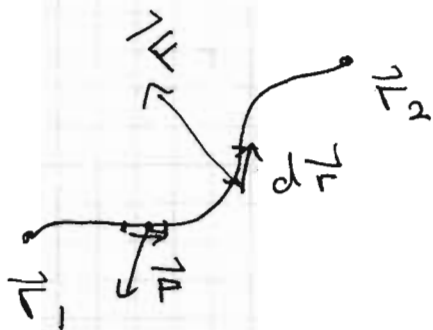


In 3D:

$$W_{1 \rightarrow 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F}(t) \cdot \frac{d\vec{r}}{dt} dt$$

$$= \int_{t_1}^{t_2} \vec{F}(t) \cdot \vec{v}(t) dt$$



add up  $\vec{F} \cdot d\vec{r}$  for all small segments at path followed in moving from  $\vec{r}_1$  to  $\vec{r}_2$ .

Example 1: gravity near Earth's surface.

$$\vec{F} = -mg \hat{z}$$

$$W_{1 \rightarrow 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = -mg \int_{z_1}^{z_2} dz$$

$$= -mg z \Big|_{z_1}^{z_2} = -mg (z_2 - z_1).$$

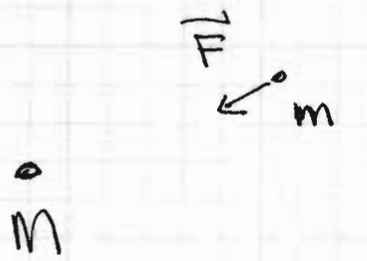
Example 2: Spring

$$\vec{F} = -kx \hat{x} \quad (1D)$$

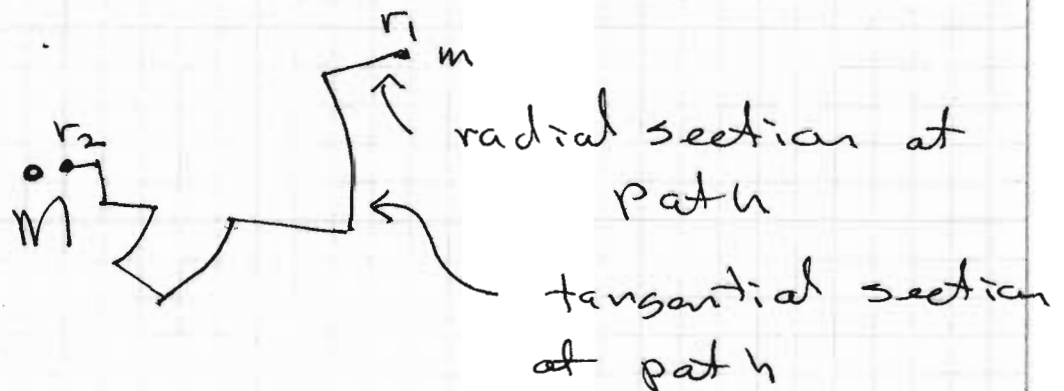
$$W_{1 \rightarrow 2} = \int_{x_1}^{x_2} -kx dx = -\frac{1}{2} kx^2 \Big|_{x_1}^{x_2}$$

$$= -\frac{1}{2} k(x_2^2 - x_1^2).$$

### Example 3: Newton's Grav. Law


$$\vec{F} = -G \frac{Mm}{r^2} \hat{r}$$

Consider some path from  $r_1$  to  $r_2$ :



Only radial sections count!  $\vec{F} \cdot d\vec{r} = 0$   
for tangential sections. Therefore,

$$W_{1-2} = \int_{r_1}^{r_2} -G \frac{Mm}{r^2} dr = +G \frac{Mm}{r} \Big|_{r_1}^{r_2}$$

$$W_{1-2} = G \frac{Mm}{r_2} - G \frac{Mm}{r_1}$$

Look at

$$W_{1 \rightarrow 2} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt,$$

Newton's 2<sup>nd</sup> law tells us  $\vec{F} = m\vec{a}$ .

But  $\vec{a} = \frac{d\vec{v}}{dt}$ . Therefore

$$\begin{aligned} \vec{F} \cdot \vec{v} &= m\vec{a} \cdot \vec{v} = m \frac{d\vec{v}}{dt} \cdot \vec{v} \\ &= m(v_x \dot{v}_x + v_y \dot{v}_y + v_z \dot{v}_z). \end{aligned}$$

Note  $\frac{d}{dt}(v_x^2) = 2v_x \dot{v}_x$ . Etc for y, z.

$\Rightarrow$

$$\vec{F} \cdot \vec{v} = \frac{1}{2} m \frac{d}{dt} (v_x^2 + v_y^2 + v_z^2)$$

$$= \frac{1}{2} m \frac{d}{dt} v^2$$

$$W_{1 \rightarrow 2} = \frac{1}{2} m \int_1^2 \frac{d}{dt} (v^2) dt$$

$$W_{1 \rightarrow 2} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$\frac{1}{2} m v^2 = \text{kinetic energy}$

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Short hand:

$$W_{1 \rightarrow 2} = KE_2 - KE_1 = \Delta KE.$$

## Potential Energy

Suppose that  $F_x = -\frac{dU}{dx}$

$U(x)$  = potential energy

$$\begin{aligned} W_{1 \rightarrow 2} &= \int_{x_1}^{x_2} F_x dx = - \int_{x_1}^{x_2} \frac{dU}{dx} dx \\ &= U(x_1) - U(x_2) \end{aligned}$$

Then

$$U(x_1) - U(x_2) = KE_2 - KE_1$$

$$PE_1 + KE_1 = PE_2 + KE_2$$

$$U(x_1) + \frac{1}{2}mV_{1x}^2 = U(x_2) + \frac{1}{2}mV_{2x}^2$$

## Conservation of Energy

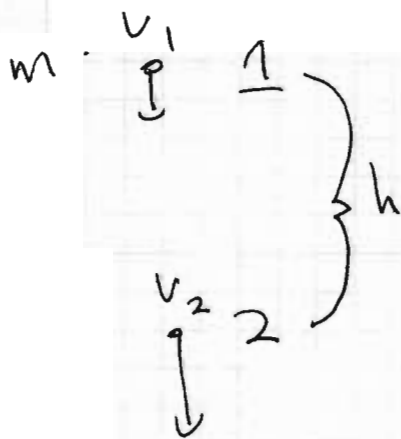
"energy is neither created nor destroyed"

Friction: energy (work) is converted into heat

Example: gravity near Earth's surface

$$\vec{F} = -mg \hat{z} = -mg \frac{dz}{dz} \hat{z}$$

so  $U(z) = mgz$  = gravitational potential energy.



$$\begin{aligned} \frac{1}{2} m v_2^2 + mg z_2 \\ = \frac{1}{2} m v_1^2 + mg z_1 \end{aligned}$$

$$\boxed{\frac{1}{2} m v_2^2 = \frac{1}{2} m v_1^2 + mgh}$$

Also:  $z(t) = z_1 - v_1 t - \frac{1}{2} g t^2$

$$v_2 = -v_1 - g t$$

If  $z(t) = z_2$ , then:

$$z_1 - z_2 = h = v_1 t + \frac{1}{2} g t^2$$

$$= \frac{1}{2g} (g^2 t^2 + 2v_1 g t + v_1^2) - \frac{v_1^2}{2g}$$

$$gh = \frac{1}{2} (v_1 + g t)^2 - \frac{1}{2} v_1^2$$

$$mgh = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \quad \checkmark$$

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In 3D:  $U = U(x, y, z) = U(\vec{r})$

$$\vec{F} = - \left( \frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z} \right)$$

Can show

$$U(\vec{r}_1) + \frac{1}{2} m \vec{v}_1^2 = U(\vec{r}_2) + \frac{1}{2} m \vec{v}_2^2$$

KE + PE constant.

Demos:

- bowling ball
- pivot pendulum
- pendulum velocity
- stairs

Potential energy functions:

$$U(z) = mgz \quad \text{gravity near earth}$$

$$U(x) = \frac{1}{2} kx^2 \quad \text{Spring}$$

$$U(r) = -G \frac{Mm}{r} \quad \text{gravity}$$

$$\vec{F} = - \frac{dU}{dr} \hat{r} = G M m \frac{d}{dr} \left( \frac{1}{r} \right) \hat{r}$$

$$= -G \frac{Mm}{r^2} \hat{r}$$

## Conservation of Momentum

Newton's 2<sup>nd</sup> law:  $\vec{F} = M\vec{a}$ .

Consider isolated object  $M$ :  $\vec{F} = 0$ .

$$\Rightarrow M\vec{a} = 0 \Rightarrow M\vec{v} = \text{constant}.$$

Now imagine that the object consists of a number of "particles"  $m_1, \dots, m_N$  with positions  $\vec{r}_1, \dots, \vec{r}_N$ .

$$m_i \vec{a}_i = \sum_{j \neq i} \vec{F}_{ij} \quad \vec{F}_{ij} = \text{force on } i \text{ due to } j.$$

Newton's 3<sup>rd</sup> law:  $\vec{F}_{ji} = -\vec{F}_{ij}$ .

Therefore 
$$\sum_{i=1}^N \sum_{j \neq i} \vec{F}_{ij} = 0.$$
  
(all pairs  $\vec{F}_{ij}, \vec{F}_{ji}$  cancel)

so

$$\sum_{i=1}^N m_i \vec{a}_i = 0$$

Therefore

$$\sum_{i=1}^N m_i \vec{v}_i = \text{constant}$$

Conservation of momentum

How do we relate this to our conception of the object as a single entity?

In that case we found  $M \vec{v} = \text{const.}$

Therefore we might expect

$$M \vec{v} = \sum_{i=1}^N m_i \vec{v}_i \quad M = \sum_i m_i$$

This leads us to define the center of mass position

$$\vec{r}_{cm} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

For a continuous mass density  $\rho(\vec{r})$ , we would write

$$\vec{r}_{cm} = \frac{1}{M} \int_V \vec{r} \rho(\vec{r}) dx dy dz$$

mass  $m dx dy dz$

Volume over which  $\rho \neq 0$ .

where  $M = \int_V \rho(\vec{r}) dx dy dz$ .

$$M \vec{v}_{cm} = M \dot{\vec{r}}_{cm} = \sum_i m_i \dot{\vec{r}}_i = \sum_i m_i \vec{v}_i$$

Note that CM moves at constant velocity (for isolated system,  $\vec{F}_{ext} = 0$ )

Therefore, we can always work in an inertial frame for which  $\vec{V}_{cm} = 0$ . This is called the "center of mass" frame, or the "center of momentum" frame.

To move into the CM frame, we perform a Galilean transformation:

$$\vec{V}'_i = \vec{V}_i - \vec{V}_{cm}$$

Velocity in  
CM frame

Velocity in  
original  
frame

$\vec{V}_{cm}$  calculated  
in original  
frame

The CM velocity in the CM frame is

$$\begin{aligned} \vec{V}'_{cm} &= \frac{1}{M} \sum_i m_i \vec{V}'_i = \frac{1}{M} \left[ \sum_i m_i \vec{V}_i - \sum_i m_i \vec{V}_{cm} \right] \\ &= \vec{V}_{cm} - \vec{V}_{cm} = 0. \quad \checkmark \end{aligned}$$

Let's look at the kinetic energy:

$$T = \sum_i \frac{1}{2} m_i \vec{V}_i^2 = \sum_i \frac{1}{2} m_i \left( \vec{V}'_i + \vec{V}_{cm} \right)^2$$

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$$T = \frac{1}{2} \sum_i m_i ( \vec{v}_i^2 - 2 \vec{v}_i \cdot \vec{v}_{cm} + \vec{v}_{cm}^2 )$$

$$\rightarrow \underbrace{\frac{1}{2} \sum_i m_i \vec{v}_i^2}_{= \text{KE in CM frame}} - \vec{v}_{cm} \cdot \underbrace{\sum_i m_i \vec{v}_i}_{= 0} + \frac{1}{2} M \vec{v}_{cm}^2$$

$$\boxed{T = T' + \frac{1}{2} M \vec{v}_{cm}^2}$$

Total KE = KE in CM frame ( $T'$ )  
+ KE of CM motion ( $\frac{1}{2} M \vec{v}_{cm}^2$ )

$T'$  can be considered to be the "internal" KE of the object. Examples include rotation of a rigid object, random motion of molecules in a gas or fluid, or both: a rotating fluid.