Erasure Coding for Real-Time Streaming

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***Abstract*—We consider a real-time streaming system where messages are created sequentially at the source, and are encoded for transmission over a packet erasure channel. Each message must subsequently be decoded at the receiver within a given delay from its creation time. We consider code design and maximum message rates when all messages must be decodable by their respective deadlines under a specified set of erasure patterns (erasure model). Specifically, we provide a code construction that achieves the optimal rate for an asymptotic number of messages, under erasure models containing a limited number of erasures per coding window, per sliding window, and containing erasure bursts of a limited length.**

I. Introduction

We consider packet erasure correction coding for a real-time streaming system where messages are created sequentially at the source, and each message must be decoded at the receiver within a given delay from its creation time. The coding scheme is designed to ensure recovery of messages under a given set of possible erasure patterns (erasure model).

In particular, we consider three erasure models: the first model considers a limited number of erasures in each coding window, the second considers a limited number of erasures in each sliding window, while the third considers erasure bursts of a limited length. For each erasure model, the objective is to find an optimal code that achieves the maximum message size among all codes that allow all messages to be decoded by their respective deadlines under all admissible erasure patterns.

We present an explicit intrasession code construction which specifies an allocation of each packet's capacity among the different messages; coding occurs within each message but not across messages. Intrasession coding is attractive due to its relative simplicity, but it is not known in general when intrasession coding suffices or when intersession coding is needed. We show that for an asymptotic number of messages, our code construction achieves the optimal rate among all codes (intrasession or intersession) for the first and second erasure models with any given number of erasures per window, and for the third erasure model with sufficiently short or long erasure bursts.

In related work, Martinian *et al.* [1], [2] provided constructions of streaming codes that minimize the delay required to correct erasure bursts of a given length. Tekin *et al.* [3] considered erasure correction coding for a non-real-time streaming system where all messages are initially present at the encoder.

We begin with a formal definition of the problem in Section II, and proceed to describe the construction of our intrasession code in Section III. We then demonstrate the optimality of this code under each erasure model in the subsequent sections. Proofs of some theorems can be found in Appendix B; complete proofs are deferred to the extended paper [4].

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II. Problem Definition

Consider a discrete-time data streaming system comprising a source and a receiver, with a directed link of normalized unit capacity from the source to the receiver. The source creates independent messages of uniform size $s > 0$ at regular intervals of $c \in \mathbb{Z}^+$ time steps, and is allowed to transmit at most a unit amount of coded data over the link at each time step. The receiver attempts to decode each message within $d \in \mathbb{Z}^+$ time steps of its creation.

More precisely, each message $k \in \mathbb{Z}^+$ is created at time step $(k-1)c+1$, and is to be decoded by time step $(k-1)c+d$. The coded data transmitted at each time step $t \in \mathbb{Z}^+$ must be a function of messages created at time step t or earlier. Let coding window W_k be the interval of d time steps between the creation time and the decoding deadline of message k, i.e.,

$$W_k \triangleq \{(k-1)c+1, \ldots, (k-1)c+d\}.$$

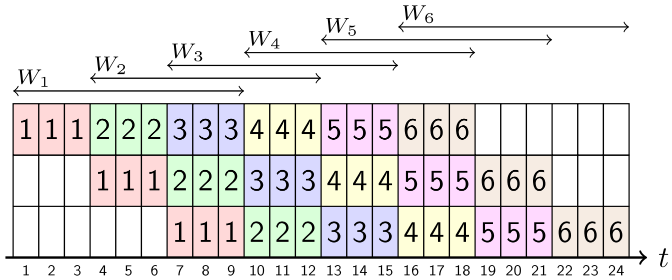
We shall assume that $d > c$ so as to avoid the degenerate case of nonoverlapping coding windows for which it is sufficient to code individual messages separately.

Consider the first n messages $\{1, \ldots, n\}$, and the union of their (overlapping) coding windows T_n, given by

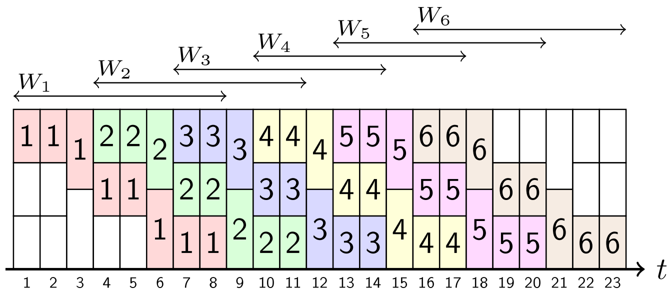
$$T_n \triangleq W_1 \cup \cdots \cup W_n = \{1, \ldots, (n-1)c+d\}.$$

An erasure pattern $E \subseteq T_n$ specifies the set of erased or lost data transmissions over the link. More precisely, if $t \in E$, then none of the data transmitted at time step t is received by the receiver; if $t \in T_n \backslash E$, then all of the data transmitted at time step t is received by the receiver at time step t. An erasure model essentially describes a distribution of erasure patterns.

For a given pair of positive integers a and b, we define the offset quotient $q_{a,b}$ and remainder $r_{a,b}$ to be the unique



(a) $(c,d) = (3,9)$



(b) $(c,d) = (3,8)$

Fig. 1. Allocation of link capacity at each time step t, in the encoding of messages $\{1,\ldots,6\}$, for (a) $(c,d)=(3,9)$ and (b) $(c,d)=(3,8)$. Each message is assigned a unique color. In (a), because d is a multiple of c, we have $q_{d,c}+1=3$ active messages at each time step. In (b), because d is not a multiple of c, we have either $q_{d,c}=2$ or $q_{d,c}+1=3$ active messages at each time step.

integers satisfying the following three conditions:

$$a = q_{a,b}\, b + r_{a,b}, \qquad q_{a,b} \in \mathbb{Z}_0^+, \qquad r_{a,b} \in \{1,\ldots,b\}.$$

Note that our definition departs from the usual definition of quotient and remainder in that $r_{a,b}$ can be equal to b but not 0.

III. Code Construction

We present an intrasession code which codes only within each message and not across different messages. We begin by specifying the amount of link capacity allocated for the encoding of each message at each time step. An appropriate code (e.g., random linear coding, MDS code) is then applied to the allocation so that each message can be decoded whenever the total amount of received data that encodes that message is at least the message size s.

The allocation of link capacity follows a simple rule: the link capacity at each time step is divided evenly among all *active* messages. We say that message k is active at time step t if and only if t falls within its coding window, i.e., $t \in W_k$. Fig. 1 shows how much link capacity at each time step is allocated to each message, for two instances of (c,d).

For a given choice of c and d, the messages that are encoded at a given time step $t \in \mathbb{Z}^+$ can be stated explicitly as follows: First, we define A_t to be the set of active messages at time step t, i.e.,

$$\begin{aligned} A_t &\triangleq \{k \in \mathbb{Z}^+ : t \in W_k\} \\ &= \{k \in \mathbb{Z}^+ : (k-1)c+1 \le t \le (k-1)c+d\} \\ &= \left\{k \in \mathbb{Z}^+ : \frac{t-d}{c}+1 \le k \le \frac{t-1}{c}+1\right\}. \end{aligned}$$

Treating nonpositive messages $0,-1,-2,\ldots$ as dummy messages, we can write

$$A_t = \left\{\left\lceil \frac{t-d}{c}+1\right\rceil, \ldots, \left\lfloor \frac{t-1}{c}+1\right\rfloor\right\}.$$

Expressing this in terms of $q_{d,c}$, $r_{d,c}$, $q_{t,c}$, $r_{t,c}$ yields

$$A_t = \left\{q_{t,c}+1-q_{d,c}+\left\lceil\frac{r_{t,c}-r_{d,c}}{c}\right\rceil, \ldots, q_{t,c}+1\right\}.$$

It follows that the number of active messages $|A_t|$ varies over time depending on the value of $r_{t,c}$; specifically, two cases are possible:

Case 1: If $r_{t,c} \le r_{d,c}$, then

$$-1 < \frac{1-c}{c} \le \frac{r_{t,c}-r_{d,c}}{c} \le 0,$$

which implies that $\left\lceil\frac{r_{t,c}-r_{d,c}}{c}\right\rceil = 0$, and

$$A_t = \{q_{t,c}+1-q_{d,c}, \ldots, q_{t,c}+1\}.$$

The $q_{d,c}+1$ messages of A_t are therefore encoded at time step t, with each message allocated $\frac{1}{q_{d,c}+1}$ amount of link capacity.

Case 2: If $r_{t,c} > r_{d,c}$, then

$$0 < \frac{r_{t,c}-r_{d,c}}{c} \le \frac{c-1}{c} < 1,$$

which implies that $\left\lceil\frac{r_{t,c}-r_{d,c}}{c}\right\rceil = 1$, and

$$A_t = \{q_{t,c}+1-(q_{d,c}-1), \ldots, q_{t,c}+1\}.$$

The $q_{d,c}$ messages of A_t are therefore encoded at time step t, with each message allocated $\frac{1}{q_{d,c}}$ amount of link capacity.

Note that when d is a multiple of c, we have $r_{t,c} \le r_{d,c} = c$ for any t, which implies that $q_{d,c}+1$ messages are encoded at every time step.

In our subsequent performance analysis of this code, we make repeated use of two key code properties; these are presented as technical lemmas in Appendix A.

IV. Performance under z Erasures per Coding Window

For the first erasure model, we look at erasure patterns that have a limited number of erasures per coding window. Consider the first n messages $\{1,\ldots,n\}$, and the union of their (overlapping) coding windows T_n. Let $\mathcal{E}_n^{\text{cw}}$ be the set of erasure patterns that have z or fewer erasures in each coding window W_k, i.e.,

$$\mathcal{E}_n^{\text{cw}} \triangleq \left\{E \subseteq T_n : |E \cap W_k| \le z \;\forall\, k \in \{1,\ldots,n\}\right\}.$$

The objective is to construct a code that allows all n messages $\{1,\ldots,n\}$ to be decoded by their respective deadlines under any erasure pattern $E \in \mathcal{E}_n^{\text{cw}}$. Let s_n^{cw} be the maximum message size that can be achieved by such a code, for a given choice of n, c, d, and z.

We observe that over a finite time horizon (i.e., when the number of messages n is finite), intrasession coding can be suboptimal. For example, given $(n, c, d, z) = (3, 1, 3, 1)$, the optimal intrasession code (which can be found by solving a linear program) achieves a message size of $\frac{6}{7}$, which is strictly smaller than the maximum achievable message size s_n^{CW} of 1 obtained by intersession coding, e.g., transmitting messages $(1, 2, 1{\oplus}2, 3, 3)$ at time steps $(1, 2, 3, 4, 5)$, respectively.

However, it turns out that the intrasession code constructed in Section III is *asymptotically* optimal; the gap between the maximum achievable message size s_n^{CW} and the message size achieved by the code vanishes as the number of messages n goes to infinity:

Theorem 1. *The code constructed in Section III is asymptotically optimal in the following sense: the code achieves a message size of*

$$\sum_{j=1}^{d-z} y_j,$$

which is equal to the asymptotic maximum achievable message size $\lim_{n\to\infty} s_n^{\text{CW}}$, *where* $\mathbf{y} = (y_1, \ldots, y_d)$ *is defined as*

$$\mathbf{y} \triangleq \Bigg(\overbrace{\underbrace{\frac{1}{q_{d,c}+1}, \ldots, \frac{1}{q_{d,c}+1}}_{(q_{d,c}+1)r_{d,c} \text{ entries}}, \underbrace{\frac{1}{q_{d,c}}, \ldots, \frac{1}{q_{d,c}}}_{q_{d,c}(c-r_{d,c}) \text{ entries}}}^{d \text{ entries}} \Bigg).$$

We prove the above theorem by considering a cut-set bound corresponding to a specific "worst-case" erasure pattern in which exactly z erasures occur in every coding window. This erasure pattern is chosen with the help of Lemma 2; specifically, the erased time steps are chosen to coincide with the larger blocks allocated to each message in the constructed code.

V. Performance under z Erasures per Sliding Window of d Time Steps

For the second erasure model, we look at erasure patterns that have a limited number of erasures per sliding window of d time steps. Consider the first n messages $\{1, \ldots, n\}$, and the union of their (overlapping) coding windows T_n. Let sliding window L_t denote the interval of d time steps beginning at time step t, i.e.,

$$L_t \triangleq \{t, \ldots, t+d-1\}.$$

Let $\mathcal{E}_n^{\text{SW}}$ be the set of erasure patterns that have z or fewer erasures in each sliding window L_t, i.e.,

$$\mathcal{E}_n^{\text{SW}} \triangleq \big\{ E \subseteq T_n : |E \cap L_t| \leq z \;\forall\, t \in \{1, \ldots, (n-1)c+1\} \big\}.$$

The objective is to construct a code that allows all n messages $\{1, \ldots, n\}$ to be decoded by their respective deadlines under any erasure pattern $E \in \mathcal{E}_n^{\text{SW}}$. Let s_n^{SW} be the maximum message size that can be achieved by such a code, for a given choice of n, c, d, and z.

We note that since $\mathcal{E}_n^{\text{SW}} \subseteq \mathcal{E}_n^{\text{CW}}$, we therefore have $s_n^{\text{SW}} \geq s_n^{\text{CW}}$. For the special case of $c = 1$, each sliding window is also a coding window, and so this sliding window erasure model reduces to the coding window erasure model of Section IV, i.e., $\mathcal{E}_n^{\text{SW}} = \mathcal{E}_n^{\text{CW}}$. Over a finite time horizon, intrasession coding can also be suboptimal for this erasure model; the illustrating example from Section IV applies here as well.

Surprisingly, the constructed intrasession code also turns out to be asymptotically optimal over all codes; the omission of erasure patterns in $\mathcal{E}_n^{\text{SW}}$ compared to $\mathcal{E}_n^{\text{CW}}$ has not led to an increase in the maximum achievable message size (cf. Theorem 1):

Theorem 2. *The code constructed in Section III is asymptotically optimal in the following sense: the code achieves a message size of*

$$\sum_{j=1}^{d-z} y_j,$$

which is equal to the asymptotic maximum achievable message size $\lim_{n\to\infty} s_n^{\text{SW}}$.

Proving the above theorem requires a different approach from that of Theorem 1. When d is a multiple of c, we need only consider a cut-set bound corresponding to an obvious "worst-case" erasure pattern in which exactly z erasures occur in every sliding window, specifically, a periodic erasure pattern with alternating intervals of z erasures and $d - z$ nonerasures. When d is not a multiple of c, we consider a "base" erasure pattern E' chosen with the help of Lemma 2, and inductively compute an upper bound for the conditional entropy

$$H\big(X[W_n \cap E'] \,\big|\, M[\{1, \ldots, n\}], X[\{1, \ldots, (n-1)c\}]\big)$$

by using erasure patterns derived from the base, where X_t is a random variable representing the coded data transmitted at time step t, M_k is a random variable representing message k, $X[A] \triangleq (X_t)_{t\in A}$, and $M[A] \triangleq (M_k)_{k\in A}$. The nonnegativity of this expression leads us to a bound for s_n^{SW} that matches the message size achieved by the constructed code in the limit $n \to \infty$.

VI. Performance under Erasure Bursts of z Time Steps

For the third erasure model, we look at erasure patterns that contain erasure bursts of a limited number of time steps. Consider the first n messages $\{1, \ldots, n\}$, and the union of their (overlapping) coding windows T_n. Let $\mathcal{E}_n^{\text{B}}$ be the set of erasure patterns in which each erasure burst is z or fewer time steps in duration, and consecutive bursts are separated by a gap of $d - z$ or more time steps, i.e.,

$$\mathcal{E}_n^{\text{B}} \triangleq \Big\{ E \subseteq T_n : \\ (t{\in}E \;\wedge\; t{+}1{\notin}E) \Rightarrow |E \cap \{t{+}1, \ldots, t{+}d{-}z\}| = 0, \\ (t{\notin}E \;\wedge\; t{+}1{\in}E) \Rightarrow |E \cap \{t{+}1, \ldots, t{+}z{+}1\}| \leq z \Big\}.$$

The objective is to construct a code that allows all n messages $\{1,\dots,n\}$ to be decoded by their respective deadlines under any erasure pattern $E \in \mathcal{E}_n^{\mathrm{B}}$. Let s_n^{B} be the maximum message size that can be achieved by such a code, for a given choice of n, c, d, and z.

Using the proof technique of Theorem 2, we can show that the constructed intrasession code is asymptotically optimal when the erasure bursts are sufficiently short or long:

Theorem 3. *If*

$$z \le c - r_{d,c}$$

or

$$z \ge q_{d,c}\, c = d - r_{d,c},$$

then the code constructed in Section III is asymptotically optimal in the following sense: the code achieves a message size of

$$\sum_{j=1}^{d-z} y_j,$$

which is equal to the asymptotic maximum achievable message size $\lim_{n\to\infty} s_n^{\mathrm{B}}$.

Appendix A
Code Properties

The first property describes when it is possible to decode each message:

Lemma 1 (Message Decodability)**.** *Consider the code constructed in Section III for a given choice of c and d. If message size s satisfies the inequality*

$$s \le \sum_{j=1}^{\ell} y_j,$$

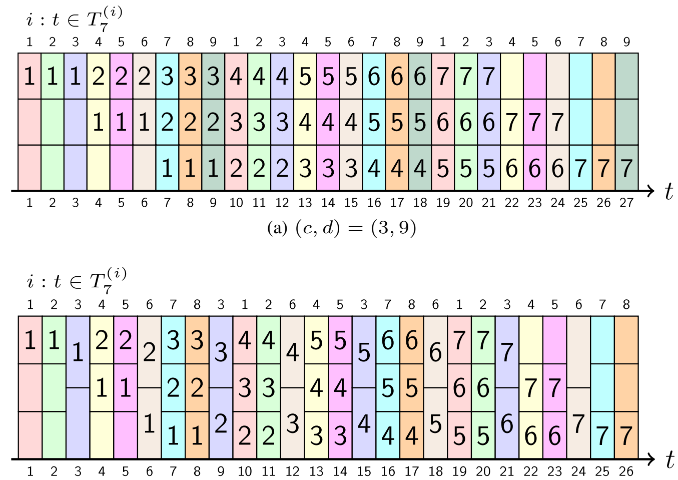
where $\mathbf{y} = (y_1,\dots,y_d)$ is as defined in Theorem 1, then each message $k \in \mathbb{Z}^+$ can be decoded from the data at any ℓ time steps in its coding window W_k.

The second property describes a way of partitioning time steps into sets with certain specific properties, which are used in our specification of the worst-case erasure patterns:

Lemma 2 (Partition of Coding Windows)**.** *Consider the code constructed in Section III for a given choice of c and d. Consider the first n messages $\{1,\dots,n\}$, and the union of their (overlapping) coding windows T_n. The set of time steps T_n can be partitioned into d sets $T_n^{(1)},\dots,T_n^{(d)}$, given by*

$$T_n^{(i)} \triangleq \begin{cases} \Big\{\big(j(q_{d,c}+1)+q_{i,c}\big)c + r_{i,c} \in T_n : j \in \mathbb{Z}_0^+\Big\} & \\ \qquad\qquad\qquad\qquad \text{if } r_{i,c} \le r_{d,c}, & \\ \Big\{\big(j\,q_{d,c}+q_{i,c}\big)c + r_{i,c} \qquad \in T_n : j \in \mathbb{Z}_0^+\Big\} & \\ \qquad\qquad\qquad\qquad \text{if } r_{i,c} > r_{d,c}, & \end{cases}$$

with the following properties:



(a) $(c,d)=(3,9)$

(b) $(c,d)=(3,8)$

Fig. 2. Partitioning of the set of time steps T_n into the d sets $T_n^{(1)},\dots,T_n^{(d)}$, in the encoding of messages $\{1,\dots,7\}$, for (a) $(c,d)=(3,9)$ and (b) $(c,d)=(3,8)$. Each set $T_n^{(i)}$ is assigned a unique color. The number i at the top of each time step t indicates the set $T_n^{(i)}$ to which t belongs.

(i) *Over the time steps in the set $T_n^{(i)}$, each message $k \in \{1,\dots,n\}$ is allocated $\frac{1}{q_{d,c}+1}$ amount of link capacity if $r_{i,c} \le r_{d,c}$, and $\frac{1}{q_{d,c}}$ amount of link capacity if $r_{i,c} > r_{d,c}$.*

(ii) *The allocated link capacity in $T_n^{(i)}$ for each message $k \in \{1,\dots,n\}$ is contained within a single time step (as opposed to being spread over multiple time steps).*

(iii) *The total amount of link capacity over all time steps in $T_n^{(i)}$, i.e., $\big|T_n^{(i)}\big|$, has the following upper bound:*

$$\big|T_n^{(i)}\big| < \begin{cases} \dfrac{n}{q_{d,c}+1} + 2 & \text{if } r_{i,c} \le r_{d,c}, \\[2ex] \dfrac{n}{q_{d,c}} + 2 & \text{if } r_{i,c} > r_{d,c}. \end{cases}$$

Fig. 2 shows how the set of time steps T_n is partitioned into the d sets $T_n^{(1)},\dots,T_n^{(d)}$, for two instances of (c,d).

Appendix B
Proofs of Theorems

Proof of Lemma 1: Consider a given message $k \in \mathbb{Z}^+$ and its coding window

$$W_k = \big\{(k-1)c + i : i \in \{1,\dots,d\}\big\}.$$

At each time step $t \in W_k$, message k is allocated either $\frac{1}{q_{d,c}+1}$ or $\frac{1}{q_{d,c}}$ amount of link capacity; at all other time steps $t \notin W_k$, message k is allocated zero link capacity.

Let x_i be the amount of link capacity at time step $t = (k-1)c + i$ that is allocated to message k. Writing t in terms of $q_{i,c}$ and $r_{i,c}$ produces

$$t = (k-1)c + i = \underbrace{(k-1+q_{i,c})}_{q_{t,c}}\, c + \underbrace{r_{i,c}}_{r_{t,c}}.$$

It follows from the code construction that the value of x_i depends on $r_{i,c}$; specifically, two cases are possible:

Case 1: If $r_{i,c} \leq r_{d,c}$, then $x_i = \frac{1}{q_{d,c}+1}$. Since $i \in \{1, \ldots, d\}$, this condition corresponds to the case where $q_{i,c} \in \{0, \ldots, q_{d,c}\}$ and $r_{i,c} \in \{1, \ldots, r_{d,c}\}$. Therefore, message k is allocated $\frac{1}{q_{d,c}+1}$ amount of link capacity per time step for a total of $(q_{d,c}+1)r_{d,c}$ time steps in the coding window W_k.

Case 2: If $r_{i,c} > r_{d,c}$, then $x_i = \frac{1}{q_{d,c}}$. Since $i \in \{1, \ldots, d\}$, this condition corresponds to the case where $q_{i,c} \in \{0, \ldots, q_{d,c}-1\}$ and $r_{i,c} \in \{r_{d,c}+1, \ldots, c\}$. Therefore, message k is allocated $\frac{1}{q_{d,c}}$ amount of link capacity per time step for a total of $q_{d,c}(c - r_{d,c})$ time steps in the coding window W_k.

Observe that $\mathbf{y}$ is simply a vector containing the elements of $\{x_i\}_{i=1}^{d}$ sorted in ascending order. Since

$$\sum_{i \in U} x_i \geq \sum_{j=1}^{|U|} y_j \quad \forall\, U \subseteq \{1, \ldots, d\},$$

it follows that over any ℓ time steps in the coding window W_k, the total amount of link capacity allocated to message k is at least $\sum_{j=1}^{\ell} y_j$. Therefore, as long as the message size s does not exceed $\sum_{j=1}^{\ell} y_j$, message k can always be decoded from the data at any ℓ time steps in W_k. ∎

Proof of Theorem 1: Consider the code constructed in Section III for a given choice of c and d. According to Lemma 1, if message size s satisfies the inequality

$$s \leq \sum_{j=1}^{d-z} y_j,$$

then any message $k \in \{1, \ldots, n\}$ can be decoded from the data at any $d-z$ time steps in its coding window W_k. Therefore, the code achieves a message size of $\sum_{j=1}^{d-z} y_j$, by allowing all n messages $\{1, \ldots, n\}$ to be decoded as long as there are z or fewer erasures in each coding window W_k, or equivalently, under any erasure pattern $E \in \mathcal{E}_n^{\text{CW}}$. We will proceed to show that this message size matches the maximum achievable message size s_n^{CW} in the limit, i.e.,

$$\lim_{n \to \infty} s_n^{\text{CW}} = \sum_{j=1}^{d-z} y_j. \tag{1}$$

To obtain an upper bound for s_n^{CW}, we consider the cut-set bound corresponding to a specific erasure pattern E' from $\mathcal{E}_n^{\text{CW}}$. Let $\{1, \ldots, d\}$ be partitioned into two sets $V^{(1)}$ and $V^{(2)}$, where

$$V^{(1)} \triangleq \{i \in \{1, \ldots, d\} : r_{i,c} \leq r_{d,c}\},$$
$$V^{(2)} \triangleq \{i \in \{1, \ldots, d\} : r_{i,c} > r_{d,c}\}.$$

Let $\mathbf{v} = (v_1, \ldots, v_d)$ be defined as $\mathbf{v} \triangleq (\mathbf{v}^{(1)} \mid \mathbf{v}^{(2)})$, where $\mathbf{v}^{(1)}$ is the vector containing the $(q_{d,c}+1)r_{d,c}$ elements of $V^{(1)}$ sorted in ascending order, and $\mathbf{v}^{(2)}$ is the vector containing the $q_{d,c}(c - r_{d,c})$ elements of $V^{(2)}$ sorted in ascending order. Define the erasure pattern $E' \subseteq T_n$ as follows:

$$E' \triangleq \bigcup_{j=d-z+1}^{d} T_n^{(v_j)},$$

where $T_n^{(i)}$ is as defined in Lemma 2. Since the code allocates a positive amount of link capacity to each message $k \in \{1, \ldots, n\}$ only within its coding window W_k, it follows from Property (ii) of Lemma 2 that for each set $T_n^{(i)}$, we have

$$\left|T_n^{(i)} \cap W_k\right| = 1 \quad \forall\, k \in \{1, \ldots, n\},$$

which implies that

$$\left|E' \cap W_k\right| = z \quad \forall\, k \in \{1, \ldots, n\}$$

because $T_n^{(1)}, \ldots, T_n^{(d)}$ are disjoint sets. Therefore, E' is an admissible erasure pattern, i.e., $E' \in \mathcal{E}_n^{\text{CW}}$.

Now, consider a code that achieves the maximum message size s_n^{CW}. Such a code must allow all n messages $\{1, \ldots, n\}$ to be decoded under the specific erasure pattern E'. We therefore have the following cut-set bound for s_n^{CW}:

$$n\, s_n^{\text{CW}} \leq \left|T_n \backslash E'\right| = \sum_{j=1}^{d-z} \left|T_n^{(v_j)}\right| \implies s_n^{\text{CW}} \leq \frac{1}{n} \sum_{j=1}^{d-z} \left|T_n^{(v_j)}\right|.$$

Applying the upper bounds in Property (iii) of Lemma 2, and writing the resulting expression in terms of y_j produces

$$s_n^{\text{CW}} \leq \frac{1}{n} \sum_{j=1}^{d-z} \left|T_n^{(v_j)}\right| \leq \frac{1}{n} \sum_{j=1}^{d-z} (n\, y_j + 2).$$

Since a message size of $\sum_{j=1}^{d-z} y_j$ is known to be achievable (by the code constructed in Section III), we have the following upper and lower bounds for s_n^{CW}:

$$\sum_{j=1}^{d-z} y_j \leq s_n^{\text{CW}} \leq \frac{1}{n} \sum_{j=1}^{d-z} (n\, y_j + 2).$$

These turn out to be matching bounds in the limit as $n \to \infty$:

$$\sum_{j=1}^{d-z} y_j \leq \lim_{n \to \infty} s_n^{\text{CW}} \leq \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{d-z} (n\, y_j + 2) = \sum_{j=1}^{d-z} y_j.$$

We therefore have equation (1) as required. ∎

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