Ph1a Homework Solution 8

Fall 2015

Each homework problem is worth 5 points unless otherwise noted.

8.1 Frautschi 17.12, 17.22, and 17.24 (5 points)

The eccentricity for zero-angular-momentum orbit is 1. The trajectory in this case is a straight line along a radius.

From Kepler’s Third Law (17.23), the solar mass

\[ M = \frac{4\pi^2 a^3}{GT^2} = 1.99 \times 10^{30} \text{ kg} \]  

Also from Kepler’s Third Law, taking M as mass of the Earth and a as radius of the earth, the shortest period is then

\[ T_{\text{min}} = \sqrt{\frac{4\pi^2 R^3}{GM}} \approx 5061 \text{ s} \approx 1.4 \text{ hrs} \]  

8.2 FP4 (5 points)

8.2.a (1 point)

The total energy \( E \) is given by:

\[ E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\left(\frac{GM}{5R}\right) - \frac{GMm}{5R} \]

\[ E = -\frac{GMm}{10R} \]

Because \( E < 0 \), the orbit will be elliptical.

8.2.b (1 point)

The magnitude of the angular momentum \( \vec{L} \) is

\[ |\vec{L}| = |\vec{r} \times \vec{p}| = (5R)(mv_0)\sin 135^\circ \]

\[ |\vec{L}| = m\sqrt{\frac{5}{2}GMR} \]

8.2.c (1.5 points)

The equations for conservation of energy and angular momentum are respectively:

\[ E = -\frac{GMm}{10R} = \frac{1}{2}m\nu_p^2 - \frac{GMm}{r_p} \]

\[ |\vec{L}| = m\sqrt{\frac{5}{2}GMR} = m\nu_p r_p \]

Solving the second equation for \( r_p \) gives

\[ r_p = \sqrt{\frac{5}{2}GMR/\nu_p^2} \]
Substituting into the energy equation:

\[-\frac{GMm}{10R} = \frac{1}{2}mv_p^2 - GMm \left( \frac{v_p}{\sqrt{\frac{5}{2} GMR}} \right)\]

\[v_p^2 - 2 \left( \sqrt{\frac{2GM}{5R}} \right) v_p + \frac{GM}{5R} = 0\]

\[v_p = \sqrt{\frac{2GM}{5R}} \pm \sqrt{\frac{2GM}{5R} - \frac{GM}{5R}}\]

\[= \sqrt{\frac{2GM}{5R}} \pm \sqrt{\frac{GM}{5R}}\]

At closest approach the speed will be maximal, so

\[v_p = (1 + \sqrt{2}) \sqrt{\frac{GM}{5R}}\]

8.2.d (1.5 points)

All that is required is that we calculate the total energy with the new mass \(M' = M/2\):

\[E = K + U = \frac{1}{2}mv^2 - GM'm = \frac{1}{2}m \left( \frac{GM}{5R} \right) - GMm \left( \frac{1}{10R} \right) = 0\]

The special case \(E = 0\) describes a parabolic orbit.

8.3 FP18 (5 points)

8.3.a (1.5 points)

In the energy scale indicated, the gravitational potential energy for the satellite is \(-GMm/R\). Meanwhile, the gravitational force that pulls the satellite towards the planet is \(GMm/R^2\), which must be equal to the centripetal force \(mu^2/R\) necessary to keep the satellite on its circular orbit. Therefore

\[u = \sqrt{\frac{GM}{R}}\]

which corresponds to a kinetic energy \(mu^2/2 = GMm/(2R)\). The total energy of the satellite is

\[E = \frac{1}{2}mu^2 - \frac{GMm}{R} = \frac{GMm}{2R}\]

The angular momentum is simply given by \(L = mRu = m\sqrt{GM}R\).

8.3.b (1 point)

Immediately after the explosion the two hemispheres are still a distance \(R\) away from the center of the planet, and their velocities are still tangential. Therefore

\[L_B = \frac{m}{2} \frac{5uR}{4} = \frac{5}{8}muR\]
In order to compute $L_A$ we must first find the velocity of $A$ after the explosion. Since the explosion was internal, total linear momentum should have been conserved. This implies that the velocity of $A$ must also be tangential to the original orbit of the satellite, and that its magnitude must be $3u/4$. Then

$$L_A = m \frac{3u}{4} R = \frac{3}{8} muR$$

Notice that total angular momentum is also conserved, as expected.

The energy of $A$ is

$$E_A = \frac{1}{2} m \left( \frac{3u}{4} \right)^2 - \frac{GMm}{2R} = \frac{GMm}{R} \left( \frac{9}{64} - \frac{1}{2} \right) = \frac{-23 GMm}{64 R}$$

The energy of $B$ is

$$E_B = \frac{1}{2} m \left( \frac{5u}{4} \right)^2 - \frac{GMm}{2R} = \frac{GMm}{R} \left( \frac{25}{64} - \frac{1}{2} \right) = \frac{-7 GMm}{64 R}$$

8.3.c (1 point)

Only the kinetic energy of the satellite parts changes during the explosion. The initial kinetic energy is simply $mu^2/2$, while the final energy is $(5u/4)^2m/4 + (3u/4)^2m/4 = 17mu^2/32$. Therefore the energy of the system has increased by

$$\frac{mu^2}{32} = \frac{GMm}{32R}$$

which is the amount of work done by the explosion.

The same result may be obtained from $W = E_A + E_B - E$, using the results from parts (a) and (b).

8.3.d (0.5 point)

Using the energies calculated in part (b) and the equation for the length of the semi-major axis length in terms of the energy of the orbiting body, we obtain that

$$a_A = \frac{GMm/2}{2E_A} = \frac{16R}{23}$$

and $a_B = 16R/7$

8.3.e (1 point)

Since the velocity of both $A$ and $B$ at the point of the explosion is entirely tangential, the orbits for the hemispheres must have either their apogee or their perigee at that location.

Notice that $A$ has less energy than is necessary to keep it in the circular orbit of radius $R$. It will then describe a smaller ellipse with semi-major axis $a_A = 16R/23$. The planet with be at the focus furthest from the point of the explosion. (Notice that if the perigee distance of the orbit is smaller than the radius of the planet, $A$ would crash into it.) Meanwhile, $B$ has more energy than necessary to keep it on the circular orbit, but not enough for it to escape, because the energy is still negative. Therefore it will describe an elliptical orbit with the planet at the focus nearest from the point of the explosion. The semi-major axis of this orbit is the $a_B$ computed in part (d).
Figure 1: Sketch of the orbits of the hemispheres of the satellite after the explosion