

Physics Ia Lecture 11

11/2/11

①

Oscillatory motion

Equations of motion

↳ constant force (non-osc.)

↳ $F \propto -x$

↳ F_{drag}

Constant force

[Motivational example]

$$ma = F, \quad \& \quad F = \text{const.}, \quad \text{perhaps } mg$$

$$m\ddot{x} = mg$$

⇓

$$\boxed{\ddot{x} = g}$$

⇒

$$\boxed{x(t) = \frac{1}{2}gt^2 + v_0t + x_0}$$

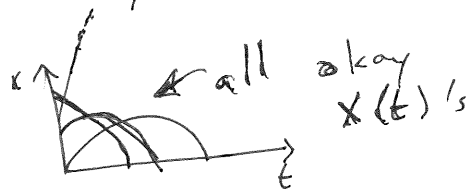
↑ "Equations of motion" ↑

→ contains all possible $x(t)$ for the system

→ two constants (x_0, v_0) determined by initial conditions

→ Sol'n works for any constant force (perhaps electrostatic? etc...)

→ all parabolas in (x, t) space



Simple harmonic motion

Same approach...

$$ma = F \quad \text{w/} \quad F = -kx \quad (\text{spring, etc...})$$



$$m\ddot{x} = -kx$$



$$\boxed{\ddot{x} = -\frac{k}{m}x}$$

Equ. of motion => done!

But would like to know what $x(t)$ looks like...

Can't just integrate this one: "Differential equation"

[In general: tough to solve analytically (often impossible)]
[valid approach: guess $x(t)$ & see if it works.]

Guess:

$$x(t) = \cos(\omega_0 t)$$

$$\hookrightarrow \omega_0 \equiv \sqrt{\frac{k}{m}}$$

as soln to $\ddot{x} = -\omega_0^2 x$ $\rightarrow \frac{k}{m}$

$$\dot{x} = -\omega_0 \sin(\omega_0 t)$$

$$\ddot{x} = -\omega_0^2 \underbrace{\cos(\omega_0 t)}_x$$

$$\Rightarrow \ddot{x} = -\omega_0^2 x \quad \checkmark \quad \text{works}$$

Some sol'ns

$x = \cos \omega_0 t$ ✓

$x = \sin \omega_0 t$ ✓

$x = A \cos \omega_0 t + B \sin \omega_0 t \rightarrow$ check :

Define : $x_1 = \cos \omega_0 t$ $\left\{ \begin{array}{l} \text{Both} \\ \text{good} \\ \text{sol'ns} \end{array} \right\}$
 $x_2 = \sin \omega_0 t$

$x = A x_1 + B x_2$

$\ddot{x} = A \ddot{x}_1 + B \ddot{x}_2$

$= A(-\omega_0^2 x_1) + B(-\omega_0^2 x_2)$

$= -\omega_0^2 (A x_1 + B x_2)$

$\ddot{x} = -\omega_0^2 x$ ✓

⊛ Linear combinations of solutions are also solutions!

So:

$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$

from initial conditions

general sol'n, because

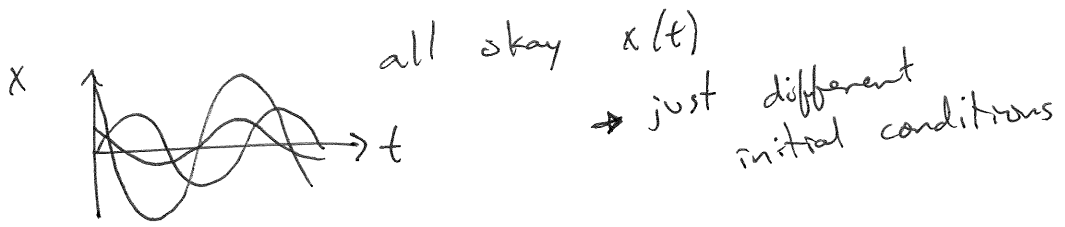
- (1) it works &
- (2) it has 2 "constants of integration" A & B

Let $x(0) \equiv x_0$
 $\dot{x}(0) \equiv v_0$ } \Rightarrow $A = x_0$
 $B = v_0 / \omega_0$

with some trig. identities:

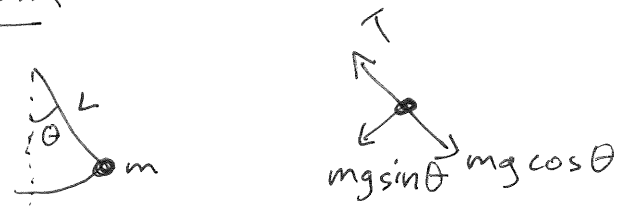
$x(t) = C \cos(\omega_0 t + \delta)$

also a general sol'n, for same reasons



Demos: $\left\{ \begin{array}{l} \text{springs on force probe} \\ \text{springs with } m \rightarrow 4m \end{array} \right.$

Pendulum



$ma = F$ look at motion \perp to rope

$m \frac{d^2(L\theta)}{dt^2} = -mg \sin\theta$
 ↙ when $\theta > 0$, F acts to reduce θ , so minus sign here

⇓

$\ddot{\theta} = -\frac{g}{L} \sin\theta$

$\ddot{\theta} = -\omega_0^2 \sin\theta$ if $\omega_0 = \sqrt{\frac{g}{L}}$

Note:

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots$

$\sin\theta \approx \theta$ if θ small

So:

$\Rightarrow \ddot{\theta} = -\omega_0^2 \theta$ for θ small \Rightarrow $\left\{ \begin{array}{l} \text{simple} \\ \text{harmonic} \\ \text{oscillator!} \end{array} \right.$

All the work was done in the spring case.
So Done!

Demos: $\begin{cases} \omega_0 = \omega_0' & \text{if } \theta \text{ small} \\ \omega_0 \neq \omega_0' & \text{if } \theta \text{ large} \\ \text{walking} \end{cases}$

Damping

$$F = \underbrace{-kx}_{\text{spring}} - \underbrace{\gamma \dot{x}}_{\text{drag, prop. to } -\dot{x}}$$

2nd law $\Rightarrow m\ddot{x} = -kx - \gamma \dot{x}$
or

$$\boxed{\ddot{x} + \beta \dot{x} + \omega_0^2 x = 0}$$

$\downarrow \rightarrow \frac{\gamma}{m} \quad \downarrow \rightarrow \frac{k}{m}$

Can we find $x(t)$?

To help our guess:

if $\beta = 0 \Rightarrow$ s.H.O. (cosine)

if $\omega_0 = 0 \Rightarrow$ exponential $\left[\begin{array}{l} \dot{v} = -\beta v \Rightarrow v \propto e^{-\beta t} \\ \Rightarrow x \propto e^{-\beta t} \end{array} \right.$

So we want a mix of exp() & cos().

Try:

$$x(t) = C e^{-bt} \cos(\omega_c t + \delta)$$

(1) Find \dot{x} , \ddot{x}

(2) Plug in to $\ddot{x} + \beta\dot{x} + \omega_0^2 x = 0$

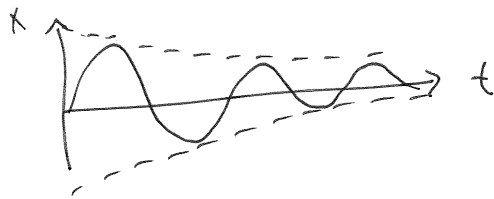
(3) It works! IF: $\begin{cases} b = \beta/2 \\ \omega_1^2 = \omega_0^2 - \frac{\beta^2}{4} \end{cases}$

So, general solution

$$x(t) = C e^{-\beta t/2} \cos(\omega_1 t + \delta)$$

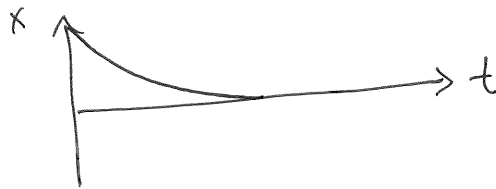
$\omega_1 = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}$

$\beta < 2\omega_0$
case



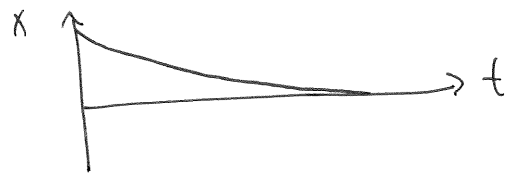
"under-damped"

$\beta = 2\omega_0$



"critically damped"
($\omega_1 = 0$)

$\beta > 2\omega_0$



"over damped"

Demos: $\begin{cases} \text{tuning fork on 'scope} \\ \text{air track with varying } \gamma \end{cases}$