Physics 1a, Fall 2011
Quiz 4 Solutions

Problem 1 (5 Points)

(a) (1 Point)
The moment of inertia for the ring about the perpendicular axis through its center is MR^2. By the parallel-axis theorem, its moment of inertia about the pivot point P is

$$I_r = MR^2 + MR^2 = 2MR^2. \tag{1}$$

Then, the angular momentum about P is

$$L_0 = 2MR^2\omega \tag{2}$$

for just the ring (ignoring the bug).

(b) (2 Points)
Initially, the angular momentum of the ring plus the bug is zero about P. Angular momentum is conserved while the bug is moving, since there are no external torques on the system. We will associate positive angular momentum with clockwise angular velocity.

When the bug reaches X and the ring is rotating about P with angular speed ω_1, the speed of the bug with respect to the table is $v - 2R\omega_1$. The angular momentum of the bug about P is

$$L_b = -2mR(v - 2R\omega_1),$$

and from equation (2), the ring has angular momentum

$$L_r = 2MR^2\omega_1.$$

Then, the total angular momentum is

$$L = 0 = 2MR^2\omega_1 - 2mR(v - 2R\omega_1).$$

Solving for ω_1 yields

$$\omega_1 = \frac{mv}{R(M + 2m)}.$$

We could have also determined the total angular momentum by using the total moment of inertia about P. Equation (1) gives the moment of inertia for the ring, and the moment of inertia about P for the bug at X is

$$I_b = m(2R)^2 = 4mR^2.$$

The total moment of inertia is

$$I = I_r + I_b = 2R^2(M + 2m),$$

so the total angular momentum is

$$L = 0 = 2R^2(M + 2m)\omega_1 - 2mRv,$$

where the last term is to account for the angular momentum from the bug's motion on the ring. Both methods of finding L are equally good; it is simply a matter of record keeping.

(c) (1 Point)
The speed of the bug with respect to the table as it passes over X is

$$V = v - 2R\omega_1 = v - 2R\frac{mv}{R(M + 2m)} = \frac{Mv}{M + 2m}.$$

(d) (1 Point)
The angular velocity about P is zero when the bug reaches P again. The situation is the same as when the bug started at P and the ring was not rotating.

Problem 2 (5 Points)

(a) (1 Point)
Let us first view the problem in the lab frame, as originally drawn in figure 1. Since particle **C** elastically collides with **A**, kinetic energy is conserved immediately after the collision:

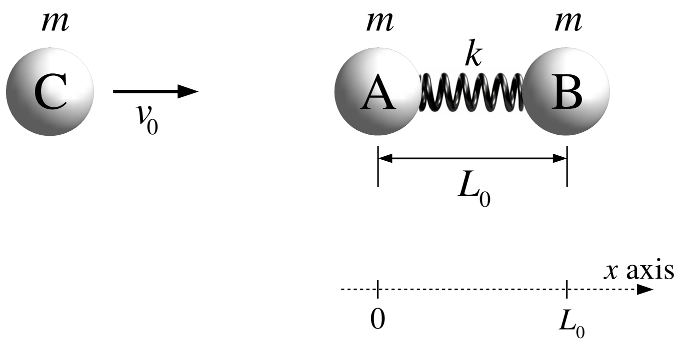

Figure 1: All particles in the lab frame at $t < 0$.

$$\begin{aligned} \text{KE}_i &= \text{KE}_f \\ \frac{1}{2}mv_0^2 &= \frac{1}{2}m\left(v_C^2 + v_A^2\right) \\ \Rightarrow \quad v_0^2 &= v_C^2 + v_A^2. \end{aligned}$$

We also have the conservation of momentum

$$\begin{aligned} \vec{p}_i &= \vec{p}_f \\ m\vec{v}_0 &= m\vec{v}_A + m\vec{v}_C \\ \Rightarrow \quad v_0^2 &= v_A^2 + v_C^2 + 2\vec{v}_A \cdot \vec{v}_C, \end{aligned}$$

but since we are working in only one dimension, $\vec{v}_A \cdot \vec{v}_C = \pm v_A v_C$. The only way to reconcile these conservation equations is to conclude that $v_C = 0$: **C** imparts all of its kinetic energy to **A**. The other possibility is that $v_A = 0$, but this answer represents the non-physical solution of **C** passing through **A** with no collision occurring. Note that **B** has no part in this interaction, so it remains stationary immediately after the collision.

The center-of-mass (CM) velocity of the **A-B** system is

$$\vec{v}_{CM} = \frac{m\vec{v}_A + m\vec{v}_B}{m+m} = \frac{\vec{v}_0}{2}.$$

In summary:

$$\begin{aligned} v_A &= v_0 \\ v_B &= 0 \\ v_C &= 0 \\ v_{CM} &= \frac{v_0}{2} \end{aligned}$$

immediately after the collision at $t = 0$.

(b) (2 Points)
After the collision between **A** and **C**, the **A-B** system does not encounter **C** again (by assumption) and does not experience any external forces. Hence, the CM velocity $v_{CM} = v_0/2$ is constant for $t \geq 0$. The CM is initially located at $L_0/2$ at the center of the spring. The CM position in the lab frame is then

$$x_{CM}(t) = \frac{L_0}{2} + \frac{v_0}{2}t. \tag{3}$$

Let us now work in the CM frame and consider only the relative motion of particles **A** and **B**. The CM is stationary in the CM frame, and the total momentum of the system is zero. Since all the particle masses m are identical, the CM frame has **A** and **B** moving with equal and opposite velocity, as shown in figure 2. Variables in the CM frame will be denoted with primes.

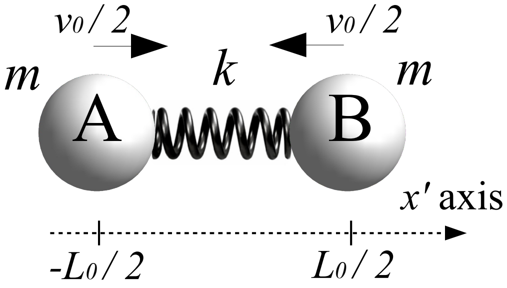

Figure 2: **A-B** system in the CM frame at $t = 0$.

Imagine cutting the spring in half and attaching the cut ends to a thin, stationary wall at $x' = 0$. The particles will undergo the exact same oscillatory motion as the original system in the CM frame. Note that the particles are each on half-springs of spring constant $2k$. See figure 3.

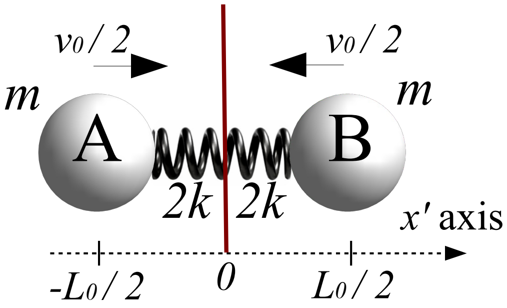

Figure 3: **A-B** system cut in half and separated by a thin wall (in red) at $t = 0$.

At $t = 0$, **A** and **B** are in their equilibrium positions

$$\begin{aligned} x'_A(t=0) &= -\frac{L_0}{2} \\ x'_B(t=0) &= +\frac{L_0}{2} \end{aligned} \tag{4}$$

with initial velocities

$$\begin{aligned} \dot{x}'_A(t=0) &= +\frac{v_0}{2} \\ \dot{x}'_B(t=0) &= -\frac{v_0}{2}, \end{aligned} \tag{5}$$

where the dots over x indicate time derivatives of x. We know that the general solution of simple harmonic motion is of the form $\mathcal{A}_1 \sin(\omega t) + \mathcal{A}_2 \cos(\omega t)$. From the initial conditions in equation (4), we know that only $\sin(\omega t)$ will contribute to the motion. The spring is stretched a maximum of ΔL, so the amplitude of oscillation for **A** and **B** each is $\Delta L/2$. From this information, we can write down general forms of the motion in the CM frame:

$$\begin{aligned} x'_A(t) &= -\frac{L_0}{2} + \frac{\Delta L}{2}\sin(\omega t) \\ x'_B(t) &= +\frac{L_0}{2} - \frac{\Delta L}{2}\sin(\omega t). \end{aligned} \tag{6}$$

Combining equations (6) and (3) gives us the positions of **A** and **B** in the lab frame:

$$\begin{aligned} x_A(t) &= x'_A(t) + x_{CM}(t) = \frac{\Delta L}{2}\sin(\omega t) + \frac{v_0}{2}t \\ x_B(t) &= x'_B(t) + x_{CM}(t) = L_0 - \frac{\Delta L}{2}\sin(\omega t) + \frac{v_0}{2}t, \end{aligned}$$

which are sketched in figure 4.

(c) (2 Points)
It is again convenient to work in the CM frame. As shown in figure 3, we can think of the particles oscillating individually on springs with spring constant $2k$. The oscillation frequency is

$$\omega = \sqrt{\frac{2k}{m}},$$

as given in equation (12.9) in *The Mechanical Universe.*

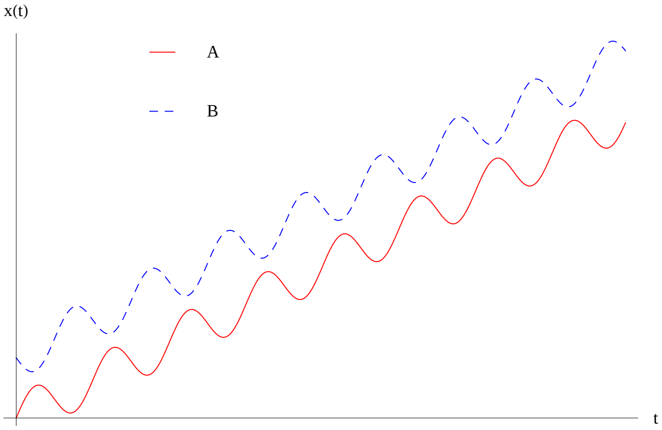

Figure 4: Positions of **A** and **B** in the lab frame for $t \geq 0$.

In order to determine ΔL, we need to use the initial conditions in equation (5). By taking a time derivative of the particle positions in the CM frame, we get

$$\dot{x}'_A(t) = \frac{\Delta L}{2}\omega\cos(\omega t)$$
$$\dot{x}'_B(t) = -\frac{\Delta L}{2}\omega\cos(\omega t).$$

Matching to the initial conditions gives

$$\Delta L = \frac{v_0}{\omega} = v_0\sqrt{\frac{m}{2k}}.$$

Alternative Solution

Another approach we could take is to write down the differential equations for x'_A and x'_B. The spring forces on **A** and **B** depend on the positions of both **A** and **B**, and we have to think carefully about what is the direction of the force when each particle is displaced from its equilibrium position. The result is

$$\begin{aligned} m\ddot{x}'_A &= -k[(x'_A + L_0/2) - (x'_B - L_0/2)] \\ m\ddot{x}'_B &= -k[(x'_B - L_0/2) - (x'_A + L_0/2)]. \end{aligned} \tag{7}$$

These equations are a set of coupled differential equations, which you will learn more about in Ph 2a or Ph 12a. Luckily, this is a simple enough system that we will not need more advanced tools past Ph 1. If we subtract the equations in (7), we get

$$m(\ddot{x}'_B - \ddot{x}'_A) = -2k[(x'_B - x'_A) - L_0].$$

Note that the spring length is $L(t) = x'_A(t) - x'_B(t)$, so we end up with a single differential equation for an harmonic oscillator

$$m\frac{d^2}{dt^2}(L(t) - L_0) = -2k[L(t) - L_0]$$

with frequency

$$\omega = \sqrt{\frac{2k}{m}}.$$

We write the general solution as

$$L(t) - L_0 = \mathcal{A}_1 \sin(\omega t) + \mathcal{A}_2 \cos(\omega t).$$

With the initial conditions

$$\begin{aligned} L(t=0) - L_0 &= 0 \\ \frac{d}{dt}(L - L_0)\bigg|_{t=0} &= -v_0, \end{aligned}$$

we conclude that $\mathcal{A}_2 = 0$ and $\mathcal{A}_1 = -v_0/\omega$. The amplitude of oscillation ΔL is simply $|\mathcal{A}_1|$, so

$$\Delta L = \frac{v_0}{\omega} = v_0\sqrt{\frac{m}{2k}},$$

and the final expression for the solution is

$$L(t) = L_0 - \Delta L \sin(\omega t) = L_0 - \frac{v_0}{\omega}\sin(\omega t). \tag{8}$$

We could also find ΔL from the conservation of energy. The system begins with no potential energy in the spring and

$$\text{KE}_i = 2 \cdot \frac{1}{2} m \left(\frac{v_0}{2}\right)^2.$$

When the spring is fully compressed or stretched, there is no kinetic energy, but the potential energy is

$$\text{U}_f = \frac{1}{2}k(\Delta L)^2.$$

Conserving total energy gives us the expression for ΔL we found before.

From the expression for $L(t)$ in equation (8), we can find the positions of **A** and **B** in the CM frame

$$x'_A(t) = -x'_B(t) = -\frac{L(t)}{2},$$

and the positions in the lab frame are

$$\begin{aligned} x_A(t) &= x'_A(t) + x_{CM}(t) \\ x_B(t) &= x'_B(t) + x_{CM}(t). \end{aligned}$$