Ph1a Homework Solution 4

Fall 2009

Each homework problem is worth 5 points. Please disregard the point values listed on the problem itself. Use these instead.

4.1 QP20 (5 points)

4.1.a (2.5 point)

At an instant t, the rope on the right hand side has length $l + x$, mass $m_1 = \rho(l + x)$ and momentum $p_1 = m_1 v$ downward, and the rope on the left hand side has length $l - x$, mass $m_2 = \rho(l - x)$ and momentum $p_2 = m_2 v$ upward. Note that v is defined to be $\frac{dx}{dt}$ and should be of the same magnitude along the rope. Similarly,

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

is the same along the rope.

Equation of motion of the R.H.S. and L.H.S. respectively are:

$$m_1 g - T_1 = \frac{dp_1}{dt} = m_1\frac{dv}{dt} + \frac{dm_1}{dt}v$$

and

$$T_2 - m_2 g = \frac{dp_2}{dt} = m_2\frac{dv}{dt} + \frac{dm_2}{dt}v$$

We have $\frac{dm_1}{dt} = -\frac{dm_2}{dt} = \rho v$, and since the nail is negligible, $T_1 = T_2 = T$ So the equations become:

$$\rho(l + x)g - T = \rho(l + x)a + \rho v^2$$

and

$$T - \rho(l - x)g = \rho(l - x)a - \rho v^2$$

Adding these two equations, you'll get $a = \frac{g}{l}x$. So $\frac{g}{l}x\,dx = v\,dv$, $\frac{g}{l}x^2 = v^2$. (At $t = 0$, $x = 0$, $t = 0$.) Use the result for v^2 in the equation,

$$T = \rho(l - x)g - \rho(l - x)a - \rho v^2 = \rho g(l - 2\frac{x^2}{l})$$

The normal force N from the nail to the infinitesimal piece of rope in contact with it balances $2T$, and this is the external force on the whole system besides gravitational force since the tension on the rope cancels. So

$$F_{\text{net}} = 2\rho g l - N = 2\rho g l - 2T = 4\frac{\rho g}{l}x^2$$

You can check that $F_{\text{net}} = \frac{d(m_1 v - m_2 v)}{dt}$ by inserting the result for v in it.

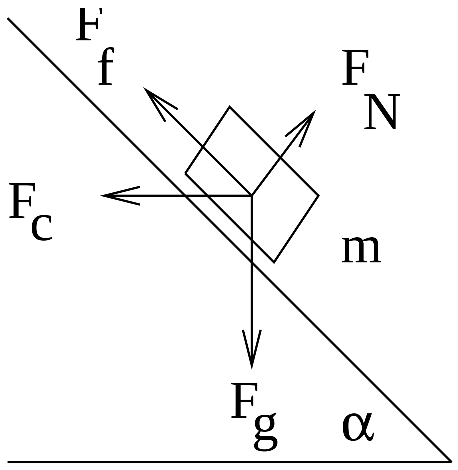
4.1.b (2.5 points)

Since at any instant t, $\frac{g}{l}x^2 = v^2$, without considering the rope's rotation around the nail in the limiting case that the nail size is negligible. When the rope comes off completely, we have $x = l$ and thus $v = \sqrt{gl}$.

4.2 QP21 (5 points)

4.2.a (2 point)

We consider the block in the rotating frame of the turntable.



Note that the frictional force could act in either direction, to oppose the motion of the block. Since in part (b) below, we are interested in the case when the block might slide down the plane, friction is drawn acting upwards.

4.2.b (3 points)

Find an expression for the minimum angular velocity, ω_c, to keep the block from sliding down the plane, in terms of g, r, μ, and the angle of the plane α.

The Newton's Law's equations in equilibrium give:

$$ma_{||} = 0 = F_g \sin\alpha - F_f - F_c \cos\alpha$$

$$ma_{\perp} = 0 = F_N - F_g \cos\alpha - F_c \sin\alpha$$

Solving the perpendicular equation gives

$$F_N = mg\cos\alpha + m\omega^2 r \sin\alpha$$

For equilibrium, we know $F_f \leq \mu F_N$, with the inequality since the full value of μ may not be required to keep the block in place. Considering the parallel equation, we see:

$$F_g \sin\alpha - F_c \cos\alpha \leq F_f = \mu F_N$$

$$mg\sin\alpha - m\omega^2 r\cos\alpha \leq \mu mg\cos\alpha + \mu m\omega^2 r \sin\alpha$$

$$\omega \geq \sqrt{\frac{g}{r}\left(\frac{\sin\alpha - \mu\cos\alpha}{\cos\alpha + \mu\sin\alpha}\right)}$$

To get the minimum value of ω_c, we realized that friction will be providing its maximum force, and hence the critical ω occurs at equality.

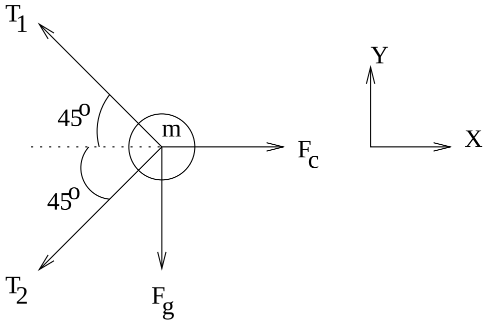
$$\omega_c = \sqrt{\frac{g}{r}\left(\frac{\sin\alpha - \mu\cos\alpha}{\cos\alpha + \mu\sin\alpha}\right)}$$

As an aside, this equation only makes sense for certain values of α and μ. In particular, if $\mu \geq \tan\alpha$, the block will stay put without any rotation at all, so then $\omega_c = 0$.

4.3 QP28 (5 points)

4.3.a (1.5 points)

In the rotating frame of the mass, the free body diagram is :



4.3.b (2 point)

Using Newton's Laws (in the rotating frame of the mass) and the above picture, we get:

$$X: \quad F_c - T_1 \cos 45 - T_2 \cos 45 = ma_x = 0$$

$$Y: \quad T_1 \cos 45 - T_2 \cos 45 - F_g = ma_y = 0$$

Using $F_c = m\omega^2 R = m\omega^2 L/\sqrt{2}$ and simplifying gives:

$$T_1 + T_2 = m\omega^2 L$$

$$T_1 - T_2 = \sqrt{2}mg$$

Adding the equations gives the solutions for T_1 and T_2.

$$T_1 = \frac{m}{2}\left(\omega^2 L + \sqrt{2}g\right)$$

$$T_2 = \frac{m}{2}\left(\omega^2 L - \sqrt{2}g\right)$$

4.3.c (1.5 point)

A string remains taut as long as $T > 0$. Since $T_1 > 0$, we only have to check T_2:

$$T_2 = \frac{m}{2}\left(\omega^2 L - \sqrt{2}g\right) > 0$$

$$\omega^2 L > \sqrt{2}g$$

$$\omega > \omega_{\min} = \sqrt{\frac{\sqrt{2}g}{L}}$$

For values of $\omega < \omega_{\min}$, the lower string becomes slack, and the angle of the top string will no longer be 45^o. This reduces to the problem of uniform rotation with the single upper string.

4.4 Frautschi 9.6 (5 points)

4.4.a (3 point)

Let $R = 10$ m and ω be the angular velociy. The length of the day is the period T.

$$T = \frac{2\pi}{\omega}$$

$$R\omega^2 = g$$

So

$$T = \frac{2\pi}{\sqrt{g/R}} = 2\pi \approx 6.28 \qquad second$$

4.4.b (2 point)

Let $h = 1.8$ m. Since $g_{head} = (R-h)\omega^2$, the difference is

$$g - g_{head} = h\omega^2 = 1.8m/s^2$$

.