

Recall:

 $M \vec{v} = \text{constant}$ isolated system

$$\sum_i m_i \vec{v}_i = \text{constant}$$

(from $\vec{F}_{ij} = -\vec{F}_{ji}$)

 $\vec{p}_i = m_i \vec{v}_i =$ momentum of i^{th} particle

$$\vec{P} = \sum_i m_i \vec{v}_i = \text{const} \quad \text{for isolated system}$$

$$= M \vec{v}_{\text{cm}} \quad M = \sum_i m_i = \text{total mass}$$

$$\Rightarrow \vec{v}_{\text{cm}} = \frac{1}{M} \sum_i m_i \vec{v}_i \quad \text{center of mass velocity}$$

$$\vec{r}_{\text{cm}} = \frac{1}{M} \sum_i m_i \vec{r}_i \quad \text{position at CM}$$

$$= \frac{1}{M} \int \rho(\vec{r}) \vec{r} \, dx \, dy \, dz \quad \begin{array}{l} \text{continuous} \\ \text{mass} \\ \text{distribution} \\ \rho(\vec{r}) \end{array}$$

Look at kinetic energy

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i^2 \quad \text{"lab frame"}$$

$$\vec{v}_i' = \vec{v}_i - \vec{v}_{cm} \quad \text{cm frame velocity}$$

$$T = \sum_i \frac{1}{2} m_i (\vec{v}_i' + \vec{v}_{cm})^2$$

$$= \sum_i \frac{1}{2} m_i (\vec{v}_i'^2 + 2 \vec{v}_i' \cdot \vec{v}_{cm} + \vec{v}_{cm}^2)$$

$$= \underbrace{\sum_i \frac{1}{2} m_i \vec{v}_i'^2}_{= T', \text{ KE in cm}} + \underbrace{\vec{v}_{cm} \cdot \sum_i m_i \vec{v}_i'}_{= 0 \text{ (momentum in cm = 0)}} + \underbrace{\frac{1}{2} M \vec{v}_{cm}^2}_{\text{KE}_{cm}}$$

$$\boxed{T = T' + \frac{1}{2} M \vec{v}_{cm}^2}$$

KE in lab frame = "internal KE"
+ "KE of cm motion"

$$T' = \frac{1}{2} m_i \vec{v}_i'^2 : \text{"internal" KE}$$

- rotation
- random motion
- combination
- etc,

Consider elastic 1D collisions:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Ph 1a 1977, Prob. 1, HW #1:

Are there any other quantities of the form $m v^n$ that are conserved?

$$m_1 (v_{1i}^2 - v_{1f}^2) = m_2 (v_{2f}^2 - v_{2i}^2)$$

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$

So:

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

unless $\underbrace{v_{1i} - v_{1f}} = \underbrace{v_{2f} - v_{2i}} = 0$

no change in
 v_1

no change
in v_2

\Rightarrow no collision!
(boring...)

so

$$2m_1 v_{1i} = (m_1 + m_2) v_{2f} + (m_1 - m_2) v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

(21)

$$\underline{m_1 = m_2} :$$

$$V_{2f} = V_{1i}$$

$$V_{1f} = V_{2i}$$

In CM frame:

$$m_1 v'_{1i} + m_2 v'_{2i} = 0 = m_1 v'_{1f} + m_2 v'_{2f} .$$

$$m_1 = m_2: \quad v'_{1i} = -v'_{2i} \quad v'_{1f} = -v'_{2f}$$

$$K_{e'} = \frac{1}{2} m (v'_{1i}{}^2 + v'_{2i}{}^2) = m v'_{1i}{}^2 = m v'_{1f}{}^2$$

$$\Rightarrow v'_{1f} = \pm v'_{1i} \quad \oplus \text{ is uninteresting; no collision.}$$

$$\Rightarrow v'_{1f} = -v'_{1i} \quad \text{and} \quad v'_{2f} = -v'_{2i} .$$

Demo - 1D collisions

2D collisions

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 = KE_{1i} = \frac{(m_1 \vec{v}_{1i})^2}{2m_1} = \frac{\vec{p}_{1i}^2}{2m_1}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \quad \text{momentum}$$

$$T = \frac{\vec{p}_{1i}^2}{2m_1} + \frac{\vec{p}_{2i}^2}{2m_2} = \frac{\vec{p}_{1f}^2}{2m_1} + \frac{\vec{p}_{2f}^2}{2m_2} \quad \text{energy (elastic)}$$

In CM frame,

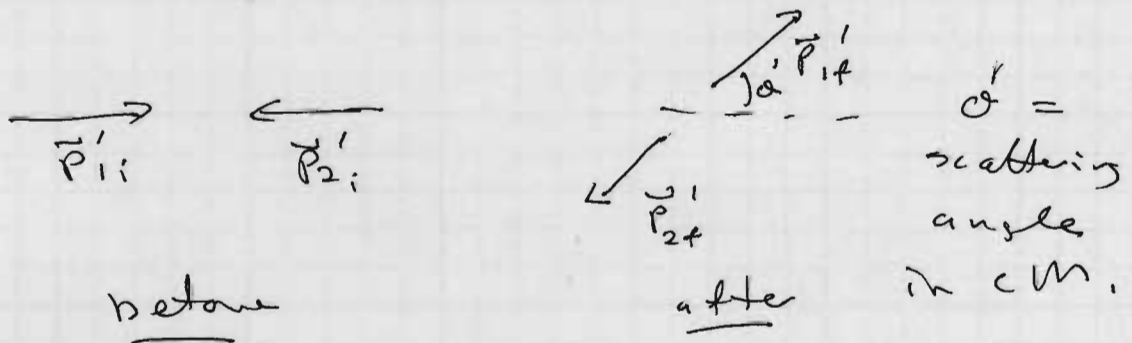
$$\vec{p}_{1i}' + \vec{p}_{2i}' = 0 = \vec{p}_{1f}' + \vec{p}_{2f}'$$

$$T' = \frac{(\vec{p}_{1i}')^2}{2m_1} + \frac{(-\vec{p}_{2i}')^2}{2m_2} = \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) (\vec{p}_{1i}')^2$$

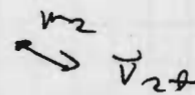
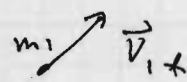
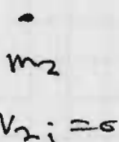
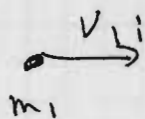
$$= \left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) (\vec{p}_{1f}')^2$$

$$\Rightarrow |\vec{p}_{1i}'| = |\vec{p}_{1f}'| \quad \text{and} \quad |\vec{p}_{2i}'| = |\vec{p}_{2f}'|$$

\Rightarrow momentum just changes direction, not magnitude



collision in lab frame



before

after

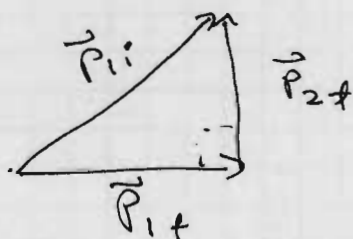
$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \quad \text{momentum}$$

$$\frac{\vec{p}_{1i}^2}{2m_1} = \frac{\vec{p}_{1f}^2}{2m_1} + \frac{\vec{p}_{2f}^2}{2m_2} \quad \text{KE}$$

$m_1 = m_2$: becomes simple.

$$\vec{p}_{1i}^2 = \vec{p}_{1f}^2 + \vec{p}_{2f}^2$$

$$\vec{p}_{1i} = \vec{p}_{1f} + \vec{p}_{2f}$$



walks to
right
triangles!

$$\vec{p}_{1f} \perp \vec{p}_{2f}$$

Proof:

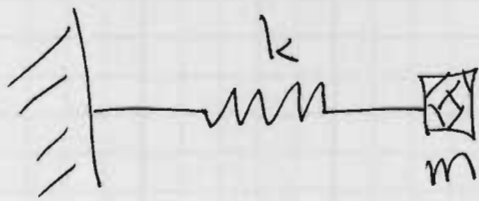
$$\begin{aligned} \vec{p}_{1i}^2 &= (\vec{p}_{1f} + \vec{p}_{2f})^2 = \vec{p}_{1f}^2 + \vec{p}_{2f}^2 + 2\vec{p}_{1f} \cdot \vec{p}_{2f} \\ &= \vec{p}_{1f}^2 + \vec{p}_{2f}^2 \end{aligned}$$

\Rightarrow

$$\vec{p}_{1f} \cdot \vec{p}_{2f} = 0$$

2D coll. laws

Oscillatory motion



$$F = -kx$$
$$= m\ddot{x}$$

$$m \frac{d^2x}{dt^2} = -kx$$

equation of motion
(second order)

Introduce $p = mv$

$$\frac{dp}{dt} = m\ddot{x} = F = -kx$$

$$\boxed{\begin{aligned} \frac{dp}{dt} &= -kx \\ \frac{dx}{dt} &= +\frac{p}{m} \end{aligned}}$$

two coupled
first-order
equations.

We can solve either.

$$[k] = N m^{-1} = kg m s^{-2} m^{-1} = kg s^{-2}$$

$$\left[\frac{k}{m}\right] = s^{-2}$$

Define

$$\boxed{\omega^2 = \frac{k}{m}}$$

$$[\omega] = s^{-1}$$

$$\{m \omega x\} = k_g \bar{s}^{-1} m = \{p\}.$$

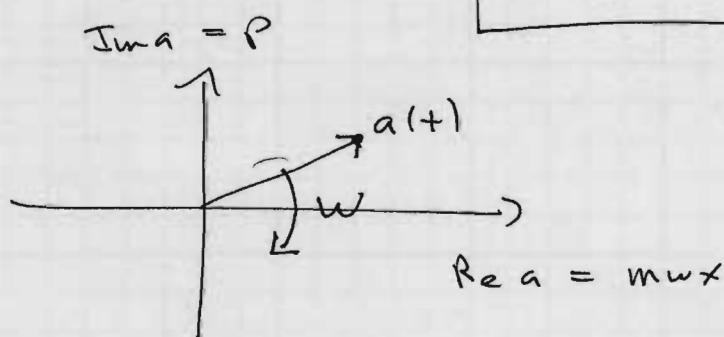
$$\frac{d}{dt} (m \omega x) = \omega p = -i \omega (i p)$$

$$\frac{d}{dt} (i p) = -i k x = -i \frac{k}{m} \cdot m x = -i \omega \cdot (m \omega x)$$

$$\frac{d}{dt} (m \omega x + i p) = -i \omega (m \omega x + i p)$$

Let $a = m \omega x + i p$. Then

$$\frac{da}{dt} = -i \omega a \Rightarrow a(t) = a(0) e^{-i \omega t}$$



$a(t)$ rotates in complex plane w/
angular velocity ω

$\Rightarrow m \omega x$ oscillates sinusoidally

p oscillates sinusoidally

Recall $e^{-i \omega t} = \cos \omega t + i \sin(-\omega t)$

Take $a(0) = a_0$ (real),

$$m\omega x + ip = a_0 (\cos \omega t - i \sin \omega t)$$

$$\begin{aligned} x(t) &= \frac{a_0}{m\omega} \cos \omega t \\ p(t) &= -a_0 \sin \omega t \end{aligned}$$

$$\frac{a_0}{m\omega} = x(0)$$

real part
at $e^{-i\omega t}$

imaginary
part at $e^{-i\omega t}$

$$\begin{aligned} x(t) &= x_0 \cos \omega t \\ p(t) &= -m\omega x_0 \sin \omega t \end{aligned}$$

More generally: $q(0) = a_0 e^{i\phi}$

$$x(t) = \frac{a_0}{m\omega} \cos(\omega t - \phi)$$

$$p(t) = -a_0 \sin(\omega t - \phi)$$

a_0, ϕ determined by $x(0), p(0)$:

$$\tan \phi = \frac{p(0)}{m\omega x(0)}$$

$$a_0^2 = p(0)^2 + m^2 \omega^2 x(0)^2$$