Binary Conflict: Theory and Experiment

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Abstract

The simplest zero-sum conflict involves an attacker with two strategies and a defender with two strategies, each being better against one attack but worse against the other. This paper reports results of an experiment with several such games that sharply contradict the Nash equilibrium comparative statics of the choice frequencies as the payoffs change. Nonparametric quantal response equilibrium comparative statics match the treatment effects in the data. Econometric estimation of the (known) payoff parameters across the treatments accurately recovers the true values of those parameters with the quantal response equilibrium model but not with the Nash model.

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Keywords: games of conflict, strategy, Nash equilibrium, randomized strategies, quantal response equilibrium

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I. Introduction

Even though an attacker may have a rich array of available actions, it is useful to consider broad classifications to simplify the analysis. Military assaults may be either frontal or indirect, as with amphibious landings farther from the front. Similarly, a tennis player may choose between an alley “pass” or a lob, a penalty kicker in soccer may aim for one of the two sides of the goal, or an offensive coordinator in football may choose between passing and running. Sometimes, one of the decisions involves more risk, e.g. for a security agency to go on unannounced low alert (instead of high), or for a terrorist organization to launch an attack instead of delaying. This paper considers asymmetric constant-sum games in which one player has two alternative attack actions, and the other has two defense actions, one of which is more effective against a particular attack.

The strategic interaction is represented by a generic binary conflict game in Table 1, with the payoffs for the defender (row player) listed on the left side of each cell. The attacker (column player) chooses between actions $a_1$ and $a_2$, which could represent a pass or run, or a terrorist attack at site 1 or site 2. The defense is most effective against the attack action with the same subscript, and the attack is most effective against a “mismatched” defense. Therefore, the attacker’s payoff (listed on the right in each cell) is highest when the subscripts for the attack and defense actions do not match, i.e. attacker payoffs $B$ and $C$ are higher than $A$ and $D$. For example, attack $a_1$ is most effective against defense $d_2$, where the positive difference $B-A$ represents the relative advantage of this attack against a mismatched defense, as compared to a matched defense.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$ ($\alpha$)</th>
<th>$a_2$ ($1-\alpha$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>$-A, A$</td>
<td>$-C, C$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$-B, B$</td>
<td>$-D, D$</td>
</tr>
</tbody>
</table>

Table 1. An Attacker-Defender Game with Choice Probabilities $\alpha$ and $\delta$
(attacker payoff, defender payoff)

Applied game theorists have been particularly interested in data from athletic contests, given the presence of video records and the belief that coaches are highly...
motivated to call plays optimally. If the main interest in terms of policy or scientific understanding were for athletics per se, the appropriate next step would involve an intricate analysis of the behavior of motivated team decisions for settings in which the observer can estimate success probabilities and payoff consequences of particular attack or defense actions. Our focus, instead, is on strategic behavior more generally in binary conflict games. The workhorse theoretical paradigm for this analysis is the Nash equilibrium in mixed strategies, and the consensus from field observations is that the data are roughly consistent with Nash predictions.\textsuperscript{1} This is reassuring given the widespread application of Nash in the study of business strategy and economic behavior. The athletic applications mentioned above involve essentially symmetric games, e.g. a score or kick to either side is equally valuable. In contrast, it is natural to use asymmetric structures in laboratory “stress tests,” which could be analogous to providing 2 points for a successful penalty kick to one side of the net, as compared with the usual single point for the other side. Such artificially-constructed payoff changes can be used to obtain sharp distinctions between the predictions of alternative theoretical models of behavior in games.\textsuperscript{2}

This paper reports a laboratory experiment where the treatments involve raising or lowering the effectiveness of one of the attacker’s actions against both defenses, e.g. an increase in both attacker payoffs $A$ and $B$ in the $a_1$ column of the table. This change, for example, could be due to a terrorist organization’s known acquisition of an improved drone that is more likely to inflict damage, whether or not the defender concentrates on a drone defense or on a perimeter defense. The motivation for investigating the effects of such changes is that Nash equilibrium makes “unintuitive” predictions about the comparative statics of such a change in the payoff matrix. Specifically, an increase in the effectiveness of one attack method does not necessarily cause that attack to be used more

\textsuperscript{1} For example, see Walker and Wooders (2001) for an analysis of serving patterns in professional tennis.

\textsuperscript{2} Asymmetric cash payoffs were implemented by Composti (2003) in a “field” penalty kick experiment done with university team soccer players and a goal divided into sectors with vertical cloth strips. Conformity to Nash predictions was observed under symmetric payoff conditions, but not when payoffs for scoring and blocking were higher in one sector of the goal than in another. In addition, results of lab experiments with asymmetric matching pennies games have produced a sharp divergence of observed decision averages from Nash predictions, see e.g. Öchs (1995), McKelvey, et al. (2000), Goeree and Holt, (2001), and Goeree et al. (2003).
often in equilibrium depending on the values of the specific payoff parameters that are changed. In fact, the Nash equilibrium probability of choosing this “enhanced” attack method will actually decrease if the relative advantage against the mismatched defense, $B - A$, increases. On the other hand, if $A$ and $B$ both increase by the same absolute amount, then there is no effect on the Nash equilibrium attacker probabilities. Our experimental treatments include both kinds of variations in payoffs.

A natural question to ask is what are the advantages of using a laboratory experiment to test for these comparative statics predictions rather than using naturally occurring field data? The first advantage is obvious. In a laboratory study, the payoffs of the game matrix are controlled and hence (1) known to the experimenter/econometrician; (2) the same for all players; and (3) commonly known to all the players in the game. In contrast, an analysis of naturally occurring games requires estimation of nearly all of the relevant parameters of the game. Moreover, unobserved payoff heterogeneity creates challenges for estimation, and heroic assumptions about the players’ beliefs about the payoffs in the game are generally required. Heterogeneity can be critical; in the case of the enhanced attack effectiveness of attack $a_1$, it is natural to assume that both $A$ and $B$ increase even though the magnitudes of the increase might be different for different attacker defender pairs in naturally occurring contests.

The second advantage of a laboratory experiment concerns the precision of control in the experiment rather than the homogeneity and informational properties of the payoffs. Because the experimenter is able to specify exact payoffs and subject payoff information, manipulation of payoffs to test comparative static predictions of the theory can be done with precision. In the class of attacker-defender games we study here, it is necessary to change payoffs in a precise way in order to generate the theoretical predictions we wish to test. In potentially useful naturally occurring data, such as from professional sports contests, the effect of enhanced attack effectiveness on the magnitude of the difference, $B - A$ (the relative advantage against a mismatched defense) is difficult or impossible to assess precisely a priori. In our laboratory experiment, this difference
is exactly controlled by the payoff matrix provided to subjects and is exogenously changed in our different treatments.

This brings up a third advantage of the laboratory experiment over field data, which is design. In the laboratory we can choose which payoff manipulations to employ in the design in order to provide a powerful test of the hypotheses. In the case of naturally occurring data, it is usually the case that the payoff manipulations are neither directly observable nor a choice variable for the researcher.³

While these advantages of a controlled laboratory study are clear for the specific class of conflict games considered in this paper, tradeoffs must be acknowledged. Field data provides at least two potential advantages. First, the application where the field data is analyzed might be of significant economic importance, and one can use Nash equilibrium as a behavioral theory in the context of the specific application to make recommendations that might be of use to policy makers or economic agents. This is a prime motivation for the structural estimation of game theoretic models in industrial organization and other applied fields, where estimates of the driving parameters of the model can then be used to assess various policy options using counterfactual exercises. A second advantage is that the decision makers in the natural applications are experienced in their respective domains and so one might expect them to have acquired specialized knowledge and skill that is relevant to the decision tasks. Typical subjects in laboratory experiments are naïve and their decisions reflect a short amount of experience. If the game is sufficiently complicated, this could lead to significant differences in the behavior.⁴

³ There are some exceptions. For example, natural experiments may allow for directly observable and known changes. It is also sometimes possible to use expert advice to sign the directions of exogenous factors on key parameters in order to derive testable propositions for use with naturally occurring data. For example if the naturally occurring data were from professional sports, one might be able to differentiate individual players by particular skills that they possess, which would change the payoff matrix in particular ways, as in the McGarrity and Lemon (2010) study of pass and run plays in professional football.

⁴ We see the second of these issues as more relevant in our case than the first. Our interest is in testing comparative static predictions of alternative behavioral theories of choice behavior in games of conflict, rather than aiming for policy recommendations about a specific application.
Several economic studies have recently compared the behavior of experts and “naifs,” and have obtained mixed results.5

A second natural question to ask is why a reader should be interested in another experimental evaluation of the Nash equilibrium. The importance of this topic is indicated by the Holt and Roth (2004) perspective on its widespread applications:

“In the last 20 years, the notion of a Nash equilibrium has become a required part of the tool kit for economists and other social and behavioral scientists, so well known that it does not need explicit citation, any more than one needs to cite Adam Smith when discussing competitive equilibrium. There have been modifications, generalizations, and refinements, but the basic equilibrium analysis is the place to begin (and sometimes end) the analysis of strategic interactions, not only in economics but also in law, politics, etc.” (Holt and Roth, 2004).

In response to mounting contradictory evidence from experiments, economists have devised several plausible alternatives to the Nash analysis that is based on twin assumptions of perfectly rational choice (sharp responses to small payoff differences) and equilibrium (action distributions are consistent with players’ beliefs). Some of these alternatives have been proposed in the context of specific types of games. For example, in committee voting with multi-stage agendas, there is experimental evidence that voters fail to account for later agenda stages, and instead vote myopically based on current-stage outcomes (Plott and Levine, 1978, Eckel and Holt, 1989). Another approach is to specify rationality “types,” with some players being random (level 0), others (level 1) making a best response to random play, and others (level 2) making best responses to level 1

5 A recent comparison of students and adults in a variety of classic experimental games documented some differences and conclude that student behavior is more rational and selfish: “Experiments using students are likely to overestimate the extent of selfish and rational behavior in the general population.” (Belot et al, 2015, p. 26). Perhaps more relevant to a comparison of student and adult subjects is a consideration of the behavior of relevant professionals. Fréchette (2015) provides a review of studies that compare student and relevant professional groups in parallel settings, and concludes that there is no systematic difference in terms of conformity of observed behavior with theoretical predictions, although other differences are noted in some cases. Weitzel et al. (2020) provides some evidence that banking professionals generate lower asset price bubbles in laboratory experiments than is the case for students, although both groups do generate bubbles, with the same qualitative treatment effects.
behavior, etc. This “level-k” approach has been useful for explaining observed behavior in games that are played once, with no opportunities for learning from experience.

One alternative to Nash equilibrium that is widely used in experimental economics is the quantal response equilibrium (QRE: McKelvey and Palfrey, 1995; Goeree, Holt, and Palfrey, 2016). This approach is general in that it is a unified equilibrium framework that can be applied to all types of games, including extensive form games, normal form games, and games with continuous strategies. QRE maintains the equilibrium assumption (belief distributions match choice distributions), but relaxes perfect rationality by adding “noise” into decision making, e.g. with decisions determined by probabilistic choice, e.g. logit, probit, etc. The intuition is that large mistakes are less likely than small ones, which could be due to distraction, miscalculation, or small random payoff disturbances. In economics, such disturbances are typically folded into an “error” term, which was the original formulation of QRE. Thus, QRE provides a generalization of Nash in the sense that sharp best responses are replaced by smoothed “quantal responses.” QRE is not an alternative to Nash; it is a generalization that includes Nash as a limiting case, in the sense of converging to Nash equilibrium in the limit as the noise is reduced and quantal response functions approach best response functions.

Any equilibrium theory, Nash, QRE, competitive equilibrium, or whatever, could be problematic for the analysis of games played only once, in which players have no opportunity to learn. With experience and repetition, observed behavior can be analyzed in the framework of learning models. Economists, however, are generally concerned with longer run properties of market or strategic interactions, which is why most economic theory is based on some notion of equilibrium.

Some lingering skepticism of QRE, however, resulted from a focal and very sharply worded critique in the American Economic Review that concludes:

“The quantal response equilibrium (QRE) notion of Richard D. McKelvey and Thomas R. Palfrey (1995) has recently attracted considerable attention, due in

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6 This approach is quite similar in spirit to the widely used discrete choice models in applied econometrics (McFadden 1976) as well as the more recent surge of theoretical work on rational inattention initiated by Sims (2003).
part to its widely documented ability to rationalize observed behavior in games played by experimental subjects…. However, even with strong a priori restrictions on unobservables, QRE imposes no falsifiable restrictions: it can rationalize any distribution of behavior in any normal form game.” (Haile, Hortaçsu, and Kosenok, 2008)

The no-empirical-content argument is based on QRE in its most general form, for which a very general existence proof in the original 1995 paper was established with essentially no restrictions on the joint distribution of payoff disturbances. However, all applications of QRE either explicitly or implicitly assume independent and identically distributed (i.i.d.) disturbances, or even weaker regularity conditions on the quantal response function, as is the case in the econometric application of discrete choice models. The key regularity condition for application is that higher payoff actions are chosen with higher probability, or monotonicity, which holds for i.i.d disturbances. Monotonicity leads to strong falsifiable restrictions both within a single game and across games.

Before proceeding, it is useful to provide intuition for the observation of “no falsifiable restrictions” without i.i.d, (or similar) assumptions. The observation can be illustrated with a simple binary attacker-defender game, e.g. when the attacker payoff in Table 1 goes down by 1 when the defender matches the subscript ($A = D = -1$), and the attacker gains 1 with a mismatch ($B = C = 1$). This is a “matching pennies” game in which the attacker (column) loses a penny if the pennies match, and gains a penny if they do not. The Nash equilibrium in this symmetric game is for each player to play each action with probability 0.5.

QRE adds an error term to each expected payoff. If expected payoffs are $\pi_{a1}$ and $\pi_{a2}$ for the attacker’s two actions and the associated disturbances are $\epsilon_1$ and $\epsilon_2$, then the disturbed payoff for action $a_2$ is larger if $\pi_{a2} + \epsilon_2 > \pi_{a1} + \epsilon_1$, or equivalently if $\epsilon_2 - \epsilon_1 > \pi_{a1} - \pi_{a2}$. This condition is that the disturbance for attack 2 is high enough to overcome the payoff pull toward the other attack action. Since all payoffs in this game are plus or

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7 This regularity condition and the sufficiency of i.i.d. disturbances to guarantee monotonicity, is clearly identified in McKelvey and Palfrey (1995, 1996) and expanded on in Goeree, Holt, and Palfrey (2005, 2016).
minus 1, the expected payoff difference, $\pi_{a1} - \pi_{a2}$ will be less than 2 for any non-zero probability associated with the defender’s action. So the trick to generating a high probability of the attacker choosing action 2 is to have the disturbance difference of $\epsilon_2 - \epsilon_1$ be greater than or equal to 2 with high probability. This can be done in many ways if the disturbances are not i.i.d. Suppose that both disturbances are (identically) distributed uniformly on the interval $[-100, 100]$, but are not independent, and that $\epsilon_2 = \epsilon_1 + 2$ whenever $\epsilon_1 < 98$, otherwise it “wraps around” to the bottom: $\epsilon_2 = -100 + (\epsilon_1 - 98)$. In this case, the probability the attacker chooses action 2 is at least 0.99 regardless of the defender’s choice behavior, even though both disturbances have the same uniform distribution with mean zero.\(^8\) The key is that these disturbance distributions are not independent.\(^9\) Notice that this observation has nothing to do with the “E” of QRE; rather the construction is simply a way to rig the quantal responses to be non-monotonic. The observation applies equally to any additive random utility model of choice. Hence this critique is not relevant for any actual application of QRE because all such applications use monotonic quantal response functions, e.g. logit, probit, etc. When the disturbances for each action are i.i.d., then the distribution of the difference between the disturbances to the expected payoffs for $a_1$ and $a_2$ is symmetric around 0, with mean zero, and the action with a higher expected payoff is chosen with a probability above 0.5, irrespective of the particular parametric form of the disturbances, e.g. extreme-valued, Gaussian, uniform, or whatever. This general nonparametric property of QRE with i.i.d. disturbances ensures empirical restrictions, i.e. it is not possible to explain any observed decision proportion.

Monotonic quantal responses for the defender in the symmetric matching pennies version of the game in Table 1 with $A = D = -1$ and $B = C = 1$ must pass through $(0.5, 0.5)$ in Figure 1, as shown by the downward sloping curved line.

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\(^8\) The probability of attack 1 can be made arbitrarily high in a similar fashion.

\(^9\) A similar trick can be used to make either one of the attack probabilities arbitrarily large if the modeler is free to specify non-identical distributions of the disturbances.
Figure 1. Best and Quantal Responses for a Symmetric Matching Pennies Game

In the absence of noise, the defender’s best response line would have sharp corners as shown by the solid line “step” down in Figure 1, and noise has the effect of smoothing off the corners (thin curved line). Similarly, the attacker’s dashed best response line in the figure would be replaced by a smoothed noisy response line with a positive slope (thin dashed line), which would also pass through the midpoint (0.5, 0.5). In this case, QRE (intersection of noisy response lines) does have empirical content; in fact, it has exactly the same empirical content as the Nash equilibrium, with both making a sharp prediction about equally likely actions for each player, irrespective of the particular functional forms of the quantal response functions. In contrast, QRE predictions will differ from Nash predictions in the presence of asymmetries in the games considered in this paper.

The next section analyzes the Nash equilibrium comparative statics of increases (or decreases) in the payoff effectiveness of one of the attack modes against either defense. Surprisingly, an attack strategy with increased effectiveness is not necessarily used more frequently in a Nash equilibrium. These unintuitive comparative static predictions provide an initial motivation for our experimental treatments. In sharp
contrast, we show that QRE generates *opposite* – and more intuitive – predictions, which are robust to the particular parametric specification of the quantal response function. Section III details the design of the laboratory experiment with financially motivated human subjects in which the treatments implement an exogenous increase in the effectiveness of one of the attack actions. One treatment change involves equal absolute changes in this method’s payoff for the attacker against either defense, and the other involves equal proportional changes. This second treatment change is particularly interesting, since the Nash prediction is that an equal proportional increase (e.g. doubling) in the effectiveness of one attack will result in a *reduction* in attack rates for that attack method. The experiment results presented in Section IV include cases where the observed behavior is quite close to Nash predictions (in a baseline low-effectiveness treatment). The most salient feature of the experiment data, however, is that attack frequencies are increasing in the attack effectiveness, regardless of whether the increase is proportional or absolute, which is intuitive but contrasts sharply with the unintuitive Nash predictions of no effect or even of a reduced use of the attack with enhanced payoffs (in the equal proportional increase treatment). The closeness of data averages to Nash predictions in some treatments and not in others is shown to be consistent with the comparative static QRE. These predictions are formally derived in Section V, based on a nonparametric specification of the quantal response functions, i.e. *without* explicit dependence on specific logit, probit, or other distributions.

A rigorous estimation and model comparison between QRE and Nash is reported in section VI. This is done in two different ways. The first, which is the more familiar approach used to analyze data produced in game theoretic laboratory experiments, is a maximum likelihood estimation of a logit QRE model that restricts the free parameter (precision) to be constant across all three treatments. The resulting estimates of choice frequencies produce a close match to the data, while the Nash equilibrium choice frequencies are far from the data in all but one treatment. The second approach, which is the more familiar approach in the applied industrial organization literature, estimates the payoff parameters in the games, treating them as unknowns. Because those payoff
parameters are laboratory-controlled constants that are known both to the experimenter and to the subjects, we can then test whether the estimates based on the QRE and Nash models successfully recover those known values. The QRE model accurately recovers the concealed payoff parameters, while the Nash equilibrium does not.

II. Nash Equilibrium for an Attacker-Defender Model of Binary Conflict

Consider a zero-sum game, with two possible actions for the column player (attacker), $a_1$ and $a_2$, and two possible actions for the row player (defender), $d_1$ and $d_2$. All parameters, $A, B, C,$ and $D$ are positive, which are payoffs for each outcome, with the defender’s payoff shown as negative on the left in each cell in Table 1 above. The assumption that the attacker decision does better against the “wrong” defense (i.e., with mismatched subscript) is indicated by the two inequalities in (i) below. The game is only interesting when there are no dominated strategies, which is ensured by assuming that the highest attacker payoff for each decision is greater than the lowest payoff for the other one, as specified in (ii):

i) $B > A$ and $C > D$

ii) $C > A > 0$ and $B > D > 0$.

The Nash equilibrium is unique and is in mixed strategies, i.e. a pair of choice probabilities strictly between 0 and 1, one for each player. Let $\alpha$ denote the probability that Column chooses attack $a_1$, and let $\delta$ denote the probability that Row chooses defense $d_2$; these probabilities are shown in parentheses next to the corresponding action in Table 1. The attacker’s expected payoff difference for $a_1$ versus $a_2$, as a function of the defender’s mixed strategy, $\delta$, is:

$$\pi_{a_1}(\delta) - \pi_{a_2}(\delta) = A - C + (C - D + B - A)\delta.$$ 

This expected payoff difference must be 0 in a mixed strategy Nash equilibrium, which determines the equilibrium probability $\delta^*$ for the defender:

$$\delta^* = \frac{C - A}{C - D + B - A} \quad \text{(Nash equilibrium probability of } d_2).$$
Note that this expression has been conveniently organized so that the various parameter differences \((B-A), (C-A),\) and \((C-D)\) are positive by the initial inequality assumptions.

The equilibrium mixing probability for the attacker is obtained similarly. The expected payoff difference for defense \(d_2\) versus \(d_1\), as a function of the attacker’s mixed strategy, \(\alpha\), is:

\[
\pi_{d_2}(\alpha) - \pi_{d_1}(\alpha) = (C - D) - (C - D + B - A)\alpha,
\]

which yields:

\[
\alpha^* = \frac{C - D}{C - D + B - A} \quad \text{(Nash equilibrium probability of } a_1).\]

Now consider an exogenous change that increases the effectiveness of attack \(a_1\), by raising both \(A\) and \(B\). Notice that these parameters only appear as a difference \((B-A)\) in the denominator of (2). This model, therefore, yields the unintuitive prediction that equal absolute increases in the effectiveness parameters \(A\) and \(B\) for attack \(a_1\) will have no effect on the equilibrium probability of choosing that attack. This prediction is due to a countervailing increase in the equilibrium probability of defending against that attack. For example, suppose that attacks \(a_1\) and \(a_2\) correspond to passing and running in football, and defenses \(d_1\) and \(d_2\) correspond to defending against a pass or against a run respectively. In this case, the treatment change implies that a pass has enhanced effectiveness against either defense, with the enhancement being of the same absolute magnitude. The Nash equilibrium prediction, however, is that the more effective pass attack is not used any more than it was prior to the enhancement.

On the other hand, it is clear from the formula for \(\alpha^*\) in (2) that unequal increases in both \(A\) and \(B\) do not necessarily leave the Nash choice rate for \(a_1\) unchanged. In particular, consider the effects of proportional changes in \(A\) and \(B\), to \(\gamma A\) and \(\gamma B\), where \(D/B < \gamma < C/A\), which ensures that the inequality restrictions in (ii) are not violated. With \(\gamma > 1\), for example, the proportional increases in the effectiveness parameters \(A\) and \(B\) for attack \(a_1\) actually cause an increase in the denominator of (2), so the Nash probability of attack \(a_1\) would decrease, which is a clearly unintuitive prediction. The reason is that the
equilibrium response of an increase in the defense $d_1$ more than offsets the attractiveness of attack $a_1$. The design is based on the unintuitive Nash predictions resulting from equal absolute or equal proportional increases in the effectiveness of one attack method.

### III. Experimental Design and Procedures

The experimental design is summarized in Table 2, which has rows for two treatment intensities of attack effectiveness (high and low) for the column player’s first attack, $a_1$, relative to that of the other attack $a_2$. The Low effectiveness treatment in bottom row of the table corresponds to relatively low column payoffs for attack $a_1$, and the High treatment corresponds to high column payoffs for $a_1$.

**Table 2.** Treatments, Predictions, and Data Averages for Equal Absolute or Equal Proportional Changes in the Effectiveness of Attack $a_1$

<table>
<thead>
<tr>
<th>Treatment of Attack $a_1$</th>
<th>Equal Absolute Changes A, B, C, D</th>
<th>Nash $a_1$ Rate</th>
<th>Equal Proportional Changes A, B, C, D</th>
<th>Nash $a_1$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Effectiveness</td>
<td>6, 12, 7, 4</td>
<td>0.33</td>
<td>6, 12, 7, 4</td>
<td>0.33</td>
</tr>
<tr>
<td>Low Effectiveness</td>
<td>2, 8, 7, 4</td>
<td>0.33</td>
<td>3, 6, 7, 4</td>
<td>0.50</td>
</tr>
</tbody>
</table>

The design on the left side of the table involves equal absolute changes in the two attack effectiveness parameters, $A$ (against defense $d_1$) and $B$ (against defense $d_2$). The High effectiveness treatment is shown in the top row of Table 2, with a parameter set for $A, B, C, D$ of 6, 12, 7, 4. The low absolute effectiveness treatment in the bottom row on the left side results from a reduction of both $A$ and $B$ by 4 in each case, i.e. to 2 and 8, respectively. With these equal absolute reductions in $a_1$ effectiveness, the Nash equilibrium probability of attack $a_1$ is the same, 0.33, in both cases, as shown in column 3 of Table 2.

The right side of Table 2 involves equal proportional changes $A$ and $B$, which are reduced by 50% from 6 and 12 in the top row to 3 and 6 in the bottom row, respectively. Recall that these equal proportional reductions cause a reduction in the $B–A$ difference in the denominator of the Nash equilibrium $\alpha$ probability calculation in (2). This reduction in the denominator, in turn, causes the Nash equilibrium value of $\alpha$ to actually increase.
(instead of staying constant). The last column of Table 2 shows the increase in the Nash equilibrium $a_1$ attack rate from 0.33 for the High effectiveness treatment to 0.50 in the bottom row on the right. Notice that the parameters for the High Effectiveness treatments (6, 12, 7, 4) are the same on the left and right sides in the top row of Table 2, and that the Low Effectiveness Treatments in the bottom row involve equal absolute reductions (“LowAb”) on the left or equal proportional reductions (“LowProp”) on the right.

The experiment was conducted with all payoffs (in dollars) reduced by a factor of 10, e.g. from 6 to 0.60, 12 to 1.20, etc. In addition, we added a fixed amount of 1.20 to all row player (defense) payoffs in order to ensure that none of the payoffs were negative. These payoff adjustments are useful to generate reasonable earnings levels and prevent loss aversion effects, and they have no effect on the Nash equilibrium predictions. We will continue to discuss the treatments in terms of integer amounts, e.g. 6, 12, 7, 4, which now refer to 10-cent payoff units.

The experiment is based on a between-subjects design, with 20 rounds for each of the two attack effectiveness treatments, High or Low. Roles stayed the same (attacker or defender), and matchings were fixed, both within each 20-round part and across parts. Both treatment orders (Low-High and High-Low) were used, with Low-High in half of the sessions and High-Low in the other half of the sessions. We conducted 4 sessions (48 subjects) with the equal absolute change design on the left side of Table 2, and another 4 sessions (48 subjects) with the equal proportional change on the right side. Subjects were recruited from the University of Virginia student population and were paid $6 plus all earnings. Total earnings averaged $30 for a session that lasted about 45 minutes, which included the reading of instructions aloud. The experiment was run with the Veconlab matrix game software. The game was presented with minimal context, i.e. with decisions labeled Top or Bottom for the row player, and Left or Right for the column player. Verbatim instructions from one of the treatments are provided in the Appendix.

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10 A scale change in all payoff parameters does not affect the Nash equilibrium choice frequencies for the attacker, which are ratios of linear functions in (1) and (2). Similarly, adding a constant to all parameters will not alter either of the equilibrium mixed strategies.

11 Fixed matching does not invite repeated game effects, since the payoffs are constant-sum.
IV. Results

Figure 2 shows the time series of $a_1$ attack frequency averages ($\bar{a}$) for the first paired treatment that changes the attack effectiveness parameters $A$ and $B$ by equal absolute amounts. Recall that this change does not affect the Nash equilibrium prediction for the $a_1$ attack rate, which is 0.33 for both treatments, as indicated by the horizontal dark line at that level. The solid lines in the figure are averages for High Effectiveness treatment, and the dashed lines are for LowAb Effectiveness. The averages for sessions with High Effectiveness in the first 20 rounds are displayed in gray, and averages for the reverse order are displayed in black. For example, the black dashed line on the left shows that the observed $a_1$ frequencies center around the Nash prediction of 0.33 with low effectiveness for the first 20 rounds. In contrast, the black solid line on the right indicates that $a_1$ attack frequencies almost doubled in the final 20 rounds after the treatment switch for the same subjects and the same Nash prediction. Average $a_1$ attack rates are obviously greater in the High treatment, with an overall average of 0.59 (for both treatment orders), than in the Low treatment, with an overall rate of 0.33. This difference holds true for
pairwise comparisons in all rounds but one. Moreover, there are no clear sequence or trend effects in terms of average attack frequencies.\textsuperscript{12}

Figure 3 shows a similar plot of $a_1$ frequencies for the case of equal proportional changes in the $A$ and $B$ effectiveness parameters for this attack method. The treatment change from the LowProp treatment (3, 6) to the High treatment (6, 12) reduces the Nash equilibrium value of $\alpha$ from 0.5 (horizontal line on left side of Figure 3) to 0.33 (horizontal line on right), or vice versa for the opposite sequencing of the treatments. Data averages for the 20 rounds in the LowProp treatment are shown a dotted line on the left side of Figure 3, regardless of whether it was done first or second, and the 20 rounds of data with the High treatment are displayed on the right. Observed $a_1$ attack frequencies increase from 0.49 to 0.63 as $A$ and $B$ increase proportionally, which is the opposite of the Nash equilibrium downward shift at the midpoint of the figure.

\textbf{Figure 3.} Average $a_1$ Attack Frequency by Round, with Equal Proportional Changes: Dashed Line for Low Effectiveness (3, 6), Solid Line for High Effectiveness (6, 12)

\textsuperscript{12} Sequence effects are evaluated by using a permutation test to compare average attack rates a given treatment, e.g. (2, 8) when it is done first with the same treatment when it is done second. There are 12 attacker-defender pairs in each order, so there are 12 independent $a_1$ attack rates for each order. The difference in treatment order is not significant for either the (2, 8) treatment ($p = 0.844$) or for the (6, 12) treatment ($p = 0.61$).
If the enhanced effectiveness of the $a_1$ attack is anticipated by the defenders, then the incidence of the matched $d_1$ defense should rise and the incidence for $d_2$ should fall. Table 3 shows a breakdown of the overall frequencies of both attack ($a_1$) and defense ($d_2$) decisions, with a row of treatment averages in bold and a row of Nash predictions in italics. First consider the left side of the table, for the equal absolute change, from the LowAb treatment with $A = 2$ and $B = 8$ to the High treatment with $A = 6$ and $B = 12$. When the $A$ and $B$ effectiveness parameters for attack $a_1$ increase by equal amounts in this manner, the targeted defense $d_1$ against this attack is predicted to increase, and the propensity $\delta$ for the other defense $d_2$ is predicted to decline sharply from 0.56 to 0.11, as shown in the bottom row on the left. The corresponding increase in $1 - \delta$ is what causes the Nash equilibrium value of $\alpha$ for attack $a_1$ to stay constant at 0.33 even in the presence of the doubled effectiveness of this attack method. The data show a less dramatic decline in the $d_2$ frequency, from 0.41 to 0.19 percent, coupled with the $a_1$ frequency that almost doubles, from 0.33 to 0.59 percent, which leads to our first result.

Table 3. Treatments, Nash Predictions, and Observed Data Proportions

<table>
<thead>
<tr>
<th>Treatment Payoffs</th>
<th>Equal Absolute Change</th>
<th>Equal Proportional Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A, B, C, D)$:</td>
<td>LowAb ($2\ 8\ 7\ 4$)</td>
<td>HighAb ($6\ 12\ 7\ 4$)</td>
</tr>
<tr>
<td>Attack Proportion $\alpha$:</td>
<td>0.33</td>
<td>0.59</td>
</tr>
<tr>
<td>Nash prediction:</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>Defense Proportion $\delta$:</td>
<td>0.41</td>
<td>0.19</td>
</tr>
<tr>
<td>Nash prediction:</td>
<td>0.56</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Result 1a: An equal absolute increase in effectiveness parameters for attack $a_1$ (from 2 and 8 to 6 and 12) causes a significant and large (nearly doubled) increase in the observed frequency of this attack, despite the Nash prediction of no change.

Support: The observed overall treatment difference is about 26 percentage points ($0.5917 - 0.3292 = 0.2625$). To assess significance, note that there were 24 column players
(attackers) who made sequences of decisions in 20 periods of each treatment, while being matched repeatedly with a single row player. The $a_1$ frequency increased in the High treatment for all but 6 of 24 column players. (There were 3 relatively small reversals and 3 ties with equal pass rates in each treatment.) Since each attacker/defender pair interacted with each other in all 20 rounds, we use an exact matched-pairs permutation test to calculate the proportion of permutations of treatment labels that yield a treatment difference as great as or greater than the observed difference of 0.2625. The resulting $p$-value is 0.0004 for a 2-tailed test and 0.0002 for a 1-tailed test, so the null hypothesis of no effect is rejected at the 1% level.

The increase in $a_1$ attack rates in the HighAb treatment is consistent with the observation that the $d_1$ defense against this attack was observed less frequently than predicted in the Nash equilibrium. The increase in $a_1$ attack effectiveness resulted in a decline in the $d_2$ defense frequency from 0.41 to 0.19, which is a substantially smaller decline than the predicted decline from 0.56 to 0.11, as shown in the bottom row of Table 2, left side, a difference that is highly significant:

**Result 1b:** An equal **absolute** increase in effectiveness parameters for attack $a_1$ (from 2 and 8 to 6 and 12) results in a tendency to over-defend against the other attack $a_2$ relative to the Nash prediction of a sharper reduction in $d_2$ defense rates.

**Support:** The observed overall reduction in the $d_2$ defense from 0.41 to 0.19 is about 22 percentage points, which is much less than the predicted 45 percentage point reduction from 0.56 to 0.11. To assess significance, note that there were 24 row players (defenders) who made sequences of decisions in 20 periods of each treatment, while being matched repeatedly with a single column player. The $d_2$ frequency decreased by less than 45 points for all but 4 of 24 row players, as can be verified by considering the $d_2$ defense rate changes shown in the rows of the first page of the Data Appendix. This difference is significant at $p < 0.001$ for a 2-tailed binomial test of the null hypothesis that the reduction in $d_2$ defense rates is equally likely to be greater than or less than the Nash prediction.
Next consider the results for the equal proportional change (doubled) in the $A$ and $B$ parameters, shown on the right side of Table 3. The Nash equilibrium value of $\alpha$ no longer stays the same, but rather, declines from 0.50 to 0.33. In contrast to the decline predicted by Nash equilibrium, $\alpha$ actually increased from 0.49 to 0.63 percent, a 14 point change, yielding an $a_1$ attack frequency that is nearly double (30 percentage points higher than) the Nash equilibrium frequency of 0.33. The observed increase is significant for a 2-tailed matched pairs permutation test ($p = 0.034$). This is summarized as our second result:

*Result 2a:* An equal proportional increase in effectiveness parameters for attack $a_1$ (from 3 and 6 to 6 and 12) causes a significant increase in the observed frequency for this attack, despite the Nash prediction that the frequency will decline.

In this equal-proportional-increase treatment, however, there is no clear tendency for defenders to over-defend against the other attack method, $d_2$:

*Result 2b:* An equal proportionate increase in effectiveness parameters for attack $a_1$ (from 3 and 6 to 6 and 12) does not result in a clear tendency to over-defend against the other attack $a_2$ relative to the Nash prediction.

*Support:* The observed overall observed reduction in the $d_2$ defense from 0.66 to 0.17 is close to the predicted 56 percentage point reduction (from 0.67 to 0.11) in a Nash equilibrium. To assess significance, recall that there are 24 row players (defenders) who make sequences of decisions in 20 periods of each treatment, while being matched repeatedly with a single column player. The $d_2$ frequency decreased by less than the predicted 56 points for 16 of 24 row players and by more than 56 points for 8 of 24 row players. This difference is not significant, with $p = 0.152$ for a 2-tail binomial test of the

---

13 The matched pairs permutation test uses data averages instead of the data ranks that are used in a matched pairs Wilcoxon (“signed rank”) test. The permutation test with data averages is based on the intuition that magnitudes as well as ranks are informative. In our data, all of the large changes were in the direction of higher $a_1$ attack rates for High effectiveness (from 0.1 to 1.0, from .02 to 0.7, and from 0.15 to 0.7).

14 These data counts can be verified by considering the $d_2$ defense rate changes, row by row, from the second page of the Data Appendix.
null hypothesis that the reduction in $d_2$ defense rates is equally likely to be greater than or less than the Nash prediction.

Finally, note that the choice frequencies in the third column of Table 3 for both attackers and defenders in the LowProp treatment are almost exactly equal to the Nash predictions (and a null hypothesis of no difference cannot be rejected by standard statistical tests). Recall from the earlier discussion that Nash and QRE predictions are identical for symmetric games where the Nash point is in the center of figure 1. This suggests that in asymmetric games, Nash predictions tend to be more accurate when Nash and QRE predictions are close, and less accurate otherwise, which leads into the next section where we characterize the properties of QRE in binary conflict games.

V. Quantal Response Equilibrium

A Nash equilibrium is based on perfect rationality: players respond sharply to differences in payoffs by choosing the high-payoff action with probability 1 (even if the payoff difference is small), and players expect their opponents to be perfectly rational. In contrast, QRE incorporates probabilistic choice (logit, probit, or whatever) in which actions with higher payoffs are chosen more often, but not necessarily with probability 1. QRE retains the equilibrium assumption of rational expectations, i.e., that players correctly expect their opponents to probabilistically respond to expected payoffs. This modeling approach often provides theoretical predictions about behavior that are more intuitive and better explain experiment data than those derived from the Nash equilibrium perfect rationality assumptions.

With two decisions for each player, QRE determines the choice probabilities, $\alpha$ and $\delta$, as functions of (equilibrium) expected payoff differences, in a manner that depends on the “precision” of choice behavior, which is parameterized here by a responsiveness coefficient $\lambda$. As precision is increased (choice error is decreased), behavior gradually approaches fully rational best replies, and choice probabilities respond more sharply to expected payoff differences. Conversely, as precision is reduced, choices are less responsive to payoff differences. Thus the Nash solutions in (1) and (2) are replaced by
QRE choice probabilities, given by any continuous and strictly increasing quantal response function \( Q(\cdot) \) of the precision-adjusted expected payoff differences:

\[
\alpha = Q(\lambda [\pi_{a_1}(\delta) - \pi_{a_2}(\delta)]) = Q(\lambda [A - C + (C - D + B - A)\delta])
\]

\[
\delta = Q(\lambda [\pi_{d_2}(\alpha) - \pi_{d_1}(\alpha)]) = Q(\lambda [C - D - (C - D + B - A)\alpha])
\]

where \( Q(\Delta) \) is a quantal response function of the expected payoff difference weighted by \( \lambda \), a positive precision parameter. The quantal response function can be any continuously differentiable function defined on the real line that satisfies:\(^{15}\)

\[
\text{iii) } \frac{Q'}{\Delta} > 0, \quad \lim_{\Delta \to \infty} Q(\Delta) = 1, \quad \text{and} \quad Q(-\Delta) = 1 - Q(\Delta) \quad \text{for all } \Delta.
\]

The final part of (iii) requires that the choice probabilities are not biased for one attack action over the other, and (because of continuity) implies that the actions are chosen with equal probability when the expected payoff difference is 0. A steep slope of this function at a payoff difference of 0 corresponds to high precision. The limit condition in (iii) ensures that choices are approximately deterministic for extreme payoff differences.

A QRE for a binary conflict game is a pair of choice probabilities, \( \alpha \) and \( \delta \), that solve the nonlinear equations in (3) and (4). The intuition is that the \( \alpha \) and \( \delta \) probabilities on the right sides of these equations can be thought of as representing “beliefs” about the other’s decisions, and the rational QRE expectations assumption requires that these beliefs match the equilibrium choice probabilities that emerge as a result of probabilistic choice on the left sides of (3) and (4). QRE is a generalization of Nash equilibrium in the sense that it includes the perfect rationality assumption implied by Nash as a limiting case for \( \lambda \to \infty \). To see the intuition, note that the left equation in (3) can be rewritten in terms of the inverse of the quantal response function as:

\[
Q^{-1}_\lambda(\alpha) = \pi_{a_1}(\delta) - \pi_{a_2}(\delta).
\]

---

\(^{15}\) The results in this section apply generally, for any functional form of the quantal response function, \( Q \). The comparative statics results are, therefore, non-parametric in nature. In the next section, for analytical convenience we will conduct a parametric QRE analysis and estimation using the Logit model, but readers should keep in mind throughout the paper that the qualitative implications of QRE in this class of games are non-parametric.
For any $\alpha$ such that $0 < \alpha < 1$, the left side goes to zero as precision goes to infinity, and (5) becomes the standard equal-expected-payoff condition used previously to determine a mixed-strategy Nash equilibrium.

Notice that different forms of the quantal response function in (3) and (4) will provide different QRE outcomes, and here we consider general comparative statics of an increase in the effectiveness of a particular attack method, for any admissible $Q$ function. This approach is analogous to analyzing the comparative statics of a supply and demand model, which can yield sharp qualitative predictions, even though the exact magnitude of the effect of an exogenous demand shock on price would depend on the elasticity of the supply and demand functions.

**QRE for Equal Absolute Changes in Attack Effectiveness Parameters**

Recall that equal changes in the effectiveness parameters $A$ and $B$ will have no effect on the Nash equilibrium $a_1$ attack probability in (2). Even though the analogous QRE attack proportion will converge to the Nash equilibrium in (2) as precision goes to infinity, a quick inspection of the QRE equilibrium equations (3) and (4) indicates that equal changes in the $A$ and $B$ parameters will generally have an effect on QRE predictions. Although the $B-A$ difference terms on the right sides of (3) and (4) will not be altered by equal absolute changes, there is an additional stand-alone $A$ term on the right side of (3), which is highlighted in bold. With equal changes in $A$ and $B$, this stand-alone term will generate a higher $a_1$ attack response for any given value of the defense choice probability, $\delta$, on the right-hand side of (3). In contrast, there is no separate stand-alone term with $A$ or $B$ on the right side of (4), so the $d_2$ response to a given attack choice probability, $\alpha$, will not shift after equal changes in $A$ and $B$. Using an analogy with market supply and demand curves, the “supply” of attack $a_1$ in (3) will shift, but the “demand” for defense $d_2$ in (4) will not, and therefore, the QRE intersection of these functions will move along the graph of (4) in a $(\alpha, \delta)$ space. As a result, the QRE equilibrium $a_1$ attack rate, $\alpha$, will be affected by equal changes in $A$ and $B$ that do not alter the Nash prediction. The
following proposition shows that equal increases in $A$ and $B$ will increase the $a_1$ attack proportion in any QRE, i.e., any $Q$ and $\lambda$:

**Proposition 1.** Under assumptions (i), (ii), and (iii), if the effectiveness parameters $A$ and $B$ increase by equal absolute amounts to $A + \gamma$ and $B + \gamma$, then the Nash equilibrium probability of attack $a_1$ is unchanged, but the QRE probability $\alpha$ of this attack method is increased. The QRE probability of defending against this attack $(1-\delta)$ increases. This result holds for any quantal response function, $Q$ and for any positive precision, $\lambda$.

**Proof.** The comparative statics prediction for the effect of equal absolute shifts in $A$ and $B$ by the same amount $\gamma$ can be derived by differentiating (3) and (4) with respect to $\gamma$ under the assumption that $dA/d\gamma = dB/d\gamma = 1$, so there is no effect on the $(B-A)$ terms in these equations. The result of this differentiation is a pair of equations in $d\alpha/d\gamma$ and $d\delta/d\gamma$:

\[
\begin{align*}
\frac{d\alpha}{d\gamma} &= \lambda Q'(\Delta) \frac{dA}{d\gamma} + \lambda Q'(\Delta) [C - D + B - A] \frac{d\delta}{d\gamma} \\
\frac{d\delta}{d\gamma} &= -\lambda Q'(\Delta) [C - D + B - A] \frac{d\alpha}{d\gamma}
\end{align*}
\]

Then substituting the $d\delta/d\gamma$ expression on the right into the first equation in (6) and using the fact that $dA/d\gamma = 1$, one obtains:

\[
\frac{d\alpha}{d\gamma} = \frac{\lambda Q'(\Delta)}{1 + Q'(\Delta) Q'(\Delta) \lambda^2 [C - D + B - A]^2} > 0 \quad \text{(equal absolute changes in } A \text{ and } B).\]

The ratio in (7) is positive since the $Q'(\Delta)$ derivatives are positive, even though the arguments of these functions (not shown) are different. This implies that an equal absolute increase in the $A$ and $B$ parameters will increase the QRE $a_1$ attack probability, and vice versa, for any quantal response function $Q$. The second equation in (6) then implies that $d\delta/d\gamma < 0$, so the probability of defending against attack $a_1$ (i.e., $1-\delta$) increases.

Instead of using the general non-parametric quantal response function $Q(\Delta)$ in (3) and (4), it is convenient and standard for estimation and experimental design to consider a parameterization of $Q$ based on a standard logit model, e.g. $Q(\Delta) = \frac{1}{1 + \exp(-\lambda \Delta)}$, where
λ again represents a positive precision parameter and Δ is a payoff difference. This distribution function has the desired property in (iii) and hence the probability of choosing the decision with the higher expected payoff is greater than 0.5. Moreover, the quantal response function limits to 0 as Δ→−∞ and to 1 as Δ→∞, in which case the logistic distribution function becomes a step function (with a step from 0 to 1 at Δ = 0). The logit function of the difference in expected payoffs between the two attack methods, \( \frac{1}{1+\exp(-\lambda[\pi_{a1}-\pi_{a2}])} \), can be used to express the quantal responses as ratios of exponential functions of precision-weighted expected payoffs: 

\[
\frac{\exp(\lambda\pi_{a1}(\delta))}{\exp(\lambda\pi_{a1}(\delta)) + \exp(\lambda\pi_{a2}(\delta))}
\]

for attack \( a_1 \), and analogous ratios of exponentials for each defense. It is useful to express expected payoffs in these ratios of exponentials in terms of the model parameters:

\[
\alpha = \frac{\exp(\lambda[ A + (B-A)\delta ])}{\exp(\lambda[ A + (B-A)\delta ]) + \exp(\lambda[ D + (C-D)(1-\delta)])}
\]

(logit attack \( a_1 \))

\[
\delta = \frac{\exp(\lambda[-D-(B-D)\alpha])}{\exp(\lambda[-D-(B-D)\alpha]) + \exp(\lambda[-C+(C-A)\alpha])}
\]

(logit defense \( d_2 \))

A simple interpretation of these Logit equations is that the log odds of choosing a particular action, e.g., \( \ln(\alpha/(1-\alpha)) \), is proportional to the expected payoff difference, where \( \lambda \) is the proportionality factor. The logit QRE for a given precision consists of values of \( \alpha \) and \( \delta \) that solve these equations, i.e., a fixed point in which the belief probabilities on the right-hand sides of (8) and (9) match the choice probabilities that emerge on the left-hand sides. These QRE probabilities will generally depend on all parameters, but for simplicity will be denoted by \( \alpha^*(\lambda) \) and \( \delta^*(\lambda) \). Proposition 1 implies that equal absolute increases in the \( a_1 \) attack effectiveness parameters \( A \) and \( B \) will increase the equilibrium \( a_1 \) probabilities for any given value of \( \lambda \). This property, which holds for any QRE, would necessarily hold for the logit form. Figure 4 plots the locus of logit QRE points for precision values that range from 0 on the left side to 2 on the right, using a log scale. The thick line locus was generated with values corresponding to the High treatment, with \( A = 6, B = 12, C = 7, \) and \( D = 4 \). The dotted line is for the Low treatment where \( A \) and \( B \) are
each reduced by 4, to 2 and 8. With 0 precision, the choice probabilities in (8) and (9) are 0.5, which is the point on the left axis of the figure where both lines originate.

![Graph showing Logit QRE Pass Proportions vs Logit Precision](image)

**Figure 4.** Locus of QRE $a_1$ Attack Frequencies as Precision Increases from Left to Right, with Attack Effectiveness Reduced in Equal Absolute Amounts from High (Solid) to LowAb (Dotted)

Notice that both lines in Figure 4 converge to the same Nash equilibrium mixed strategy of 0.33 as precision increases on the right side of the figure, as implied by the theory. In contrast to the fact that they both converge to the same point, the QRE correspondence exhibits the intuitive tendency for $a_1$ attack proportions to be higher, *for all values of $\lambda$* in the High treatment compared with the LowAb treatment. Furthermore, the convergence is from above for both treatments: i.e., QRE predicts $\alpha > .33$ for both treatments, regardless of the precision parameter. Although the logit form of the quantal response function was used to generate the curved correspondences in Figure 4, it is worth emphasizing that these properties are quite general in the sense that they will hold for any $Q$ satisfying the regularity condition (iii).

Figure 5 illustrates, in a slightly different way, the equilibrium effects of equal shifts in the attack effectiveness parameters $A$ and $B$. The figure is constructed with $\alpha$ (the $a_1$ attack rate) on the horizontal axis and $\delta$ (the $d_2$ defense rate) on the vertical axis.
For the LowAb parameters used \((A = 2, B = 8, C = 7, D = 4)\), column is indifferent between the two attack methods along the horizontal solid line at a height of \(\delta = 0.55\). If there is no noise, the resulting best response lines have sharp corners and intersect at \(\alpha = 0.33\) and \(\delta = 0.55\), as shown by the upper Nash equilibrium diamond at the intersection of the vertical and horizontal lines in the figure. With some noise (\(\lambda = 0.7\) used in the figure), the resulting curved solid and dashed quantal response curves intersect at a point a little lower and to the right, but still close to the Nash equilibrium.

Next consider the effect of equal absolute increases in \(A\) and \(B\), from 2 and 8 to 6 and 12, respectively. Notice that the quantal response function in (4) specifies \(\delta\) to be a decreasing function of \(\alpha\), which is invariant to equal absolute changes in \(A\) and \(B\), so the curved downward-sloping dark line in the figure does not shift. In contrast, (3) shows that \(\alpha\) is an increasing function of \(\delta\) (the curved upward-sloping dashed lines), which shifts to the right (higher \(\alpha\) for each given \(\delta\)) in response to equal absolute increases in \(A\) and \(B\). A rightward shift in this positively sloped function is akin to a shift in supply, which would move the QRE (supply/demand) intersection to the right, since the downward sloping function analogous to demand did not shift. The curved dotted line in Figure 5 shows the shift in column’s quantal response with balanced increases in \(A\) and \(B\) (to 6 and 12). Thus the QRE \(a_1\) attack probability \(\alpha\) would increase as a result of equal absolute increases in \(A\) and \(B\), and vice versa. Notice that this argument does not depend on the shape of the quantal response functions, only that they are monotone and take a value of 0.5 when the two actions have equal expected payoff. It is in this sense that the comparative static predictions are non-parametric.
The invariance of the Nash prediction is due to the fact that the vertical solid line at $\alpha = 0.33$ is not affected by equal increases in $A$ and $B$. Therefore a downward shift in the horizontal thin line from 0.56 to the horizontal dotted line at 0.11 would push the Nash equilibrium diamond down along the vertical line to the lower (dotted) diamond, leaving the Nash equilibrium value of $\alpha$ unchanged at 0.33.

**Equal Proportional Changes in Attack Effectiveness**

There is no a priori reason to expect that the attack effectiveness reductions in $A$ and $B$ would be equal against each defense. In fact, when the reductions are not equal, then Nash equilibrium can produce even more unintuitive predictions. That is the reason the experimental design included the treatment with proportional changes in $A$ and $B$, to $\gamma A$ and $\gamma B$, where $\gamma > 0$. We assume:

iv) $D/B < \gamma < C/A$,

which ensures that the inequality restrictions in (ii) are not violated.

With $\gamma > 1$, for example, the proportional increases in the $a_1$ attack effectiveness parameters $A$ and $B$ lead to an increase in the denominator of (2), so the Nash probability of this attack decreases, which is a clearly unintuitive prediction. As precision becomes
higher, the QRE equilibria will converge to Nash equilibria, so the QRE predictions would also inherit the unintuitive properties of Nash predictions, e.g. that a proportional increase in $a_1$ attack effectiveness would decrease the $a_1$ attack rate in the presence of high precision values. However, this is not the case for a wide range of plausible lower precisions, as can be seen in Figure 6, which compares the logit QRE correspondence for our LowProp treatment parameters, $A = 3$, $B = 6$, $C = 7$, $D = 4$, with the correspondence for the High treatment (corresponding to two values of $\gamma$, equal to 1 (LowProp) and 2 (High), respectively). The effects of equi-proportional changes in $A$ and $B$ are reversed as the precision increases above a level of about $\ln(1+\lambda)=1$. In particular, the “intuitive” pattern on the left side (with higher effectiveness resulting in higher $a_1$ attack proportions) is reversed on the right as dictated by the analogous reverse pattern in the Nash predictions of 0.33 for the High ($A = 6$, $B = 12$) treatment and 0.50 for the LowProp ($A = 3$, $B = 6$) treatment.

The effect of equal proportional changes in $A$ and $B$ shown for the example in Figure 6 is a general property of any QRE (not only logit):

**Proposition 2.** Under assumptions (i), (ii), and (iv), if the $a_1$ attack effectiveness payoffs, $A$ and $B$, increase by equal proportional amounts (to $\gamma A$ and $\gamma B$, where $\gamma > 1$), then for any quantal response function $Q$ satisfying (iii), there exists a positive precision parameter $\lambda_0$ such that for $0 < \lambda < \lambda_0$, such that the QRE $a_1$ attack rate with high effectiveness is higher, while the analogous Nash attack rate is lower, $p^*(\gamma) < p^*(\gamma')$. Furthermore, the frequency of defending against $a_1$ increases in $\gamma$ for all values of $\lambda$. 

Figure 6. QRE Locus Predicted $a_1$ Attack Proportions as Precision Increases from Left to Right, with Attack Effectiveness Reduced in Equal Proportions from High (Solid Line) to Low (Dotted)

**Proof.** Replace $A$ and $B$ with $\gamma A$ and $\gamma B$, respectively, in the quantal response functions for the attack and defense rates, $\alpha$ and $\delta$. Denoting $x = C-D+\gamma(B-A)$, the quantal response functions in (3) and (4) are:

$$\alpha = Q(\lambda[\gamma A - C + x\delta]) \quad \text{and} \quad \delta = Q(\lambda[C - D - x\alpha])$$

Since the derivative of $x$ with respect to the proportionality parameter $\gamma$ is $B-A$, the total derivative of the QRE conditions with respect to $\gamma$ are:

$$\alpha' = \lambda Q'(\cdot)[A + \delta(B - A) + x\delta'] \quad \text{and} \quad \delta' = -\lambda Q'(\cdot)[\alpha(B - A) + x\alpha']$$

where the arguments of the $Q'(\cdot)$ functions are suppressed. These 2 equations can be solved by substitution to determine $\alpha'$ and $\delta'$. In particular, the comparative statics effect of the scale parameter $\gamma$ on the defense rate $\delta$ is:

$$\delta' = \frac{-\lambda Q'(\cdot)[\alpha(B - A) + x\lambda Q'(\cdot)[A + \delta(B - A)]]}{1 + \lambda^2 Q'(\cdot)Q'(\cdot)x^2}$$

Even though the arguments of the $Q'$ functions are omitted, these derivatives of quantal response functions are always positive, and it follows that the $\delta'$ comparative statics derivative is negative, as stated in the proposition. The comparative statics condition of
the proportionality parameter $\gamma$ on the attack rate $\alpha$ is $\alpha' = \lambda Q'(\cdot)[A + \delta(B - A) + \chi \delta']$. The first term, $\lambda Q'(\cdot)$ is strictly positive for all $\lambda > 0$ and the second term, $A + \delta(B - A) - \chi \delta'$, is also positive for sufficiently small $\lambda$ because $\delta'$ vanishes. Hence, the probability the attacker chooses $a_1$ increases for sufficiently low precision. It is easily verified that the comparative static in the Nash equilibrium goes in the opposite direction.$\blacksquare$

The QRE has empirical content in the sense that the effects of both additive and proportional changes in $a_1$ attack effectiveness derived in the proofs of Propositions 1 and 2 hold for any quantal response function, not just a logit form. Moreover, comparisons of multiple treatments that vary the intensity of the effectiveness parameters offer clear and falsifiable ranking predictions. In the case of additive changes, these rankings are independent of the precision used. For the proportional changes in Proposition 2, the intuitive rank order for relatively low precision values would be inconsistent with ranked Nash predictions. Moreover, the vertical ranges spanned by the curves in the figures show that there are regions of high or low attack proportions that cannot be explained by any quantal response function, even if the researcher has full freedom to specify $\lambda$. The empirical content of QRE in this context stems from the standard properties of quantal response functions in (iii). In particular, $Q$ is a continuous increasing function of expected payoff differences, which implies that $Q(0) = \frac{1}{2}$, i.e. that the choice probability is $\frac{1}{2}$ when expected payoffs for two decisions are equal. This is satisfied by the logit QRE in (8) and (9) and by any other formulation that is based on i.i.d., e.g. probit (Gaussian disturbances). A separate issue, which is relevant to the present exercise, is that overfitting can be a problem if one allows the precision parameter to vary across similar games. For this reason, in the analysis of data that follows, we restrict the precision parameter to be the same across all the treatments.
VI. Estimation and Model Comparison of QRE and Nash

In this section we propose and estimate two different statistical methods to compare the QRE and Nash equilibrium models of choice behavior in this class of games. The first approach is the standard method for estimating the logit responsiveness parameter, $\lambda$, from data generated by controlled laboratory experiments of finite games. Every non-negative value of $\lambda$ implies a profile of six QRE choice frequencies for the row and column player in the three treatments. Because each such QRE profile is uniquely associated a single value of $\lambda$ in this class of games, it is straightforward to obtain a maximum likelihood estimate of $\lambda$ given the data (Goeree et al. 2016, Ch. 6).

The second approach is less standard in the analysis of experimental data and more commonly used in structural estimation of game theoretic models in empirical industrial organization studies based on historical data or on observations in the field rather than the laboratory. Such data sets typically do not have direct measurements of the key structural parameters of the game theoretic model the econometrician is using as the theoretical foundation for the empirical study. Thus, the econometric analysis of the data necessarily requires simultaneous estimation of those key parameters as well as any nuisance parameters of the model.

This second approach is equally well-suited for the structural estimation of data generated by laboratory experiments, and is often used to estimate parameters of the model that are impossible to control for directly in the experimental design. However, the method can also be used to estimate game parameters that are known and directly controlled in the experiment simply by concealing those parameters and estimating them as if they were unknown. We use this method by concealing the two game payoff parameters ($C$ and $D$) that are constant across our treatments and then compare how well QRE and Nash recover the revealed parameters.

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16 For example, the theoretical model might include preference parameters such as risk aversion coefficients or social preference parameters. See Goeree et al. (2002, 2003).

17 See Bajari and Hortaçsu (2005) and Merlo and Palfrey (2018).
VI. 1. Logit QRE Estimation

The previous analysis was based on qualitative data comparisons with patterns predicted by a Nash equilibrium or with general QRE comparative static predictions that hold non-parametrically. In this section, we report estimation results based on the parametric logit specification. The estimated value of $\lambda$ was constrained to be the same for all treatments: Using the logit quantal response functions (8) and (9), the equilibrium values of the attack and defense rates $\alpha$ and $\delta$ for each treatment were calculated for all possible values of $\lambda$. Then we evaluated the likelihood function for each $\lambda$ and determined that it is maximized at $\lambda = 0.70$.18

The maximum likelihood estimate of $\lambda$ was, in fact, used to construct the quantal response lines shown earlier in Figure 5, for the LowAb and HighAb treatments. The fitted QRE choice predictions are at the intersections of the decreasing defense quantal response curve with the solid and dotted attack (increasing) quantal response curves in Figure 5. The data averages for each treatment are superimposed as black dots. The QRE predictions track the data closely in this equal absolute change treatment for both the defender and the attacker, including when it is near the Nash prediction in the Low condition as well as when $a_1$ attack frequencies increase sharply in the HighAb condition. This QRE treatment effect prediction contrasts sharply with the Nash attack rate invariance.

Similarly, the estimated precision (for all treatments) was used to generate the logit quantal response functions that are depicted in Figure 7 as the curved solid lines for the Low treatment and the curved dotted lines for the High treatment, with QRE at the respective intersections of these lines. As before, the large dark dots indicate data averages for the two treatments. The QRE and Nash predictions are both close to the data for the LowProp treatment (marked with a solid dot in the upper part of the figure). Both QRE and Nash equilibrium are consistent with the sharp reduction in the $d_2$ frequency in the High treatment, but QRE correctly accounts for the most salient feature of the data,

18 Since there is twice as much data for the $A=6, B=12$ treatments, we divided the data counts and number of observations for this treatment by 2 in the likelihood function to avoid over-weighting this treatment.
the increase in $a_1$ attack frequency, the opposite of the Nash prediction of a decrease, as shown by the large data dot near the intersection of the dotted quantal response lines.

![Graphical representation](image)

**Figure 7.** Effects of Equal Proportional Increases in $A$ and $B$

The graphical representation in Figure 7 shows the intuition behind the nonparametric QRE predictions for changes in defender behavior, which were characterized in the previous section. In particular, consider the effect of an equal proportional increase in both $A$ and $B$, e.g. a doubling of $a_1$ effectiveness parameter. It is apparent from the defender’s quantal response function in (4) that proportionate increases in both $A$ and $B$ will increase the $B-A$ difference on the right, which reduces $\delta$ for any given level of $\alpha$. This explains the downward shift of the “defense” quantal response function from the dark downward-sloping solid curve to the dark dotted curve. The rightward shift from the upward-sloping light solid curve to the light dotted quantal response function for the $a_1$ attack rate $\alpha$ follows from an analysis in Proposition 2 of the attack quantal response function (3). The combined effect of both of these shifts is to reduce the QRE $d_2$ defense rate, $\delta$, in an analogous way that a reduction in demand and

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19 The estimated value of $\lambda = 0.7$ is sufficiently small that the comparative statics in Proposition 2 hold.
an increase in supply would lower the equilibrium price in a market. The net effect of
these shifts is to generate a clear increase in the QRE predicted $a_1$ attack rate, as observed
in the data. In contrast, the Nash predictions for this treatment change are indicated by
the switch from the solid, straight-line best responses in Figure 7 to the dotted-line best
responses, with the post-shift Nash equilibrium indicated by the dotted diamond shape at
a decreased attack rate, 0.33.

The main message of this subsection is that incorporation of a single estimated
precision parameter results in a set of logit QRE predictions that do a much better job of
tracking the overall pattern of data averages across treatments, with an intuitive backstory
based on shifts in quantal response lines. Nevertheless, it is useful to look at similarities
and divergences in more detail. Table 4 presents a numerical comparison of the treatment
data averages with the theoretical predictions. The data points in the top row, in bold, are
listed as ($\alpha$, $\delta$) pairs for $a_1$ and $d_2$ choice rates. The middle and bottom rows show
analogous QRE and Nash point predictions by treatment. For the equal absolute change
treatments on the left side of the table, the fitted logit QRE predictions in the middle row
imply that the $a_1$ attack frequency should increase from 0.36 in the LowAb treatment ($A = 2, B = 8$) to 0.59 in the HighAb treatment ($A = 6, B = 12$). This strong treatment effect
in attack rates is inconsistent with the Nash prediction of no effect, i.e. an unchanged
attack rate of 0.33 in the bottom row, left side. A similar pattern can be observed for
attack rates in response to equal proportional increases in attack effectiveness (from
LowProp to HighProp) on the right side of Table 4. This treatment change increased the
$a_1$ attack frequency observed in the data from 0.49 to 0.63 (top row, right side), a change
that was reasonably well predicted by logit QRE, from a predicted attack frequency
increase from 0.41 to 0.59, in stark contrast to the Nash prediction (bottom row, right
side) of a decline from 0.50 to 0.33.
Table 4. Data, Nash, and Logit QRE: \((a_1 \text{ Attack Rate}, d_2 \text{ Defense Rate})\)

<table>
<thead>
<tr>
<th>Treatment Condition:</th>
<th>Equal Absolute Change</th>
<th>Equal Proportional Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LowAb</td>
<td>HighAb</td>
</tr>
<tr>
<td>Data Avg.:</td>
<td>(0.33, 0.41)</td>
<td>(0.59, 0.19)</td>
</tr>
<tr>
<td>Logit QRE for (\lambda = 0.7):</td>
<td>(0.36, 0.46)</td>
<td>(0.59, 0.17)</td>
</tr>
<tr>
<td>Nash Equilibrium:</td>
<td>(0.33, 0.56)</td>
<td>(0.33, 0.11)</td>
</tr>
</tbody>
</table>

Turning next to the defense behavior, the Nash equilibrium over-predicts the reduction in \(d_2\) frequencies with equal absolute changes, which fell from 0.41 to 0.19. This observed \(d_2\) defense frequency of 0.19 after the treatment change (top row, left side) is close to and not significantly different from the logit QRE prediction of 0.17 (middle row). But the observed 0.19 defense frequency rate in the HighAb treatment is somewhat greater than the 0.11 Nash prediction of 0.11 shown below it in the bottom row, left side, although this latter difference is not significant either. As was the case with equal absolute changes in \(A\) and \(B\), there is a sharp reduction in \(d_2\) rates from an equal proportional increase in \(A\) and \(B\), from 0.66 to 0.17. This observed reduction in \(d_2\) defense rates is also over-predicted by Nash, which predicts a reduction to 0.11, but not by QRE, which predicts the reduction to 0.17 that was observed. As before, neither of these differences between observed and either Nash or QRE predictions are statistically significant.

VI. 2. Concealed Payoff Parameter Estimation

The method used above for QRE estimation and comparison with Nash equilibrium was only possible because, in a laboratory setting, the payoff parameters of

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20 Even though statistical tests of attack-effectiveness treatment effects are provided in the support for Results 1 and 2, we include several more tests here for descriptive purposes. First consider whether the observed post-treatment defense rate of \(\delta = 0.19\) is different from the QRE prediction of a 0.17 in the HighAb treatment. The far right column of the first page of the Data Appendix can be used to document that 11 of the 24 subjects in defender roles had \(d_2\) defense rates above QRE prediction (0.17), and the other 13 had lower defense rates. This difference is not significant at standard levels with a binomial test. For comparison, 15 the 24 defenders had defense rates above the Nash prediction of 0.11, with 9 defense rates below, which is again not significant.
the games for which the data is generated are known, fixed numbers. This contrasts with structural estimation methods used in applied field settings, for example in Industrial Organization, where payoff parameters are unknown to the econometrician and need to be estimated using the error structure of the model.

In this subsection, we follow this traditional field-data approach by ignoring the fact that the payoff parameters in the experiment are known, and instead blinding ourselves to those parameters that are common to all treatments. Then we directly estimate those payoff parameters for both QRE and Nash equilibrium models. Since the true values of the estimated parameters are known with certainty, we can then conduct a simple statistical test, for each of the two models, to see whether the concealed parameters are accurately recovered by the estimation.

There are two parameters that we fixed throughout the four treatments of the experiment: \( C=7 \) and \( D=4 \). We use maximum likelihood to estimate these two parameters using the QRE model with \( \lambda \) fixed at 0.7, yielding \( \hat{C}_{QRE} \) and \( \hat{D}_{QRE} \). For comparison, we also use maximum likelihood to estimate these two parameters using the Nash mixed strategy equilibrium model, yielding \( \hat{C}_{Nash} \) and \( \hat{D}_{Nash} \). We then conduct a likelihood ratio test for each of the two models, QRE and Nash, with \( C=7 \) and \( D=4 \) as the null hypothesis. The other parameters in the game, \( A \) and \( B \), vary across the three treatments, and are not estimated. Both the QRE and the Nash models are identified because we have three different \( (A, B) \) pairs in our treatment, each of which generates different QRE and Nash equilibrium strategy profiles for all values of \( C \) and \( D \) that satisfy inequalities (i) and (ii) in section II.21

The results of the estimation are given in Table 5. We conduct a likelihood ratio test for both the QRE and the Nash models, where the test statistic is twice the likelihood ratio and is chi-square distributed with two degrees of freedom. For the QRE model, we obtain an estimate of \( \hat{C} = 7.1, \hat{D} = 3.7 \), which are not significantly different from \( (7, 4) \).

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21 The estimates of \( C \) and \( D \) for both models lie in the interior of the region specified by the inequalities. If these inequalities are not satisfied, then there is a pure strategy Nash equilibrium, which would lead to zero likelihood under the Nash model.
at the 5% significance level ($\chi^2=4.602$). For the Nash model, we obtain an estimate of $\hat{C} = 7.6, \hat{D} = 3.1$, and the test statistic ($\chi^2=69.94$) is sufficiently high to strongly reject the null of (7, 4) at the 1% level ($p<0.001$). The bottom line is that the QRE model successfully recovers the concealed parameters, $C$ and $D$, while the Nash model does not.

Table 5. Concealed Parameter Estimation ($a_1$ Attack Rate, $d_2$ Defense Rate)

<table>
<thead>
<tr>
<th>Treatment</th>
<th>LowAb</th>
<th>High</th>
<th>LowProp</th>
<th>−logLikelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Attack and Defense Data:</strong></td>
<td>(0.33, 0.41)</td>
<td>(0.61, 0.18)</td>
<td>(0.49, 0.66)</td>
<td></td>
</tr>
<tr>
<td><strong>QRE estimates:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>with $C = 7, D = 4$</td>
<td>(0.36, 0.46)</td>
<td>(0.59, 0.17)</td>
<td>(0.41, 0.58)</td>
<td>1832.675</td>
</tr>
<tr>
<td>with $\hat{C} = 7.1, \hat{D} = 3.7$</td>
<td>(0.38, 0.47)</td>
<td>(0.60, 0.18)</td>
<td>(0.45, 0.59)</td>
<td>1830.374</td>
</tr>
<tr>
<td><strong>Nash estimates:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>with $C = 7, D = 4$</td>
<td>(0.33, 0.56)</td>
<td>(0.33, 0.11)</td>
<td>(0.50, 0.67)</td>
<td>1924.214</td>
</tr>
<tr>
<td>with $\hat{C} = 7.6, \hat{D} = 3.1$</td>
<td>(0.43, 0.53)</td>
<td>(0.43, 0.15)</td>
<td>(0.60, 0.61)</td>
<td>1889.242</td>
</tr>
</tbody>
</table>

The difference between the performances of the two models in recovering the payoff parameters is also reflected in the difference between the constrained and unconstrained estimates of the mixed strategy attack and defend frequencies in the three treatments. The average difference in the estimate for the attack rate $\alpha$ is 0.10 for the Nash model and 0.02 for the QRE model, while the average difference in the defense rate $\delta$ estimate is 0.04 for the Nash model and 0.01 for the QRE model.

VII. Conclusions

This paper presented the results of an experiment that was motivated in three distinct ways. First, the types of games being investigated are based on simple models of binary conflict between two opposing interests, a classic strategic problem that arises in wide range of different competitive settings and applications. Besides this substantive interest, there are theoretical and behavioral motivations as well. On the behavioral side, the standard theoretical approach to studying these games, Nash equilibrium, provides
non-intuitive predictions about the comparative statics of how behavior responds to changes in underlying parameters of the game, such as the relative strength of one of the attacker’s strategic options. Intuitively, when the strength or effectiveness of one option increases relative to the other, one would expect that option to be chosen more frequently, but Nash equilibrium often predicts either no change at all or change in the opposite direction in some cases. The experiment indeed finds that the intuitive change in behavior is observed, sharply contradicting Nash prediction. Third, from a theoretical perspective, the experiment is motivated by the fact that QRE theory makes predictions that are quite different from the Nash comparative statics. Moreover, QRE theoretical predictions based on regular quantal response functions or i.i.d. error disturbances have strong non-parametric empirical content, and hence the analysis of these games (both theoretical and empirical) is not subject to critique that QRE has no empirical content, which is based entirely on a technical possibility of irregular quantal response functions that are never used in QRE analysis, either in this paper or any other application of QRE to date. The observed data are found to follow the predicted cross-treatment comparative statics of QRE theory, even when these comparative static predictions are the opposite of Nash equilibrium.

The laboratory experiment reported in this paper implements the two distinct changes in the effectiveness of one of the attacker’s two strategies, equal absolute changes in effectiveness or proportional changes in effectiveness. The first case considered is the balanced change in effectiveness, e.g. where an increase in effectiveness of one of the attacker’s strategies is the same regardless of the defense encountered. In this case, Nash equilibrium predicts no change of attacker behavior. The second case is based on effect of a proportional change in the effectiveness of one of the attacker’s strategies, where the Nash prediction that an equal proportional reduction in effectiveness will (counterintuitively) result in the attacker choosing to use that strategy more often. Both of these predictions are sharply contradicted by the observed reduction in $a_1$ attack rates following a switch from the high effectiveness to low effectiveness for that action. These observations are proved to be generally consistent with regular QRE that injects noise.
into decision making in a manner that smooths out the sharp best response functions
derived from the Nash assumption of perfect rationality. The mechanism behind the more
intuitive QRE predictions is based on a supply-demand intuition in which the decrease in
the strategy’s effectiveness leads to a rightward shift (increase) in the supply of that action
for any given defense probability, as illustrated in figures 5 and 7. The qualitative
directional nature of the QRE predictions is general and does not depend on specific
functional forms (logit, probit, etc.), but estimation based on a logit probabilistic choice
function is used to derive more precise predictions.

It is enlightening to compare our findings to the McGarrity and Lemon (2010)
analysis of the pass or run decisions taken by National Football League teams. The focus
was on instances where the first-string quarterback was injured, which allows a pairwise
comparison of pass rates before and after being replaced by the second-string quarterback.
The motivation is that the second-string quarterback is likely to be less effective as a
passer against either a pass or a run defense, although direct evidence for this conjecture
was not provided. But if the injury does result in diminished pass effectiveness, the
defense should be more prone to defend against a run, which might, in turn, negate any
tendency for the offense to pass less. The authors formulate a model in which the Nash
equilibrium prediction is that the equilibrium pass rate will not change after a reduction
in pass effectiveness, i.e. the defensive response fully negates any tendency for the
offense to pass less. At an intuitive level, it seems surprising that the strategic defensive
reaction to a quarterback injury would be so complete that pass rates would not change
after a quarterback injury. The authors found some support for this unintuitive prediction
of no change by looking at overall pass rates for each team with injury-induced
quarterback replacements. However, the actual strategic setting faced by a team is clearly
more complicated than that in a simple game with binary decisions; the incentive to pass
may depend on many factors, e.g. “third down and long” or large deficits or advantages
in the total score, and game time remaining. Therefore, the authors based their main
arguments on the insignificance of a first-string quarterback dummy variable in reduced-
form pass rate regressions that controlled for “down” and several other factors. The
interpretation of this observation of no significant difference as supporting a Nash equilibrium, however, is clouded by its dependence on an untested assumption that the decline in pass effectiveness is balanced, i.e. the same against either defense.

As it turns out, the key assumption that is implicit in their theoretical model is that the change in pass effectiveness is the same against either defense, i.e. against a pass or against a run. This assumption ensures that the strategic incentives to reduce the probability of passing are exactly offset by the incentives by the opponent to defend more against the run – which is a knife-edged property in two distinct ways. The first way the assumption is special – and the main focus of this paper – is that it relies on perfect rationality of the players. That is, with equal absolute reductions in pass efficiency, in a Nash equilibrium the defend-run probability increases exactly to the point such that the pass probability will remain unchanged. But evidence from previous experiments shows that for a wide range of games, including two-player games with unique mixed strategies, sharp unintuitive predictions of Nash equilibrium are highly suspect. QRE relaxes the perfect rationality assumption of Nash equilibrium with an assumption of soft optimization, without jettisoning the basic concept of equilibrium. Players are assumed to have correct expectations on average, but do not choose best replies with 100% accuracy. Rather, it is assumed that they choose better strategies more often than worse strategies, in a continuous way. The equilibrium notion of QRE incorporates the idea that players in the game not only make occasional mistakes themselves, but also take into account that their opponent can also fail to optimize.

For the pass-run interpretation of the game considered in this paper, the curvature of quantal responses for each player causes the QRE pass ($a_1$) probability to decline in response to a balanced decline in pass effectiveness. In fact, the comparative static effects of changes in pass effectiveness are essentially always non-zero and quite intuitive. An even more striking difference between the Nash equilibrium comparative static and the QRE comparative static in these games is exhibited by a proportional decrease in pass effectiveness. That is, the decline in payoff from passing against a pass defense is proportional rather than equal to the decline in payoff from passing against a run defense.
In this case, Nash equilibrium predicts the decrease in pass effectiveness to cause an increase in passing, while QRE still predicts a decrease in passing rates, unless the players are highly rational.

Equilibrium theories (like QRE and Nash) are not only valued for the generation of comparative statics predictions associated with exogenous factors. Such theories are also used with naturally occurring data to estimate preference and payoff parameters of interest. In particular, the Nash equilibrium has widely been used by industrial organization economists to provide a structural framework for estimation, often with data from auctions. As a check, some econometricians have had some success in applying the same techniques to data generated from laboratory experiments in settings where a key structural parameter, experimentally controlled and known with certainty to the experimenter and the subjects, is treated as unobserved and estimated from the observed choices in the experiment. This concealed parameter recovery method was used in the previous section to estimate the two payoff parameters that stayed fixed across the various treatments in the attacker-defender game. In particular, estimation based on quantal response equilibrium provided accurate parameter recovery while estimation based on the Nash equilibrium model failed.

We close with two cautionary notes. First, QRE is an equilibrium theory, and as such may be difficult for players to achieve in more complex games or without sufficient opportunities to learn. Behavior in one-shot games with no feedback will likely be more influenced by framing, focal points, and rules of thumb borrowed from experience. With repeated interactions, the presence of significant trends in data averages would suggest that learning is taking place, i.e. expectations are adjusting to observed choices made by others. Unless the equilibrium is unstable, learning dynamics have been found to converge toward equilibrium across a broad range of settings, e.g., in markets (Smith,

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22 The first paper to exploit this approach was Bajari and Hortaçsu (2005), who used the Nash equilibrium bidding functions to estimate the (known) distribution used to generate private values in laboratory auctions. More recently, Merlo and Palfrey (2018) used a QRE framework to estimate the (known) distribution of voting costs in a voting experiment to test the validity of several competing behavioral models of voter turnout.
coordination games (Van Huyck et al, 1990), and beauty contest games (Nagel, 1995).

Second, the noisy behavior incorporated into quantal response is not the whole story, especially if the game structures tend to trigger concerns about risk aversion, loss aversion, fairness, or other factors that have been identified in the behavioral game theory literature. These types of behavioral biases can also drive outcomes away from Nash equilibrium. Our experiment tries to neutralize the effects of these factors, by adding a fixed amount to eliminate losses and by ensuring that all actions involve significant risk. Another factor that can generate sharp deviations from Nash predictions is the presence of payoff inequities and players’ aversions to such inequities. Such factors probably have little effect in games of direct conflict, especially constant sum games used here, since it is not possible to associate positive or negative intentions with others’ actions in these games. Nevertheless, in games where these other behavioral factors and biases are important, QRE provides a natural structural framework the estimation of such factors.23

References


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23 Goeree, Holt and Palfrey (2016, Chapter 6) discuss methods for using QRE as a structural framework for the estimation of altruism, risk aversion, and other behavioral parameters. For example, see Goeree, Holt, and Palfrey (2003) and Holt, Sahu, and Smith (2020) for a case where risk aversion helped explain subjects’ choices in games in which one strategy is inherently more risky than another, e.g. a switch from a high-alert to a low-alert defense posture.


## Average Attack and Defense Rates for Equal Absolute Increases in Attack Effectiveness

<table>
<thead>
<tr>
<th>Session</th>
<th>First Treatment</th>
<th>Second Treatment</th>
<th>Attacker ID</th>
<th>Decision Rates $A = 2, B = 8$</th>
<th>Decision Rates $A = 6, B = 12$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>7</td>
<td>$a_1$ rate: $0.2$, $d_2$ rate: $0.7$</td>
<td>$a_1$ rate: $0.25$, $d_2$ rate: $0.75$</td>
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<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>8</td>
<td>$a_1$ rate: $0.15$, $d_2$ rate: $0.6$</td>
<td>$a_1$ rate: $0.7$, $d_2$ rate: $0$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>9</td>
<td>$a_1$ rate: $0.55$, $d_2$ rate: $0.45$</td>
<td>$a_1$ rate: $0.25$, $d_2$ rate: $0.1$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>10</td>
<td>$a_1$ rate: $0.1$, $d_2$ rate: $0.65$</td>
<td>$a_1$ rate: $0.1$, $d_2$ rate: $0.75$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>11</td>
<td>$a_1$ rate: $0.3$, $d_2$ rate: $0.2$</td>
<td>$a_1$ rate: $0.45$, $d_2$ rate: $0.3$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>12</td>
<td>$a_1$ rate: $0$, $d_2$ rate: $0$</td>
<td>$a_1$ rate: $0.45$, $d_2$ rate: $0.2$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>7</td>
<td>$a_1$ rate: $0.45$, $d_2$ rate: $0.5$</td>
<td>$a_1$ rate: $0.6$, $d_2$ rate: $0.3$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>8</td>
<td>$a_1$ rate: $0.35$, $d_2$ rate: $0.55$</td>
<td>$a_1$ rate: $0.95$, $d_2$ rate: $0$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>9</td>
<td>$a_1$ rate: $0.4$, $d_2$ rate: $0.45$</td>
<td>$a_1$ rate: $0.45$, $d_2$ rate: $0.15$</td>
</tr>
<tr>
<td></td>
<td>$A = 2, B = 8$</td>
<td>$A = 6, B = 12$</td>
<td>10</td>
<td>$a_1$ rate: $0.55$, $d_2$ rate: $0.6$</td>
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Average Attack and Defense Rates for Equal Proportional Increases in Attack Effectiveness

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<th>Session</th>
<th>First Treatment</th>
<th>Second Treatment</th>
<th>Attacker ID</th>
<th>Decision Rates $A = 3, B = 6$</th>
<th>Decision Rates $A = 6, B = 12$</th>
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<td>$a_1$ rate</td>
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<td>prqu4</td>
<td>$A = 3, B = 6$</td>
<td>$A = 6, B = 12$</td>
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<td>$A = 6, B = 12$</td>
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<td>0.55</td>
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<td>prqu4</td>
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<td>$A = 6, B = 12$</td>
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<td>0.45</td>
<td>0.85</td>
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<td>$A = 6, B = 12$</td>
<td>7</td>
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<td>0.85</td>
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<td>0.55</td>
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<td>0.75</td>
<td>0.85</td>
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<td>$A = 6, B = 12$</td>
<td>$A = 3, B = 6$</td>
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<td>$A = 6, B = 12$</td>
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<td>prqu5</td>
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<td>$A = 3, B = 6$</td>
<td>12</td>
<td>0.4</td>
<td>0.75</td>
</tr>
</tbody>
</table>
Rounds and Matchings: The experiment consists of a number of rounds. Note: You will be matched with the same person in all rounds.

Interdependence: Your earnings are determined by the decisions that you and the other person make.

Roles: In each pair of people, one person will be designated as the "row" player and the other will be the "column" player. You will be a column player (or) row player in all rounds.

Continue with Instructions

The column player will press either the Left or the Right button. The row player will choose Top or Bottom. These choices determine which part of the matrix is relevant (Top Left, Top Right, Bottom Left, Bottom Right). In each cell, the row player's payoff is shown in blue and the column player's payoff is shown in red.

Payoff Matrix (Row, Column)

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>$0.80, $0.40</td>
<td>$0.40, $0.80</td>
</tr>
<tr>
<td>Bottom</td>
<td>$0.50, $0.70</td>
<td>$1.00, $0.20</td>
</tr>
</tbody>
</table>

If you are a row player, your decision buttons will be on the left side of the payoff table, and if you are a column player, your decision buttons will be above the table.

Continue

Matchings: Please remember that you will be matched with the same person in all rounds.

Earnings: Your earnings are determined by the choices that you and the other person make in the round. You begin with a fixed payment of $0, and earnings will be added to this amount (losses, if the game has negative payoffs, will be subtracted). Your total earnings will be displayed in a cumulative earnings column on the page that follows.

Rounds: There will be 20 rounds in this part of the experiment, and you are matched with the same person in all rounds.