

Ph1a Homework Solution 9

Fall 2011

Each homework problem is worth 5 points. Please disregard the point values listed on the problem itself. Use these instead.

9.1 FP4 (5 points)

9.1.a (1 point)

The total energy E is given by:

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m \left(\frac{GM}{5R} \right) - \frac{GMm}{5R} \\ E &= -\frac{GMm}{10R} \end{aligned}$$

Because $E < 0$, the orbit will be elliptical.

9.1.b (1 point)

The magnitude of the angular momentum \vec{L} is

$$\begin{aligned} |\vec{L}| &= |\vec{r} \times \vec{p}| \\ &= (5R)(mv_0) \sin 135^\circ \\ |\vec{L}| &= m\sqrt{\frac{5}{2}GMR} \end{aligned}$$

9.1.c (1.5 points)

The equations for conservation of energy and angular momentum are respectively:

$$\begin{aligned} E &= -\frac{GMm}{10R} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \\ |\vec{L}| &= m\sqrt{\frac{5}{2}GMR} = mv_p r_p \end{aligned}$$

Solving the second equation for r_p gives

$$r_p = \sqrt{\frac{5}{2}GMR/v_p^2}$$

Substituting into the energy equation:

$$\begin{aligned} -\frac{GMm}{10R} &= \frac{1}{2}mv_p^2 - GMm \left(\frac{v_p}{\sqrt{\frac{5}{2}GMR}} \right) \\ v_p^2 - 2 \left(\sqrt{\frac{2GM}{5R}} \right) v_p + \frac{GM}{5R} &= 0 \\ v_p &= \sqrt{\frac{2GM}{5R}} \pm \sqrt{\frac{2GM}{5R} - \frac{GM}{5R}} \\ &= \sqrt{\frac{2GM}{5R}} \pm \sqrt{\frac{GM}{5R}} \end{aligned}$$

At closest approach the speed will be maximal, so

$$v_p = (1 + \sqrt{2})\sqrt{\frac{GM}{5R}}$$

9.1.d (1.5 points)

All that is required is that we calculate the total energy with the new mass $M' = M/2$:

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 - \frac{GM'm}{r} \\ &= \frac{1}{2}m\left(\frac{GM}{5R}\right) - \frac{GMm}{10R} = 0 \end{aligned}$$

The special case $E = 0$ describes a parabolic orbit.

9.2 FP6 (5 points)

9.2.a (2 points)

For an object (of finite mass, m) to escape, it's total energy must be non-negative ($E \geq 0$). For the event horizon (distance R), this implies the limiting condition $E = 0$.

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 - GMm/R = 0 \\ \frac{1}{2}mv^2 &= GMm/R \\ R &= 2GM/v^2 \end{aligned}$$

Since the speed of light is $v = c$, the event horizon is at a distance $R = 2GM/c^2$ from the center of the black hole. For a solar-mass black hole:

$$R = \frac{2(6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2})(2.0 \times 10^{30} \text{ kg})}{(3.0 \times 10^8 \text{ m/s})^2} = 3.0 \text{ km}$$

9.2.b (1.5 points)

The difference in acceleration a is given by

$$\begin{aligned} \delta a &= \frac{\partial a}{\partial r} \delta r \\ &= \frac{\partial}{\partial r} \left(-\frac{GM}{r^2} \right) \delta r \\ &= \frac{2GM}{r^3} \delta r \\ &= \frac{2GM}{R^3} h \\ &= \frac{2(6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2})(2.0 \times 10^{30} \text{ kg})}{(3.0 \times 10^3 \text{ m})^3} (2.0 \text{ m}) \\ \delta a &= 2.0 \times 10^{10} \frac{\text{m}}{\text{s}^2} \end{aligned}$$

(Note: Answers which solve this problem by numerically subtracting the acceleration at R from the acceleration at $(R + h)$ are invalid because they don't properly account for significant digits. Students who work the problem this way should not receive full credit. However, solution by algebraic subtraction is correct.)

This difference in acceleration is called the tidal acceleration. (The tidal accelerations induced on the oceans by the moon and sun cause the sea tides. There is an excellent discussion in J.B. Marion and S.T. Thornton, *Classical Dynamics of Particles and Systems*, 4th ed. Fortworth: Saunders, 1995. pp. 204–210.)

9.2.c (1.5 points)

We can generalize the above discussion by substituting the expression for R from part a into the expression for the tidal acceleration in part b:

$$\begin{aligned}\delta a &= \frac{2GM}{R^3} h = 2GM \left(\frac{c^2}{2GM} \right)^3 h \\ &= \frac{c^6}{4G^2 M^2} h\end{aligned}$$

This clearly shows that the tidal acceleration is inversely proportional to the square of the mass of the black hole, implying that being at the event horizon of a supermassive black hole would be safer. Checking the numbers (not required for full credit):

$$\begin{aligned}\delta a_{\text{solar}} &= \frac{(3.0 \times 10^8 \frac{\text{m}}{\text{s}})^6}{4(6.7 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2})^2 (2.0 \times 10^{30} \text{kg})^2} (2.0 \text{ m}) = 2.0 \times 10^{10} \frac{\text{m}}{\text{s}^2} \\ \delta a_{\text{super}} &= \frac{(3.0 \times 10^8 \frac{\text{m}}{\text{s}})^6}{4(6.7 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2})^2 (2.0 \times 10^{37} \text{kg})^2} (2.0 \text{ m}) = 2.0 \times 10^{-4} \frac{\text{m}}{\text{s}^2}\end{aligned}$$

9.3 FP12 (5 points)

9.3.a (1 point)

Kepler's 3rd Law gives the period of a small mass (planet) orbiting a large one in terms of a , the semi-major axis of the planet's orbit.

$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

For circular orbits $a = r$. Thus $a_1 = R$ and $a_2 = 2R$.

$$\frac{T_2}{T_1} = \left(\frac{a_2}{a_1}\right)^{3/2} = 2\sqrt{2}$$

$$T_1 = 2\pi\sqrt{\frac{R^3}{GM}}$$

9.3.b (1 point)

The energy of a planet is given by:

$$E = KE + PE_g = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

To find the velocity of the planet, recall that $v = \text{const}$ in a circular orbit, and balance gravity against the centrifugal force.

$$\begin{aligned} F_c &= F_g \\ \frac{mv^2}{r} &= \frac{GMm}{r^2} \\ v &= \sqrt{\frac{GM}{r}} \end{aligned}$$

Combined with the energy equation, we have

$$\begin{aligned} E &= \frac{1}{2}m\left(\frac{GM}{r}\right) - \frac{GMm}{r} \\ E &= -\frac{GMm}{2r} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{E_2}{E_1} &= \frac{r_1}{r_2} = \frac{1}{2} \\ E_1 &= -\frac{GMm}{2R} \end{aligned}$$

9.3.c (1 point)

Since the velocity is perpendicular to the radius vector, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is simple to compute.

$$L = rmv$$

Using the velocity found above, we have

$$L = m\sqrt{GMr}$$

Thus,

$$\begin{aligned} \frac{L_2}{L_1} &= \sqrt{\frac{r_2}{r_1}} = \sqrt{2} \\ L_1 &= m\sqrt{GMR} \end{aligned}$$

Now suppose that we send a spacecraft of mass m_s from P1 to P2 along the elliptical transfer orbit shown below. The orbit has a "periastron" (the distance of closest approach to the star about which the planets are orbiting) of R and an "apastron" (the greatest distance to the star) of $2R$.

9.3.d (2 points)

Finding a and e is straightforward. In terms of the periastron $r_p = R$ and apastron $r_a = 2R$, we have

$$2a = r_p + r_a = 3R$$

$$a = \frac{3}{2}R$$

Similarly, the star is at a focus of the spacecraft's elliptical orbit. Recalling that c is the distance from center to a focus, we have

$$c = a - r_p = \frac{R}{2}$$

$$e = \frac{c}{a} = \frac{1}{3}$$

To find E and L , consider evaluating both at periastron and apastron. This gives two equations for v_p and v_a , the craft's velocity at each point.

$$L_p = L_a$$

$$r_p m_s v_p = r_a m_s v_a$$

$$v_p = \frac{r_a}{r_p} v_a = 2v_a$$

$$E_p = E_a$$

$$\frac{1}{2} m_s v_p^2 - \frac{GMm_s}{r_p} = \frac{1}{2} m_s v_a^2 - \frac{GMm_s}{r_a}$$

$$\frac{1}{2} (2v_a)^2 - \frac{GM}{R} = \frac{1}{2} v_a^2 - \frac{GM}{2R}$$

$$\frac{3}{2} v_a^2 = \frac{GM}{2R}$$

$$v_a = \sqrt{\frac{GM}{3R}}$$

This allows us to evaluate the energy and angular momentum.

$$L = L_a = r_a m_s v_a = 2m_s \sqrt{\frac{GMR}{3}}$$

$$E = E_a = \frac{1}{2} m_s v_a^2 - \frac{GMm_s}{r_a}$$

$$E = \frac{1}{2} m_s \left(\frac{GM}{3R} \right) - \frac{GMm_s}{2R}$$

$$E = -\frac{GMm_s}{3R}$$

9.4 FP18 (5 points)

9.4.a (1.5 points)

In the energy scale indicated, the gravitational potential energy for the satellite is $-GMm/R$. Meanwhile, the gravitational force that pulls the satellite towards the planet is GMm/R^2 , which must be equal to the centripetal force mu^2/R necessary to keep the satellite on its circular orbit. Therefore

$$u = \sqrt{\frac{GM}{R}}$$

which corresponds to a kinetic energy $mu^2/2 = GMm/(2R)$. The total energy of the satellite is

$$E = \frac{1}{2} mu^2 - \frac{GMm}{R} = -\frac{GMm}{2R}$$

The angular momentum is simply given by $L = mRu = m\sqrt{GMR}$.

9.4.b (1 point)

Immediately after the explosion the two hemispheres are still a distance R away from the center of the planet, and their velocities are still tangential. Therefore

$$L_B = \frac{m}{2} \frac{5u}{4} R = \frac{5}{8} muR$$

In order to compute L_A we must first find the velocity of A after the explosion. Since the explosion was internal, total linear momentum should have been conserved. This implies that the velocity of A must also be tangential to the original orbit of the satellite, and that its magnitude must be $3u/4$.

Then

$$L_A = \frac{m}{2} \frac{3u}{4} R = \frac{3}{8} muR$$

Notice that total angular momentum is also conserved, as expected.

The energy of A is

$$E_A = \frac{1}{2} \frac{m}{2} \left(\frac{3u}{4} \right)^2 - \frac{GMm}{2R} = \frac{GMm}{R} \left(\frac{9}{64} - \frac{1}{2} \right) = -\frac{23}{64} \frac{GMm}{R}$$

The energy of B is

$$E_B = \frac{1}{2} \frac{m}{2} \left(\frac{5u}{4} \right)^2 - \frac{GMm}{2R} = \frac{GMm}{R} \left(\frac{25}{64} - \frac{1}{2} \right) = -\frac{7}{64} \frac{GMm}{R}$$

9.4.c (1 point)

Only the kinetic energy of the satellite parts changes during the explosion. The initial kinetic energy is simply $mu^2/2$, while the final energy is $(5u/4)^2 m/4 + (3u/4)^2 m/4 = 17mu^2/32$. Therefore the energy of the system has increased by

$$\frac{mu^2}{32} = \frac{GMm}{32R}$$

which is the amount of work done by the explosion.

The same result may be obtained from $W = E_A + E_B - E$, using the results from parts (a) and (b).

9.4.d (0.5 point)

Using the energies calculated in part (b) and the equation for the length of the semi-major axis length in terms of the energy of the orbiting body, we obtain that

$$a_A = -\frac{GMm/2}{2E_A} = \frac{16R}{23}$$

and $a_B = 16R/7$

9.4.e (1 point)

Since the velocity of both A and B at the point of the explosion is entirely tangential, the orbits for the hemispheres must have either their apogee or their perigee at that location.

Notice that A has less energy than is necessary to keep it in the circular orbit of radius R . It will then describe a smaller ellipse with semi-major axis $a_A = 16R/23$. The planet will be at the focus furthest from the point of the explosion. (Notice that if the perigee distance of the orbit is smaller than the radius of the planet, A would crash into it.)

Meanwhile, B has more energy than necessary to keep it on the circular orbit, but not enough for it to escape, because the energy is still negative. Therefore it will describe an elliptical orbit with the planet at the focus nearest from the point of the explosion. The semi-major axis of this orbit is the a_B computed in part (d).

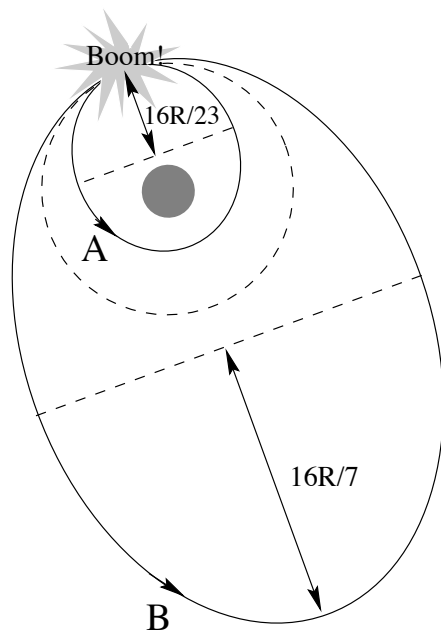


Figure 1: Sketch of the orbits of the hemispheres of the satellite after the explosion