer true. In this more complicated environment, a member who contemplates taking an action early on faces a real risk that their action will be useless because there will never be a sufficient number of members who decide to activate at later dates. In fact, we show that in such environments, while there will always be a positive probability of success, there is also always a positive probability that the dynamic process of accumulating more activists and getting closer to the goal can fizzle out. In that case, all members who have activated lose out and nobody benefits. The possibility of such failure always exists unless it is common knowledge that every group member would be willing to activate if pivotal. Hence, there are two sources of inefficiency: delay (time is costly); and the positive probability that the goal is not achieved but many members suffer their action cost.

Our second category of results explores the efficiency properties of dynamic equilibrium in large groups. For these results, we allow both the group size as well as the required threshold to grow without bound and obtain a characterization of efficiency in the limit, which depends on the relative rate at which the threshold grows relative to the group size.

If the fraction of members required for the threshold converges to zero very fast as group size increases –at a rate *faster than the cube root of the inverse of the group size*– then there is always a limiting equilibrium where the goal is instantly achieved with probability 1. There is no delay and full efficiency is achieved. The volunteers dilemma is a special case of this.

On the other hand, if the fraction of members required for the threshold converges to zero at a rate *slower than the cube root of the inverse of the group size*, then *in every limiting equilibrium* the probability of group success is 0 and action fizzles out immediately. A special case of this arises if a constant fraction of group size is required. Thus, in both cases, delay costs disappear, as does the deadweight loss incurred when some group members' action costs are wasted. The only efficiency issue in the limit is the probability of group success, which is either 0 or 1.

In collective action problems, bandwagon effects are key, and their role in the equilibrium dynamics warrant further study. We do not have detailed results in the form of the finer properties of equilibrium in our model, but this is a useful direction to pursue. If a protest movement or petition drive catches on quickly and exhibit heavy participation from the outset, this early activity snowballs and encourage others to join in because the prospects of success are higher. On the flip side, if there are only a handful of visible activists in the early stages, then other who were sitting on the sidelines might decide to just forget about it and the movement would fizzle out.

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Appendix

Proof of Lemma 1: We proceed in two steps.

Step 1. We first prove that the expected continuation value of a player who does not volunteer is convex in c and admits right and left derivatives, respectively denoted $\partial^r \left[V^1\right]^- (c, h_t^1)/\partial c$

and $\partial^l [V^1]^- (c, h_t^1)/\partial c$, with $\partial^d [V^1]^- (c, h_t^1)/\partial c > -e^{-\gamma\Delta}$ for $d = r, l$. Consider any history h_t^1 in which only one volunteer is missing for success, and denote h_{t+j}^1 any history following h_t^1 in which there is no volunteer from t to $t + j$. Let $\beta(h_t^1)$ be the probability that there is at least 1 volunteer at history h_t^1. Define $W^{1,\lambda}(c, h_t^1)$ to be the expected value to a player of type c at h_t^1 who does not volunteer at t, but instead volunteers after $\lambda \in [1, \infty)$ periods (if there is not a volunteer before):

$$\begin{aligned} W^{1,\lambda}(c, h_t^1) &= \sum\nolimits_{\tau=0}^{\lambda-1} e^{-\gamma\Delta\tau} \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \beta(h_{t+j-1}^1)\right]\right] \beta(h_{t+\tau}^1) v \\ &\quad + e^{-\gamma\Delta\cdot\lambda} \cdot \prod\nolimits_{j=0}^{\lambda} \left[1 - \beta(h_{t+j-1}^1)\right] \cdot (v - c) \end{aligned} \tag{21}$$

where we define by convention $\beta(h_{t-1}^1) = 0$. Note that all these expressions are linear in c and $\partial W^{1,\lambda}(c, h_t^1)/\partial c > -e^{-\gamma\Delta} > -1$. The value for a player who does not volunteer at h_t^1 is: $[V^1]^- (c, h_t^1) = \max_\lambda W^{1,\lambda}(c, h_t^1)$, which is convex in c and so admits right and left derivatives with $\partial^d [V^1]^- (c, h_t^1)/\partial c > -e^{-\gamma\Delta}$ for $d = r, l$.

Step 2. Note that at h_t^1, the expected utility of a type c who volunteers at t is $[V^1]^+ (c, h_t^1) = v - c$. Suppose now a type $c(h_t^1)$ is indifferent between volunteering or not, so:

$$v - c(h_t^1) = \left[V^1\right]^- (c(h_t^1), h_t^1) \tag{22}$$

Consider a type $c' > c(h_t^1)$ with $\Delta c = c' - c(h_t^1)$. We have:

$$\begin{aligned} v - c' &= v - c(h_t^1) - \Delta c = \left[V^1\right]^- (c(h_t^1), h_t^1) - \Delta c \\ &< \left[V^1\right]^- (c(h_t^1), h_t^1) - e^{-\gamma\Delta}\Delta c < \left[V^1\right]^- (c', h_t^1) \end{aligned}$$

where in the second we use (22), and in the last inequality we use the convexity of $[V^1]^- (c', h_t^1)$. The proof that $c' < c(h_t^1)$ implies $v - c > [V^1]^- (c', h_t^1)$ is analogous. ■

Proof of Lemma 2: Note that $c(h_t) \in [l_{h_t}, v)$ and suppose to the contrary that $c(h_t) = l_{h_t}$ in the PBE, i.e., no member will activate in period t. This implies $V^-(c, h_t) > V^+(c, h_t)$ for all $c(h_t) \in (l_{h_t}, v)$. If no member activates in period t, then it must be that no member activates in period $t + 1$ as well: if a member with cost $c \in [l_{h_t}, v)$ were using a strategy to activate in period $t+1$ and the member knows that no member is activating in period t, then their payoff in the continuation game is $V^-(c, h_t) = e^{-\gamma\Delta}(v - c) < v - c = V^+(c, h_t)$, a contradiction. It follows that if no player volunteers at t, $V^-(c, h_t) = 0 < v - c = V^+(c, h_t)$, implying again a contradiction. We conclude that at t the probability a player volunteers is strictly positive. To prove that $\lim_{t\to\infty} c(h_t) = v$, suppose to the contrary that $\lim_{t\to\infty} c(h_t) = \bar{c} < v$. Then $\lim_{t\to\infty} V^-(\bar{c}, h_t) = 0 < V^+(\bar{c}, h_t) = v - \bar{c} > 0$, a contradiction. ■

Proof of Lemma 3: We proceed in three steps.

Step 1. Lemma 1 already established that there is a unique PBE when $k = 1$, which is in cutoff strategies. In that equilibrium, for any history at which $k = 1$, the value for an active player who volunteers is $[V^1]^+ (c, h_t^1) = v - c$, which is linear in c with $\partial [V^1]^+ (c, h_t^1)/\partial c = -1$. The proof of Lemma 1 also established that $[V^1]^- (c, h_t^1)$ is piecewise linear, convex and hence admits right and left derivatives $\partial^r [V^1]^- (c, h_t^1)/\partial c$ and $\partial^l [V^1]^- (c, h_t^1)/\partial c$, both bounded below by $-e^{-\gamma\Delta} > -1$.

Step 2. At any history h_t^k, the continuation value of a player who has *previously* committed (i.e. $Q^k(l_{h_t^k})$) does not depend on his/her own type c; it depends only of the behavior of the other players. It follows that the value of an active player at h_t^k, with cost c, who volunteers at h_t^k, i.e. $\left[V^k\right]^+ (c, h_t^k)$, is linear in c with $\partial \left[V^k\right]^+ (c, h_t^k)/\partial c = -1$. We now proceed by induction. Assume that, for all $\kappa = 1, ..., k-1$, $[V^\kappa]^- (c, h_t^\kappa)$ is convex in c, with $\partial^d [V^\kappa]^- (c, h_t^\kappa)/\partial c \geq -e^{-\gamma\Delta}$ for $d = l, r$. (Step 1 established this property for $\kappa = 1$.) We need to prove that the same is true for $\left[V^k\right]^- (c, h_t^k)$.

To see this, first observe that $[V^\kappa] (c, h_t^\kappa) = \max\left\{[V^\kappa]^+ (c, h_t^\kappa), [V^\kappa]^- (c, h_t^\kappa)\right\}$ for any $\kappa < k$. Since $[V^\kappa]^+ (c, h_t^\kappa)$ is linear in c and $[V^\kappa]^- (c, h_t^\kappa)$ is convex in c , then $[V^\kappa] (c, h_t^\kappa)$ is convex in c for all $\kappa < k$. Define $\vartheta(h_t^k)$ to be the probability of having at least one volunteer at h_t^k. Define $\Phi^k(c, h_t^k)$ to be the expected utility of a type c player at history h_t^k conditioning on at least one volunteer at history h_t^k. Because of the properties of $[V^\kappa] (c, h_t^\kappa)$ for $\kappa < k$ proven above, $\Phi^k(c, h_t^k)$ is convex in c and $\partial^d\Phi^k(c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for any $d = l, r$. Finally, define $W^{k,\lambda}(c, h_t^k)$ to be the value of a player of type c at h_t^k who does not volunteer at t, but volunteers instead after at most $\lambda \in [1, \infty)$ periods in which there is no other volunteer, if there is no volunteer before. We can write:

$$
\begin{aligned}
W^{k,\lambda}(c, h_t^k) \;&=\; \sum\nolimits_{\tau=0}^{\lambda-1} e^{-\gamma\Delta\tau} \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \vartheta(h_{t+j-1}^k)\right]\right] \cdot \vartheta(h_{t+\tau}^k) \cdot \Phi^k(c, h_{t+\tau}^k) \\
&\quad + e^{-\gamma\Delta\cdot\lambda} \cdot \left[\prod\nolimits_{j=0}^{\lambda} \left[1 - \vartheta(h_{t+j-1}^k)\right]\right)\right] \cdot \left[V^k\right]^+ (c, h_{t+\lambda}^k) \qquad (23)
\end{aligned}
$$

where $\vartheta(h_{-1}^k) = 0$ by convention. Since $\Phi^k(c, h_{t+\tau}^k)$ and $\left[V^k\right]^+ (c, h_t^k)$ are convex $\forall\tau = 1, ..., \lambda - 1$, $W^{k,\lambda}(c, h_t^1)$ is convex. And by the same argument as before, this also implies that $\partial^d W^{k,\lambda}(c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for any $d = l, r$. Finally, note that $\left[V^k\right]^- (c, h_t^k) = \max_\lambda W^{k,\lambda}(c, h_t^k)$. It follows that $\left[V^k\right]^- (c, h_t^k)$ is convex in c and $\partial^d \left[V^k\right]^- (c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for $d = l, r$.

Step 3. Assume now a type $c(h_t^k)$ is indifferent between volunteering or not at history h_t^k. That is: $\left[V^k\right]^+ (c(h_t^k), h_t^k) = \left[V^k\right]^- (c(h_t^k), h_t^k)$. Consider a type $c' > c(h_t^k)$ with $\Delta c =$

$c' - c(h_t^k)$. We have:

$$\begin{aligned}\left[V^k\right]^+(c', h_t^k) &= \left[V^k\right]^+(c(h_t^k), h_t^k) - \Delta c = \left[V^k\right]^-(c(h_t^k), h_t^k) - \Delta c \\ &< \left[V^k\right]^-(c(h_t^k), h_t^k) - e^{-\gamma\Delta}\Delta c < \left[V^k\right]^-(c', h_t^k)\end{aligned}$$

where in the first equality we use the linearity of $\left[V^k\right]^+(c', h_t^k)$, in the second equality we use the indifference condition, in the third (inequality) we use the induction hypothesis, and finally in the last inequality we use the convexity of $\left[V^k\right]^-(c', h_t^k)$. The proof that $c' < c(h_t^k)$ implies $\left[V^k\right]^+(c', h_t^k) > \left[V^k\right]^-(c', h_t^k)$ is analogous. ■

Proof of Lemma 4: We proceed in three steps.

Step 1. We first prove that if $m_n \prec n^{2/3}$ then $\frac{F[vB(m_n-1,n-1,\alpha_n)]}{\alpha_n} \to \infty$, where $\alpha_n = m_n/n$. To establish this property, note that we can write:

$$B(m_n - 1, n-1, \alpha_n) = \binom{n-1}{m_n - 1} \frac{\left[(\alpha_n)^{\alpha_n}(1-\alpha_n)^{(1-\alpha_n)}\right]^n}{\alpha_n} \simeq \frac{1}{\sqrt{2\pi\alpha_n(1-\alpha_n)n}}$$

by Stirling's formula. Furthermore, F is approximately uniform in the neighborhood of 0, and $vB(m_n - 1, n-1, \alpha_n) \to 0$, so:

$$\frac{F\left[vB(m_n - 1, n-1, \alpha_n)\right]}{\alpha_n} \simeq \frac{f(0) \cdot vB(m_n - 1, n-1, \alpha_n)}{\alpha_n} \simeq \frac{vf(0)}{\sqrt{2\pi\left(\frac{m_n}{n^{2/3}}\right)^3\left(1 - \frac{m_n}{n}\right)}}$$

which diverges to infinity if $m_n \prec n^{2/3}$. This implies $\frac{F[vB(m_n-1,n-1,L\alpha_n)]}{L\alpha_n} \to \infty \; \forall L > 1$.

Step 2. Recall from Theorem 3 that in every equilibrium the cutpoint at the initial period $t = 1$ is strictly positive, and hence is given by (13) in the paper evaluated at $k = m_n$ and $l = 0$:

$$c_{1,n}^{m_n}(0) = e^{-\gamma\Delta}\sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c_n^{m_n}(0)\right)\right)\left[\begin{array}{c} Q^{m_n-j-1}\left(c_n^{m_n}(0)\right) \\ -\left[V^{m_n-j}\right]^+\left(c_n^{m_n}(0), c_n^{m_n}(0)\right)\end{array}\right]. \quad (24)$$

The maximal fixed-point consistent with (24) can be bounded below by $\overline{c}_n^{m_n}(0)$ defined as follows:

$$\overline{c}_n^{m_n}(0) = \max_{c\in[0,1]}\left[c \middle| c \le e^{-\gamma\Delta}\sum_{j=0}^{m_n-1} B\left(j, n-1, F(c)\right)\left[Q^{m_n-j-1}(c) - \left[V^{m_n-j}\right]^+(c,c)\right]\right] \quad (25)$$

For any arbitrary constant $L > 1$, define $\widehat{c}_n^L$ by $F\left(\widehat{c}_n^L\right) = L\frac{m_n}{n}$ for n large enough so that $L\frac{m_n}{n} < 1$. That is, given L, $\widehat{c}_n^L$ is a hypothetical cutpoint with the property that the expected fraction of types lower than or equal to $\widehat{c}_n^L$ is greater than the required threshold fraction by a factor of $L > 1$.

We next show that if we choose a sufficiently large (but still finite) value of L then there will exist a critical group size n_L such that for $n > n_L$:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right) \equiv e^{-\gamma\Delta}\sum_{j=0}^{m_n-1} B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\left[\left(Q^{m_n-j-1}\left(\widehat{c}_n^L\right)-\left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L,\widehat{c}_n^L\right)\right)\right] \tag{26}$$

$$> \varsigma B\left(m_n-1,n-1,F(\widehat{c}_n^L)\right). \tag{27}$$

where ς is a strictly positive number that does not depend on n. Notice that $\Psi_{m_n,n}\left(\widehat{c}_n^L\right)$ is the right hand side of (24) evaluated at $\widehat{c}_n^L$.

From the definition of $\widehat{c}_n^L$ we have:

$$\frac{B\left(m_n-2,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}=\frac{m_n}{n-m_n}\frac{1-F(\widehat{c}_n^L)}{F(\widehat{c}_n^L)}=\frac{\frac{n}{L}-m_n}{n-m_n}\to\frac{1}{L}$$

as $n\to\infty$. Similarly, one obtains, for $j=2,...,m_n-1$: $\frac{B\left(m_n-1-j,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F(\widehat{c}_n^L)\right)}\to\left(\frac{1}{L}\right)^j$ as $n\to\infty$. So, for large values of L, the probability of exactly j volunteers, conditional on having less than or equal to m_n-1 volunteers becomes highly concentrated on $j=m_n-1$.

Define $\overline{B}_j^L$ as the probability of exactly $j\le m_n-1$ volunteers, conditional on having less than or equal to m_n-1 volunteers, when the cutpoint is $\widehat{c}_n^L$: $\overline{B}_j^L=\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B(k,n-1,F(\widehat{c}_n^L))}$. Hence, we have

$$1 = \sum_{j=0}^{m_n-1}\overline{B}_j^L=\sum_{j=0}^{m_n-1}\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B\left(k,n-1,F\left(\widehat{c}_n^L\right)\right)} \tag{28}$$

$$= \sum_{j=0}^{m_n-1}\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}\cdot\frac{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B\left(k,n-1,F\left(\widehat{c}_n^L\right)\right)} \tag{29}$$

$$\to \sum_{j=0}^{m_n-1}\left(\frac{1}{L}\right)^j\overline{B}_{m_n-1}^L=\overline{B}_{m_n-1}^L\frac{(1-(\frac{1}{L})^{m_n})}{1-\frac{1}{L}}. \tag{30}$$

This implies that there is a n_L sufficiently large such that for $n > n_L$:

$$\overline{B}_{m_n-1}^L>\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} \tag{31}$$

for all $\epsilon > 0$. That is, $\overline{B}_{m_n-1}^L$ approaches 1 for large L.

Next observe that:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right) = e^{-\gamma\Delta}\sum_{j=0}^{m_n-1}B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\left[Q^{m_n-j-1}\left(\widehat{c}_n^L\right)-\left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L,\widehat{c}_n^L\right)\right]$$

$$\ge e^{-\gamma\Delta}\left(1-e^{-\gamma\Delta}\right)vB\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)-e^{-\gamma\Delta}\sum_{j=0}^{m_n-2}B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)v$$

since for all c: $e^{-\gamma\Delta}\left[Q^{0}(c)-[V^{1}]^{+}(c,c)\right] = v - e^{-\gamma\Delta}[V^{1}]^{+}(c,c) \geq \left(1-e^{-\gamma\Delta}\right)v$, and $e^{-\gamma\Delta}\left[Q^{m_n-j-1}(c)-[V^{m_n-j}]^{+}(c,c)\right] \geq -e^{-\gamma\Delta}v$ for $j=0,...m_n-2$. Substituting the inequality (31), it follows that for $n>n_L$:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right) \geq \sum_{j=0}^{m_n-1} vB\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\cdot\left[\begin{array}{c}\left(1-e^{-\gamma\Delta}\right)\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} \\ -e^{-\gamma\Delta}\left(1-\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}\right)\end{array}\right]$$

for all $\epsilon>0$. For any discounting parameters $\gamma\Delta$, we can choose value of L large enough so that:

$$\left(1-e^{-\gamma\Delta}\right)\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} > e^{-\gamma\Delta}\left(1-\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}\right) \Leftrightarrow \frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} > e^{-\gamma\Delta} \tag{32}$$

since the right hand side of the first line in (32) is strictly less than 1 and the left hand side converges to 1 as L increases. It follows that for such values of L we have: for $n>n_L$

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right) \geq \left(1-e^{-\gamma\Delta}\right)v\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)$$
$$= \frac{\varsigma}{1-\left(\frac{1}{L}\right)^{m_n}}B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right) > \varsigma B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)$$

for all $n>n_L$, where $\varsigma=\left(1-e^{-\gamma\Delta}\right)v(1-\frac{1}{L})$ is the desired strictly positive constant.

Step 3. From the definition of $\widehat{c}_n^L = F^{-1}(L\frac{m_n}{n})$ and Step 1, we have that for n sufficiently large:

$$\rho_{L,n} < F\left[vB\left(m_n-1,n-1,\rho_{L,n}\right)\right] \Leftrightarrow \widehat{c}_n^L < vB\left(m_n-1,n-1,F(\widehat{c}_n^L)\right) \tag{33}$$
$$\Leftrightarrow F^{-1}\left(L\frac{m_n}{n}\right) = \widehat{c}_n^L \leq \max_{c\in[0,1]}\left[c\,|c\leq\Psi_{m_n,n}(c)\right] = \overline{c}_n^{m_n}(0) \tag{34}$$

where $\rho_{L,n}=L\alpha_n$, and therefore: $F(\overline{c}^{m_n}(0)) > L\frac{m_n}{n}$.

Note now that, as proven in Theorem 2 in the online appendix, the set of possible continuation values is a non empty, closed valued and upperhemicontinuous correspondence in c, so $\Psi_{m_n,n}(c)$ has these properties as well. Let $\underline{\varphi}_{m_n,n}(c)$ and $\overline{\varphi}_{m_n,n}(c)$ be the the minimal and maximal values that can be assumed by $\Psi_{m_n,n}(c)$ in equilibrium. Since $\Psi_{m_n,n}(1)=0<1$ and we have just proven above there is a $\overline{c}^{m_n}(0)$ such that $\Psi_{m_n,n}\left(\overline{c}^{m_n}(0)\right)\geq\overline{c}^{m_n}(0)$, there must be a $c_{1,n}^{m_n}(0)\geq F^{-1}(L\frac{m_n}{n})$ such that $\underline{\varphi}_{m_n,n}\left(c_{1,n}^{m_n}(0)\right)\geq c_{1,n}^{m_n}(0)$ and $\underline{\varphi}_{m_n,n}\left(c_{1,n}^{m_n}(0)\right)\leq c_{1,n}^{m_n}(0)$. Since, as also proven in Theorem 2 in the online appendix, the set of continuation value functions is convex valued in c, we must also have that $c_{1,n}^{m_n}(0)\in\Psi_{m_n,n}\left(c_{1,n}^{m_n}(0)\right)$. We conclude that $c_{1,n}^{m_n}(0)$ solves (24) and satisfies $F\left(c_{1,n}^{m_n}(0)\right)\geq L\frac{m_n}{n}$. ■

Proof of Theorem 4: We proceed in two steps.

Step 1. Consider the case in which $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$. The case in $m_n \to m$, a constant, is analogous and omitted for brevity (details are provided in the online appendix). We now prove that in this case, $\lim_{n\to\infty} \frac{F(c_{n,1}^{m_n}(0))}{\alpha_n} \to L > 1$, where L is either bounded but strictly larger than 1 or infinite (and as, defined in the text, $\alpha_n = m_n/n$). Since $\frac{F(c_{n,1}^{m_n}(0))}{\alpha_n} \geq 1$ by Lemma 4, assume by way of contradiction that $\frac{F(c_{n,1}^{m_n}(0))}{\alpha_n} \to 1$. As in Step 1, a standard approximation gives us: $B(\alpha_n n-1, n-1, F(c_{n,1}^{m_n}(0))) \simeq \sqrt{\frac{1}{2\pi\alpha_n(1-\alpha_n)n}}$. Similarly as in Step 1, by the definition of $c_{n,1}^{m_n}(0))$ and the fact that $m_n \prec n^{2/3}$, for large n, we have: $1 \geq \frac{f(0)B(m_n-1,n-1,F(c_{n,1}^{m_n}(0)))}{c_{n,1}^{m_n}(0)} \simeq \sqrt{\frac{1}{2\pi(\alpha_n)^3(1-\alpha_n)n}} \to \infty$, a contradiction. We must therefore have that in equilibrium: $\frac{F\left(c_{n,1}^{m_n}(0)\right)}{\alpha_n} \to L > 1$, with L possibly arbitrarily large.

Step 3. We now prove that the probability of success in the first period converges to 1. Define for convenience here, $\zeta_n = \frac{F(c_{n,1}^{m_n}(0))}{\alpha_n}$. Note that the probability of failure in the first period is equal to the probability that the number of volunteers in period 1, j, is less than or equal to $\alpha_n n$ agents, which using Chebyshev's inequality can be bounded above as follows:

$$\Pr(j \leq \alpha_n n) \leq \Pr\left[\left|\frac{j}{n} - F\left(c_{n,1}^{m_n}(0)\right)\right| \geq \alpha_n(\zeta_n - 1)\right] \leq \left(\frac{\sqrt{\zeta_n(1-F\left(c_{n,1}^{m_n}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n-1)}\right)^2 \quad (35)$$

The result follows since $\zeta_n \to L > 1$, so $\left(\frac{\sqrt{\zeta_n(1-F\left(c_{n,1}^{m_n}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n-1)}\right)^2 = \lim_{n\to\infty} \frac{1}{m_n}\frac{L}{(L-1)^2} = 0.$ ■

Proof of Theorem 5: We proceed in two parts.

Part 1. We first prove that for any PBE, possibly asymmetric, there is a payoff equivalent (static) Honest and Obedient (HO) direct mechanism that achieves an expected utility for each type that is equal to that type's expected payoff in the PBE. To this goal, first recall that an HO direct mechanism is an *activity function* $\mu : [0,1]^n \to \Delta(2^I)$, that maps each profile of types, $\mathbf{c}$, to a probability distribution over subgroups of agents who volunteer. Note that for any history h_t with $k_{h_t} = k$ and $l_{h_t} = l$ and public signals $\theta_t = (\theta^1, ..., \theta^t)$, the PBE cutpoint at that history can be written as a function $c(h_t) = \{c_1(h_t), ..., c_n(h_t)\}$, where $c_i(h_t)$ is the cutpoint of player i. For any profile of types $\mathbf{c} = (c_1, ...c_n)$, vector of signals θ_t, a PBE defines a deterministic sequence of $c_t(\mathbf{c},\theta_t) = \{c_{t,1}(\mathbf{c},\theta_t), ..., c_{t,n}(\mathbf{c},\theta_t)\}$, where $c_{t,i}(\mathbf{c},\theta_t)$ is the cutpoint of agent i in period t given the history generated in correspondence to a realized profile $\mathbf{c} = (c_1, ...c_n)$, and a vector of signals θ_t up to t. At $t = 1$, $c_1(\mathbf{c},\theta_1) = c(h_1)$, where h_1 is just the realization of the public signal at $t = 1$, i.e. θ^1. Given the realization of types $\mathbf{c}$ and θ_2, $c_1(\mathbf{c},\theta_1)$ defines a unique history $h_2(\mathbf{c},\theta_2)$ in which the set of players who participate at 1 is given by the players with type $c_i \in [0, c_{1,i}(\mathbf{c},\theta_1)]$ and the public signals are θ_2. Suppose we have defined the history up to t. Then the cutpoints at t are given by $c_t(\mathbf{c},\theta_t) = c\left(h_t(\mathbf{c},\theta_t)\right)$.

Next, define a direct HO mechanism as follows. Denote $k_t(\mathbf{c},\theta_t)$ as the number of missing volunteers at the end of period t after history $h_t(\mathbf{c},\theta_t)$. For any profile $\mathbf{c}$ and corresponding PBE thresholds, $(c_t(\mathbf{c},\theta_t))_{t=0}^{\infty}$, define $T_i(\mathbf{c},\theta)$ for each i as the period at which i would volunteer in the PBE when the profile of types is $\mathbf{c}$, if there is such a period. That is:

$$\begin{aligned} T_i(\mathbf{c},\theta) &= \min\{t|c_{t,i}(\mathbf{c},\theta_t) \geq c_i\} \text{ if } \exists t \text{ such that } c_{t,i}(\mathbf{c},\theta_t) \geq c_i \\ &= \infty \text{ otherwise} \end{aligned}$$

where $\theta = (\theta^\tau)_{\tau=1}^{\infty}$; and let $S(\mathbf{c},\theta)$ denote the period at which the game ends with success if there is ever success at $\mathbf{c}$. That is:

$$\begin{aligned} S(\mathbf{c},\theta) &= \min\{t|k_t(\mathbf{c},\theta_t) = 0\} \text{ if } \exists t \text{ such that } k_t(\mathbf{c},\theta_t) = 0 \\ &= \infty \text{ otherwise} \end{aligned}$$

Denote by $I_t(\mathbf{c},\theta) = \{i|T_i(\mathbf{c},\theta) \leq t\}$ the set of agents who have activated up to and including t. We can now define the activity function μ^{DYN} for a static mechanism as follows, where, for each subset of agents, $g \subseteq I$, $\mu_g^{DYN}(\mathbf{c})$ specifies the probability that only the agents in g are activated when the reported cost profile is $\mathbf{c}$:

$$\mu_g^{DYN}(\mathbf{c}) = \begin{cases} \int_\theta \left[\sum_{\{\tau|I_\tau(\mathbf{c},\theta)=g\}} \left(1-e^{-\gamma\Delta}\right) e^{-\gamma\Delta(\tau-1)}\right] d\Pi(\theta) & |g| < m \\ \int_\theta \left[1_{\{\theta|I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta)=g\}} \cdot e^{-\gamma\Delta(S(\mathbf{c},\theta)-1)}\right] d\Pi(\theta) & \text{for } |g| \geq m \end{cases} \tag{36}$$

where $\Pi(\theta)$ is the distribution of the public signals; and $1_{\{\theta|I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta)=g\}}$ is the indicator function equal to 1 when θ is such that $I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta) = g$, and zero otherwise.

The activity function for the static mechanism, $\mu_g^{DYN}(\mathbf{c})$, is constructed from the PBE by the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals i with a type below $c_{1,i}(\mathbf{c},\theta_1) = c_i(h_1(\mathbf{c},\theta_1))$ are asked to volunteer (i.e., the set $I_1(\mathbf{c},\theta)$). If there are at least m such individuals, i.e., $k_1(\mathbf{c},\theta_1) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c},\theta) = 1$ and $\mu_{I_1(\mathbf{c},\theta)}^{DYN}(\mathbf{c}) = 1$. If $k_1(\mathbf{c},\theta_1) > 0$, i.e., $S(\mathbf{c},\theta) > 1$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c},\theta) > 1$ and $\mu_{I_1(\mathbf{c},\theta)}^{DYN}(\mathbf{c}) = 1 - e^{-\gamma\Delta}$, as in (36). With probability $e^{-\gamma\Delta}$, instead, the algorithm proceeds to Step 2. In Step 2, a public signal θ^2 is drawn; the vector of cutpoints $c_2(\mathbf{c},\theta_2) = \{c_{2,1}(\mathbf{c},\theta_2), ..., c_{2,n}(\mathbf{c},\theta_2)\}$ is determined; and an individual i with a type in the interval $(c_{1,i}(\mathbf{c},\theta_1), c_{2,i}(\mathbf{c},\theta_2)]$ is asked to volunteer and the process continues. In general, at any step t at which the algorithm has not yet stopped, any individual i with a type in the interval $(c_{t-1,i}(\mathbf{c},\theta_{t-1}), c_{t,i}(\mathbf{c},\theta_t)]$ is asked to volunteer. If there are at least $k_{t-1}(\mathbf{c},\theta_t)$ such individuals, i.e., $k_t(\mathbf{c},\theta_t) = 0$ and $S(\mathbf{c},\theta) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c},\theta)$ without proceeding to Step $t+1$. If $k_t(\mathbf{c},\theta_t) > 0$, i.e.,

$S(\mathbf{c}, \theta_t) > t$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability $e^{-\gamma\Delta}$ the algorithm proceeds to step $t+1$. In all cases, the probabilities are given at each step by (36). Thus, the static mechanism mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1 - e^{-\gamma\Delta}$ after any step at which the threshold m has not yet been achieved.

From the above construction of μ_g^{DYN}, we can represent the probability of success and that a player i is asked to volunteer by:

$$P(\mathbf{c}) = \int_{\theta} e^{-\gamma\Delta(S(\mathbf{c},\theta)-1)} d\Pi(\theta), \; A_i(\mathbf{c}) = \int_{\{\theta | S(\mathbf{c},\theta) \geq T_i(\mathbf{c},\theta)\}} e^{-\gamma\Delta(T_i(\mathbf{c},\theta)-1)} d\Pi(\theta)$$

where $P(\mathbf{c})$ is the probability of obtaining the public good at profile $\mathbf{c}$, and $A_i(\mathbf{c})$ is the probability that i is asked to volunteer at $\mathbf{c}$. The expected utility for an individual with type c at profile $\mathbf{c} = (c, \mathbf{c}_{-i})$ is $U_i(\mathbf{c}) = vP(\mathbf{c}) - c_i A_i(\mathbf{c})$. This is exactly equal to the expected utility for an individual with type c_i at profile $\mathbf{c}$ in the corresponding PBE of the dynamic game. We only need to prove that this direct mechanism is Honest and Obedient (Myerson, 1982). We need to show that every type c is weakly better off reporting c and obeying all recommendations, than they would be reporting c and disobeying some recommendations or reporting $c' \neq c$ and then following some optimal strategy in terms of obedience/non-obedience of the subsequent recommendation. Suppose there is a player i of type c who is strictly better off reporting to be a type $c' > c$. The analysis of the case in which i reports be a type $c' < c$ is analogous and omitted. There are two cases, corresponding to the two information sets in which i can find himself/herself: when the recommendation is to volunteer; and when it is to not volunteer.

Case 1. Consider first the case in which the recommendation is to volunteer. We prove here that reporting $c' > c$ and obeying to a recommendation to volunteer is not a strictly optimal deviation for an agent i. The proof that reporting $c' > c$ and disobeying to a recommendation to volunteer is not a strictly optimal deviation is similar and omitted (details are available in the online appendix). We first observe that if by reporting c' the recommendation is to volunteer, then the same recommendation must be received by reporting c. Since c' has received a recommendation to volunteer, it must be that $\mathbf{c}_{-i}$ is such that $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t) \geq c'$ for some some $t \leq S(\mathbf{c}, \theta)$ and a sequence of cutoffs $c_{t,i}(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the smallest period in which $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, then $c \in I_{t'}(c', \mathbf{c}_{-i}, \theta)$. Note, moreover, that by definition $c_{t'',i}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and $c_{t',i}(\widetilde{c}, \mathbf{c}_{-i}, \theta_{t'})$ is the same if $\widetilde{c} = c'$ or $\widetilde{c} = c$, so $c_{t',i}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t',i}(c, \mathbf{c}_{-i}, \theta_{t'})$ and $I_{t'}(c', \mathbf{c}_{-i}, \theta) = I_{t'}(c, \mathbf{c}_{-i}, \theta)$. Since $t' \leq t$, we have

$I_{t'}(c, \mathbf{c}_{-i}, \theta) = I_{t'}(c', \mathbf{c}_{-i}, \theta) \subseteq I_t(c', \mathbf{c}_{-i}, \theta)$: we conclude that a recommendation to volunteer to a type c', implies the same recommendation to a player who reports to be a type c as well. When the recommendation is to volunteer and the player obeys, then reporting $c' > c$ cannot be strictly superior than reporting c.

Case 2. The case in which the recommendation is to abstain in analogous to the cases above. Details are in the online appendix. Since there is no scenario in which the player finds it strictly optimal to report to be a type $c' > c$, we conclude that the player is never strictly better off by reporting $c' > c$, no matter what obedience policy s/he follows afterwards.

Part 2. From Part 1 we know that the expected utility in the PBE is equal to the expected utility in a specific direct, static mechanism that is honest and obedient. Battaglini and Palfrey (2024) have proven that the expected utility of a player in the best direct, static mechanism that is honest and obedient converges to 0 as $n \to \infty$ when $m_n \succ n^{2/3}$, i.e. when $\alpha_n / \sqrt[3]{1/n} \to \infty$ (See Theorem 4). The same must be true in the PBE. ■