Ph1a Homework Solution 3

Fall 2009

Each homework problem is worth 5 points. Please disregard the point values listed on the problem itself. Use these instead.

3.1 QP3 (5 points)

3.1.a (0.5 points)

In terms of the lengths in the diagram, we have:

$$l_1 = (p_1 - x_1) + \pi R + (p_1 - p_2) = 2p_1 - x_1 - p_2 + \pi R$$
$$l_2 = p_2 + \pi R + (p_2 - x_2) = 2p_2 - x_2 + \pi R$$

3.1.b (1 point)

The free body diagrams are in Fig. 1.

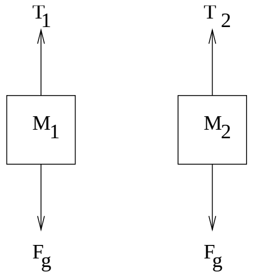

Figure 1: QP3.b

In the vertical direction, we have Newton's Laws:

$$m_1 a_1 = T_1 - m_1 g$$
$$m_2 a_2 = T_2 - m_2 g$$

3.1.c (1 point)

Taking two time derivatives of the equations in part a gives:

$$\frac{d^2 l_1}{dt^2} = 0 = -a_1 - \frac{d^2 p_2}{dt^2}$$
$$\frac{d^2 l_2}{dt^2} = 0 = 2\frac{d^2 p_2}{dt^2} - a_2$$

From this, we get $a_2 = -2a_1$, or $a_1 = ka_2$ where $k = -\frac{1}{2}$.

3.1.d (1 point)

We can get the relationship between the tensions by considering Newton's Laws applied to the (massless) 2$^{\text{nd}}$ pulley. For massless objects, the condition is that net force must be zero to avoid the unphysical result of infinite acceleration.

The force diagram is Fig. 2

$$m_{P2} a_{P2} = T_1 - 2T_2 - m_{P2} g$$

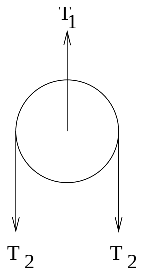

Figure 2: QP3.d

$$T_1 - 2T_2 = 0$$

So we get $T_1 = 2T_2$, or $T_1 = k_2 T_2$ with $k_2 = 2$.

3.1.e (1 point)

Solving for a_1 and T_2 is straight-forward:

$$T_2 = m_2\left(\frac{a_1}{k}\right) + m_2 g = m_2\left(\frac{a_1}{k} + g\right)$$

$$m_1 a_1 = (k_2 T_2) - m_1 g$$

$$m_1 a_1 = m_2 k_2 (\frac{a_1}{k} + g) - m_1 g$$

$$a_1 = \frac{(m_2 k_2 - m_1) k g}{k m_1 - k_2 m_2}$$

$$T_2 = \frac{m_1 m_2 g (k-1)}{k m_1 - k_2 m_2}$$

Using the correct values of $k = -1/2$ and $k_2 = 2$, we get

$$a_1 = \frac{2m_2 - m_1}{m_1 + 4m_2} g \qquad T_2 = \frac{3 m_1 m_2 g}{m_1 + 4m_2}$$

Using the incorrect assumption of $k = -1$ and $k_2 = 1/3$, we get

$$a_1 = \frac{m_2 - 3m_1}{3m_1 + m_2} g \qquad T_2 = \frac{6 m_1 m_2 g}{3m_1 + m_2}$$

3.1.f (0.5 point)

If $m_1 = m_2$ the formula for a_2 reduces to:

$$a_2 = \frac{a_1}{k} = \frac{(k_2 - 1) g}{k - k_2}$$

Thus $a_2 = -\frac{2}{5} g$ (or $a_2 = \frac{1}{2} g$ under the wrong assumption). The condition for no acceleration of the masses is that a_1 (and hence a_2 also) be zero.

$$a_1 = \frac{2m_2 - m_1}{m_1 + 4m_2} g = 0 \quad \Rightarrow \quad m_1 = 2m_2$$

Alternatively, under the wrong assumption:

$$a_1 = \frac{m_2 - 3m_1}{3m_1 + m_2} g = 0 \quad \Rightarrow \quad m_1 = \frac{1}{3} m_2$$

3.2 QP4 (5 points)

3.2.a (1.5 points)

The free body diagrams are Fig. 3. The normal forces N_{ij} are the force of the i^{th} block on the j^{th} block, or just N_i for the force of the ground on block i.

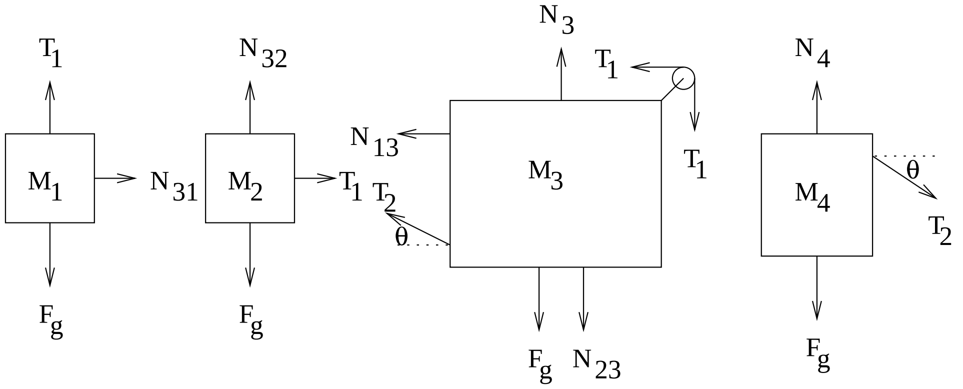

Figure 3: QP4.a

3.2.b (1 point)

Newton's Second Law for m_1 and m_2:

$$m_1 a_{1X} = N_{31} \qquad m_2 a_{2X} = T_1$$

$$m_1 a_{1Y} = T_1 - m_1 g \qquad m_2 a_{2Y} = N_{32} - m_2 g$$

3.2.c (1 point)

Since there is no motion of m_2 relative to m_3, we must have $a_{2X} = a_{3X}$. Also, we know $a_{1Y} = 0$. Thus:

$$T_1 = m_1 g$$

$$a_{3X} = a_{2X} = \frac{T_1}{m_2} = \frac{m_1}{m_2} g$$

3.2.d (0.5 points)

The whole system moves with the same horizontal acceleration, $a = a_{3X}$. The only external force is F, so we have:

$$F = m_{\text{sys}} a_{\text{sys}}$$

$$F = (m_1 + m_2 + m_3 + m_4) \frac{m_1}{m_2} g$$

3.2.e (0.5 points)

Newton's Second Law for m_4 yields:

$$m_4 a_{4X} = m_4 a = T_2 \cos\theta \quad \Rightarrow \quad T_2 = \frac{m_4 a}{\cos\theta}$$

$$m_4 a_{4Y} = 0 = N_4 - m_4 g - T_2 \sin\theta$$

$$N_4 = m_4 g + T_2 \sin\theta$$

$$N_4 = m_4(g + a\tan\theta) = m_4 g\left(1 + \frac{m_1}{m_2}\tan\theta\right)$$

3.2.f (0.5 point)

Since pulley is massless, the net force is zero. See Fig. 4.

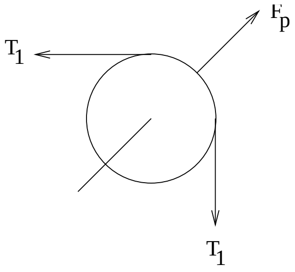

Figure 4: QP4.f

Clearly, $\vec{\mathbf{F}}_\mathbf{P} = T_1(\hat{\mathbf{x}} + \hat{\mathbf{y}})$. So $|F_P| = \sqrt{2}T_1 = \sqrt{2}m_1 g$, and it is directed at $\phi = \tan^{-1}[F_{PY}/F_{PX}] = 45°$.

3.3 Frautschi 7.16 (5 points)

3.3.a (2 points)

Let $\vec{F}_{12}$ be the gravitational force m_1 feels because of m_2, $\vec{F}_{13}$ be the gravitational force m_1 feels because of m_3, and $\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$ be the total force. Use the coordinate system as in the problem,

$$\vec{F}_{12} = Gm_1m_2\left(-\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)/a^2$$

$$\vec{F}_{13} = Gm_1m_3\left(\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)/a^2$$

$$\vec{F}_1 = Gm_1\left((-m_2+m_3)\hat{i} - \sqrt{3}(m_2+m_3)\hat{j}\right)/2a^2$$

Similarly for m_2,

$$\vec{F}_{21} = Gm_2m_1\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)/a^2$$

$$\vec{F}_{23} = Gm_2m_3\hat{i}/a^2$$

$$\vec{F}_2 = Gm_2\left((\frac{m_1}{2} + m_3)\hat{i} + \frac{\sqrt{3}}{2}m_1\hat{j}\right)/a^2$$

and m_3,

$$\vec{F}_{31} = Gm_3m_1\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)/a^2$$

$$\vec{F}_{32} = Gm_3m_2(-\hat{i})/a^2$$

$$\vec{F}_3 = Gm_3\left((-\frac{m_1}{2} - m_2)\hat{i} + \frac{\sqrt{3}}{2}m_1\hat{j}\right)/a^2$$

3.3.b (1.5 points)

Let the position of m_1 be p_1 and similarly for p_2 and p_3. Call the point through which the axis should pass be Z, the center of mass of the whole system be M, and the center of mass of m_2 and m_3 be M_{23}. Notice that the three points, p_1, M, and M_{23} must line up, so $\overrightarrow{p_1M_{23}}//\overrightarrow{p_1M}$.

Let the centripetal force of m_1 be $\vec{F}_1^c$. We want to use the condition that $\vec{F}_1^c$ is parallel with $\vec{F}_1$, denoted again by $\vec{F}_1^c//\vec{F}_1$, to find out where Z is. We can prove that Z=M, because we know for this special case the three masses are arranged equidistance to each other,

$$\vec{F}_1//\overrightarrow{p_1M_{23}}$$

$$\vec{F}_1^c//\overrightarrow{p_1Z}$$

So that Z must lie on the line $\overrightarrow{p_1M_{23}}$.

Use similar reasoning for m_2 and m_3, we conclude that Z must lie on both $\overrightarrow{p_2M_{13}}$ and $\overrightarrow{p_3M_{12}}$. The intersection of the three lines, $\overrightarrow{p_1M_{23}}$, $\overrightarrow{p_2M_{13}}$, and $\overrightarrow{p_3M_{12}}$, is the center of mass, M. So Z=M.

If you like, you can find the equation of the line p_1M, which has the slope of $-\sqrt{3}\frac{m_2+m_3}{m_3-m_2}$, as can be seen from the expression of $\vec{F_1}$ in part(a), and it passes the point $p_1 : (0, \frac{\sqrt{3}a}{2})$. So

$$p_1M : y - \frac{\sqrt{3}a}{2} = -\sqrt{3}\frac{m_2+m_3}{m_3-m_2}x$$

Similarly for the equation of line p_2M and p_3M,

$$p_2M : y = \sqrt{3}\frac{m_1}{m_1+2m_3}(x+a/2)$$

$$p_3M : y = -\sqrt{3}\frac{m_1}{m_1+2m_2}(x-a/2)$$

And the intersection is $\left(\frac{a(m_3-m_2)}{2(m_1+m_2+m_3)}, \frac{\sqrt{3}am_1}{2(m_1+m_2+m_3)}\right)$

Later, in Chapter 11.3, we will recognize this as the center of mass.

3.3.c (1.5 points)

Let $\vec{r}_M$ be the position vector of the center of mass and $vecr_1$ be the position vector of m_1, then

$$\vec{F}_1^c = m_1\omega^2(\vec{r}_M - \vec{r}_1)$$

Using the $x-y$ coordinate, we get

$$\vec{r}_M - \vec{r}_1 = \frac{a(m_3-m_2)}{2(m_1+m_2+m_3)}\hat{i} + \frac{\sqrt{3}am_1}{2(m_1+m_2+m_3)}\hat{j} - \frac{\sqrt{3}a}{2}\hat{j},$$

So

$$\vec{F}_1^c = m_1\omega^2\Big(\frac{a(m_3-m_2)}{2(m_1+m_2+m_3)}\hat{i} + (\frac{\sqrt{3}am_1}{2(m_1+m_2+m_3)} - \frac{\sqrt{3}a}{2})\hat{j}\Big)$$

Since $\vec{F}_1^c = \vec{F}_1$, we can compare with the expression for $\vec{F}_1$ in part(a) to get

$$\frac{\omega^2 a}{m_1+m_2+m_3} = \frac{G}{a^2}$$

Therefore,

$$\omega = \sqrt{\frac{G}{a^3}m_1+m_2+m_3}$$

The same answer should be derived from $\vec{F}_i^c = \vec{F}_i$ for $i = 2, 3$.

3.4 Frautschi 7.17 (5 points)

3.4.a (1.5 point)

Let the tension at the lowest point be T_b, and it is positive when pointing inward, or upward in this case. Also let the mass be m, and the radius be r. We have

$$T_b - mg = m\frac{v^2}{r}$$

$$T_b = mg + m\frac{v^2}{r}$$

$$T_b = 0.5 \times \big(9.8 + 3^2/1\big) = 9.4 Newton$$

3.4.b (1.5 point)

Let the tension at the top be T_t and it is positive when pointing inward, or downward in this case.

$$T_t + mg = m\frac{v^2}{r}$$

$$T_t = -mg + m\frac{v^2}{r}$$

Since it moves with constant speed, then T_b has a larger absolute value then T_t.

$$|T_b| > |T_t|$$

Therefore, the rod is more likely to break when the mass is at the bottom.

3.4.c (2 point)

If the rod remains under tension at the top, that means it is pointing inward, or downdard. So T_t has to be positive.

$$\begin{aligned} T_t = -mg + m\frac{v^2}{r} &\geq 0 \\ \frac{v^2}{r} &\geq g \\ v \geq \sqrt{rg} = \sqrt{1 \times 9.8} &= 3.13m/s \end{aligned}$$

So the minimum speed is $3.13m/s$