Comparison of Network Coding and Non-Network Coding Schemes for Multi-hop Wireless Networks

Jia-Qi Jin, Tracey Ho
California Institute of Technology
Pasadena, CA
Email: {jin,tho}@caltech.edu

Harish Viswanathan
Lucent Technologies, Bell Labs
Murray Hill, NJ
Email: harishv@lucent.com

***Abstract*— Network coding has been shown to be useful for throughput and reliability in various network topologies, under a fixed-rate, point-to-multipoint wireless network model. We study the effect of introducing a wireless network model where link capacity depends on the network geometry and the signal to interference and noise ratio. In particular, we compare strategies with and without network coding on a multicast network with and without fading, and on single-user multiple path networks with fading. For the multicast network without fading, we find that the network geometry affects which scheme attains higher throughput. For the case with fading, we compare the throughput-outage probability curves achieved by network coding and repetition schemes. For the multiple path networks, we further consider the case where multiple simultaneous transmissions of identical information signals can be combined at a receiver. We find that the relative performance of the schemes we consider depends on the network geometry, the ratio of signal to noise power, whether multiple simultaneous transmissions can be combined, and the operating point on the throughput-outage probability curve.**

I. Introduction

In this paper we examine the effect of network coding on throughput and outage probability in multi-hop wireless networks. Network coding has been shown to offer exciting advantages in throughput [1], reliability [2] and distributed operation [3] for wired networks. These advantages have been translated into the wireless setting using simple fixed-rate, point-to-multipoint wireless link models, e.g. [4], [5], which generalize the wired model by including wireless multicast and interference-based restrictions on which subsets of links can be simultaneously activated.

For more complex wireless models, it is not obvious when and to what extent such network coding advantages still apply. For instance, in a delay-constrained fading environment, a particular strategy gives a throughput-outage probability curve when different transmission rates are considered, whereas with a fixed transmission rate it gives a single point on the curve. In multiple path topologies, for a fixed link transmission rate, coding information across different paths gives a higher throughput but also higher outage probability compared to repeating the same information on each path. On the other hand, when the link transmission rate is not fixed, it is less clear how the throughput-outage curves of different coding strategies compare with repetition. Are some strategies always better than others, or if not, what affects their relative performance?

These are among the questions are examined in this paper, which represents a first step towards understanding the wireless models and scenarios under which coding is useful in terms of throughput/reliability. More specifically, we take into account the dependence of link rates and outage probability on factors such as trasmission power and interference, and consider both the case of point-to-multipoint transmissions as well as the case where multiple transmissions of the same information can be combined at a node.

For various wireless network configurations, we show how the usefulness of coding depends on factors such as the network geometry, the signal-to-noise ratio, whether multiple transmissions can be combined, and the operating point on the throughput-outage probability curve.

II. Wireless Network Model

We consider a wireless network with a set $\mathcal{V}$ of nodes. Each node can transmit and receive wireless signals, but not simultaneously, i.e. we consider half-duplex operation. The distance between two nodes $v_1, v_2 \in \mathcal{V}$ is denoted $d_{v_1v_2}$. We assume each node transmits at the same power P and has the same noise power σ^2. In this paper, we consider transmission schemes where each link transmits at a common (network dependent) rate R.

Our wireless network model is similar to that in [6], but with the added consideration of interference and physical layer combining of identical information signals. In particular, we consider two types of wireless links: the first involves a single transmitter; the second involves multiple transmitters sending identical information signals. In both cases, all other simultaneous transmissions are treated as noise at each receiver. We do not consider joint design of physical layer code books across nodes or multiuser detection of different signals, which requires more complex receivers.

In the remainder of this paper, the term "coding" refers to network coding, not to be confused with the physical layer coding on each link.

A. Single transmitter link model

Consider the simple four-node configuration in Fig 1(a), where two different signals are transmitted from A to B and from C to D, respectively, through the zero mean additive

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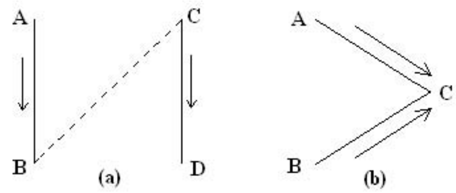

Fig. 1. Link Model with (a) Interference and (b) Physical Layer Combining

white Gaussian noise (AWGN) channels. Following [6], the capacity for link AB is taken as:

$$C_{AB} = \log\left(1 + \frac{\frac{|f_{AB}|^2 P}{d_{AB}^k}}{\sigma^2 + \frac{|f_{CB}|^2 P}{d_{CB}^k}}\right) \tag{1}$$

where k is the propagation power loss exponent. Fadings are modeled as independent identically-distributed Rayleigh random variables with $E[|f_{AB}|^2] = E[|f_{CB}|^2] = \mu$ and cumulative density function (CDF)

$$F_{|f|^2}(x) = 1 - \exp\left(-\frac{x}{\mu}\right) \qquad (x > 0) \tag{2}$$

If R is the transmission rate, for channel AB:

$$\begin{aligned} &\Pr(Outage_{AB}) \\ = \quad &\Pr\left\{\log\left(1 + \frac{\frac{|f_{AB}|^2 P}{d_{AB}^k}}{\sigma^2 + \frac{|f_{CB}|^2 P}{d_{CB}^k}}\right) < R\right\} \end{aligned} \tag{3}$$

In this paper, we assume μ and k are the same for all links. We define $\text{snr} = \frac{\mu P}{\sigma^2}$ and reliability $Rel_{AB} = 1 - \Pr(Outage_{AB})$. Integrating Equation 3 over the fading parameters, we get:

$$Rel_{AB} = \exp\left(-\frac{d_{AB}^k\left(2^R - 1\right)}{\text{snr}}\right)\frac{d_{CB}^k}{d_{CB}^k + d_{AB}^k\left(2^R - 1\right)} \tag{4}$$

Similarly, if there is more than one interfering transmitter, the link reliability can be derived as:

$$Rel = \exp\left(-\frac{l^k\left(2^R - 1\right)}{\text{snr}}\right)\prod_{i=1}^{\gamma}\frac{\tilde{l}_i^k}{\tilde{l}_i^k + l^k\left(2^R - 1\right)} \tag{5}$$

where l is the distance between the communicating nodes, $\tilde{l}_i$ is the distance between the receiver and the ith interfering transmitter, and γ is the number of interfering transmitters.

B. Multi transmitter link model

Now we consider physical layer combing of multiple identical signals. In Fig 1(b), two identical signals are transmitted from both A and B to C through AWGN channels. We approximate the link capacity by the matched filter bound:

$$C_{comb} = \log\left(1 + \left(\frac{|f_{AC}|^2}{d_{AC}^k} + \frac{|f_{BC}|^2}{d_{BC}^k}\right)\frac{P}{\sigma_\eta^2}\right) \tag{6}$$

Similarly as above, we can obtain:

1. If $d_{AC} = d_{BC} = d$, then

$$Rel = \exp\left(-\frac{\left(2^R - 1\right)d^k}{\text{snr}}\right)\left(1 + \frac{\left(2^R - 1\right)d^k}{\text{snr}}\right) \tag{7}$$

2. If $d_{AC} \neq d_{BC}$, then

$$\begin{aligned} Rel = &\frac{d_{AC}^k}{d_{AC}^k - d_{BC}^k}\exp\left(-\frac{\left(2^R - 1\right)d_{BC}^k}{\text{snr}}\right) \\ &+ \frac{d_{BC}^k}{d_{BC}^k - d_{AC}^k}\exp\left(-\frac{\left(2^R - 1\right)d_{AC}^k}{\text{snr}}\right) \end{aligned} \tag{8}$$

which reduces to Equation 7 when $d_{AC} \to d_{BC}$.

III. Comparison for Coding and Non-Coding Schemes

Using this network model, we revisit three scenarios in which coding has previously been shown to be useful under a simpler point-to-multipoint fixed rate wireless network model.

A. Multicast Network

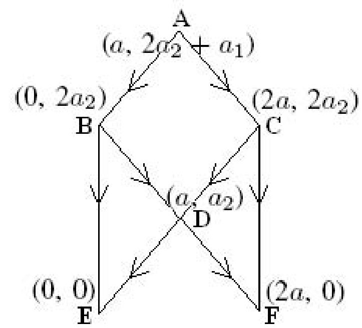

Fig. 2. Comparison for schemes with variable schedulings under Butterfly Structure by half-duplex constraint

We first consider a multicast network shown in Fig 2, where node A multicasts the same information to nodes E and F. We compare four schemes:

- scheme 1 (non-coding, interference): In time slot 1, node A broadcasts to B and C. In slot 2, data is transmitted on links BE and CF. The throughput is $R/2$, where R is the transmission rate on each link.
- scheme 2 (non-coding, no interference): In slot 1, node A broadcasts to B and C. Data is transmitted on link BE in slot 2 and on link CF in slot 3. The throughput is $R/3$.
- scheme 3 (coding, interference): In slot 1 node A transmits to B, and node C broadcasts to D and F; in slot 2 node A transmits to C, and B broadcasts to D and E; in slot 3 data is transmitted on links DE and DF. The throughput is $2R/3$.
- scheme 4 (coding, no interference): Data is transmitted on link AB in slot 1, and on link AC in slot 2. In slot 3, node B broadcasts to D and E; in slot 4, node C broadcasts to D and F. In slot 5, node D broadcasts to E and F. The throughput is $2R/5$.

1) No fading case: Suppose $|f_l|^2 = \mu$ for all links l. For each scheme, we calculate the capacity of each link using Equation 1 and take the minimum over links as the common transmission rate R. As shown in Fig 3, the multicast throughputs achieved by the four schemes depends on the network geometry, which is parameterized by the distances a, a_1, a_2 defined in Figure 2(b), as well as the snr.

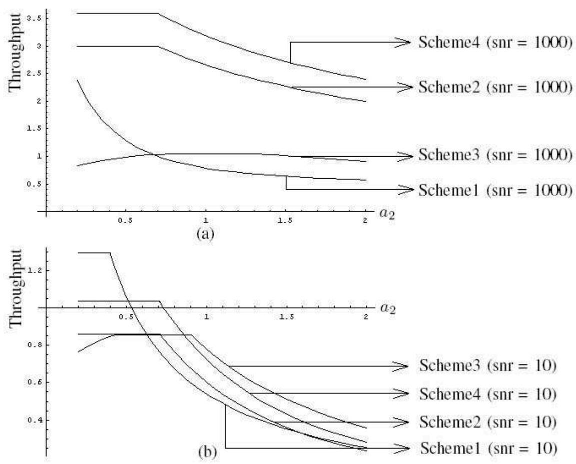

Fig. 3. The influence of network geometry and snr on the multicast throughput attained by non-coding and coding schemes in the no fading case. Here $a = a_1 = 1, k = 2$.

2) Delay-constrained fading case: Here we assume $|f_l|^2$ is distributed as in Equation 2 for all links l. Using Equation 5, we calculate the reliability, i.e. probability that node E and F receive information successfully.

The reliability-throughput tradeoff is shown in Fig 4. Fig 5 shows that for a fixed throughput, changing the network geometry changes the relative performance of the four schemes.

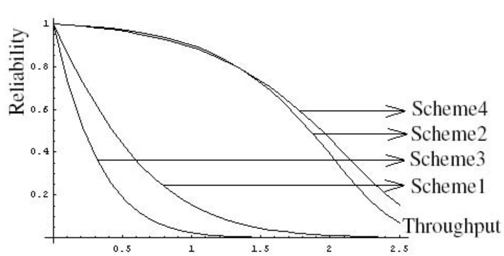

Fig. 4. Reliability-throughput tradeoff for non-coding and coding schemes. Here $a = 1,\ k = 2,$ snr = 800 (≈29 dB).

B. Multiple path Network

Next we consider a multiple path wireless network illustrated by Fig 6. We compare three schemes:

- coding scheme with interference, illustrated in Fig 6(a), where independent information transmitted on 2 paths and the binary sum of corresponding data bits is transmitted on the third.

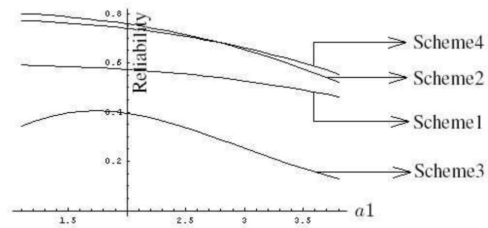

Fig. 5. The effect on reliability of modifying the geometry of the network of Fig 2. Here throughput = 0.4, $a = 1.6,\ k = 3.5$, snr = 800 (≈ 29dB).

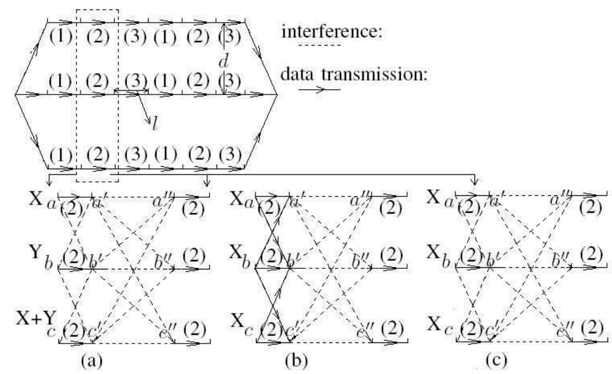

Fig. 6. Illustration of multiple path wireless network for (a): coding scheme, for (b): repetition with combining, and for (c): repetition without combining. Here p=3. Links marked (i), $(i = 1, 2, 3)$ are active on i th slot (channel). For computational tractability, when calculating the outage probabilities of transmissions within the dashed box, we only integrate over the fading parameters of the channels within the box, and approximate the interference power due to transmitters outside the box by their mean values.

- repetition with combining, illustrated in Fig 6(b), where the same information is transmitted on each path, with physical layer combining of simultaneous transmissions of the same information. Simultaneous transmissions of different information are treated as noise.
- repetition without combining, illustrated in Fig 6(c), where the same information is transmitted on each path. Simultaneous transmissions are treated as interference

Since the interference at the start and the end where the paths meet depends on the exact geometry and channel assignment, we focus on characterizing the middle section. We define the following variables: p is the number of timeslots (channels) used; l_1 is the length (number of links) of each of the three path segments; d is the distance between neighbor paths. The throughput of the coding scheme is $\frac{2R}{p}$ and that of the repetition schemes is $\frac{R}{p}$.

For finite d (the distance between neighbor paths), the results are shown in Fig 7. We observe that the relative usefulness of coding and repetition depends on the operating point on the throughput-reliability curve: coding becomes relatively advantageous as throughput increases. Furthermore, increasing d and snr increases the range of reliability over which the coding scheme offers higher throughput. This is

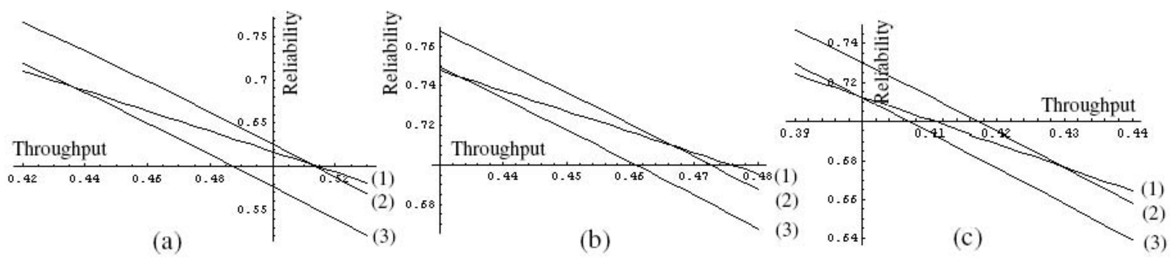

Fig. 7. Reliability-throughput tradeoff for the multiple path network of Fig 6. Here (1) is for coding, (2) is for repetition with combining and (3) is for repetition without combining. $l = 1,\ k = 2.5$, for (a): $d = 5.5, \text{snr} = 800$; for (b) $d = 8, \text{snr} = 800$; for (c): $d = 8, \text{snr} = 40$.

because when d goes up, both the effects of physical layer combining and interference decrease, making coding more advantageous. When snr goes up, the reliability of both coding and repetition increases, shifting all the curves upwards.

As d increases to ∞, the transmission schemes reduce to the multiple path wireless network with interference only within a path. For this case, we consider the coding scheme and the repetition scheme with combining under different values of p, snr and l_1, showing our results in Figures 8 and 9. As before, we observe that coding is useful in high-throughput region, and can show the following asymptotic result:

Proposition 1: For the multiple path network with $d = \infty$, when throughput$\to$ 0, $Rel_{coding},\ Rel_{repetition} \to 1$ and $Rel_{repetition} > Rel_{coding}$;
when throughput$\to \infty$, $Rel_{coding},\ Rel_{repetition} \to 0$ and $Rel_{coding} > Rel_{repetition}$.

Proof: From Equations 5, we have:

1. For repetition scheme:

$$\Pr_{succ}(Tp) = \exp\left(-\alpha\left(2^{2\beta \cdot Tp} - 1\right)\right) \prod_{i=1}^{\gamma} \frac{\tilde{l}_i^k}{\tilde{l}_i^k + l_i^k\left(2^{2\beta \cdot Tp} - 1\right)}$$

$$Rel = {\Pr_{succ}}^3 + 3{\Pr_{succ}}^2\left(1 - \Pr_{succ}\right) + 3\Pr_{succ}\left(1 - \Pr_{succ}\right)^2$$

2. For coding scheme:

$$\tilde{\Pr}_{succ}(Tp) = \exp\left(-\alpha\left(2^{\beta \cdot Tp} - 1\right)\right) \prod_{i=1}^{\gamma} \frac{\tilde{l}_i^k}{\tilde{l}_i^k + l_i^k\left(2^{\beta \cdot Tp} - 1\right)}$$

$$\tilde{Rel} = \tilde{\Pr}_{succ}^3 + 3\tilde{\Pr}_{succ}^2\left(1 - \tilde{\Pr}_{succ}\right)$$

where Tp denotes throughput, $\alpha,\ \beta$ are certain positive coefficients. $Rel,\ \tilde{Rel}$ are the network reliability, $\Pr_{succ},\ \tilde{\Pr}_{succ}$ are the success probabilities for each path.

When $Tp \to 0$, using $\exp(x) \approx x + 1$; $2^x - 1 \approx (ln2)\,x$ as $x \to 0$, we get:

$$\Pr_{succ}(Tp) \to 1 - 2\lambda \cdot Tp;\ \tilde{\Pr}_{succ}(Tp) \to 1 - \lambda \cdot Tp$$
$$Rel \to 1 - (2\lambda \cdot Tp)^3;\ \tilde{Rel} \to 1 - 3(\lambda \cdot Tp)^2$$

where λ is a certain coefficient.

So $Rel,\ \tilde{Rel} \to 1$, and since $3(\lambda \cdot Tp)^2 > (2\lambda \cdot Tp)^3$ for sufficiently small Tp, $Rel > \tilde{Rel}$.

When $Tp \to \infty$, it is easy to note that $Rel,\ \tilde{Rel} \to 0$, and we find:

$$\frac{Rel}{\tilde{Rel}} \approx \mu \frac{\exp\left(-\alpha \cdot 2^{\beta \cdot Tp}\right)\frac{1}{2^{\beta \cdot Tp}}}{\left(\exp\left(-\alpha \cdot 2^{\frac{\beta \cdot Tp}{2}}\right)\frac{1}{2^{\frac{\beta \cdot Tp}{2}}}\right)^2} \approx \mu \exp\left(-\alpha \cdot 2^{\beta \cdot Tp}\right) \to 0$$

where μ is a certain positive coefficient, so $Rel < \tilde{Rel}$. ■

Also, the range of reliability over which coding is advantageous increases as p and snr increase (see Fig 9(a) (b)), or as l_1 decreases (see Fig 9(c)), because reliability is generally improved.

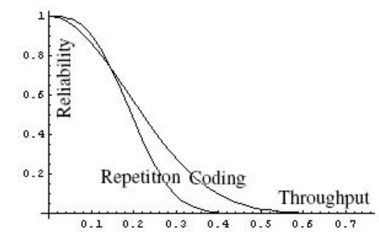

Fig. 8. Reliability-throughput tradeoff for multiple path wireless network where $d = \infty$. Here $l_1 = 64,\ p = 8,\ l = 1,\ k = 3,\ \text{snr} = 800$.

C. Generalized Multiple path Wireless Network

Finally, we consider a wireless network with multiple groups of paths, where paths within a group are physically close together while paths in different groups are far apart. The results of the previous section show that it is advantageous to send the same information on the closely-spaced paths within a group to take advantage of physical layer combining. Here we investigate the benefit of coding across different groups of paths.

Analogously to the previous section, we consider two transmission schemes for a simple network consisting of three widely-spaced groups of paths, illustrated in Fig 10, with negligible interference across groups:

- a coding scheme where independent information is transmitted on 2 groups and the binary sum of corresponding data bits is transmitted on the third

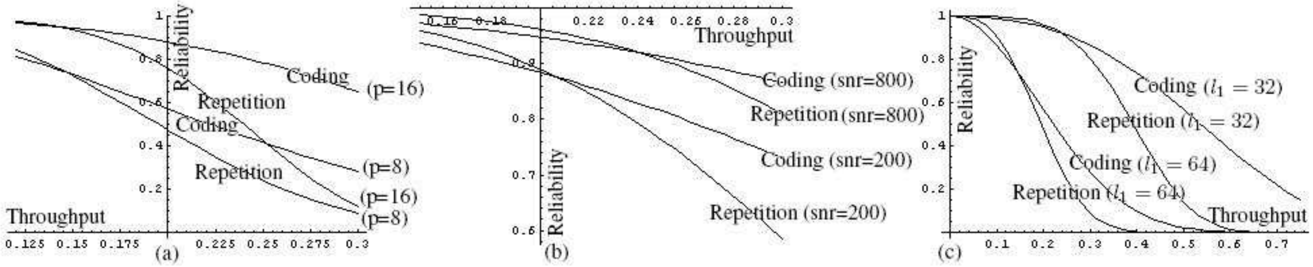

Fig. 9. Influence of network geometry (p, l_1) and SNR on the reliability-throughput tradeoff. Here $l = 1,\ k = 3,\ d = \infty$ and (a): $l_1 = 64,\ \text{snr} = 800$ (b): $p = 8,\ l_1 = 32$ (c): $p = 8,\ \text{snr} = 800$

- a repetition scheme where the same information is transmitted on each path.

By similar assumptions and calculations as for the previous section, we focus on the middle section and obtain the results in Fig 11. We observe that similar to the multiple path network of the previous section, for this network the coding scheme is advantageous for higher throughput values. However, compared to the multiple path network, for this network there is a larger range of reliability values over which the coding scheme gives higher throughput than the repetition scheme.

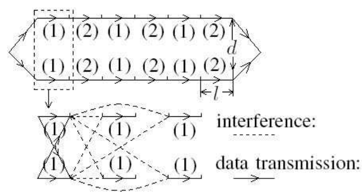

Fig. 10. Illustration of a group of paths in a generalized multiple path wireless network consisting of three such groups of paths. The groups are spaced widely enough that interference between groups can be neglected. Links marked (i), $(i = 1, 2)$ are active on the ith slot (channel).

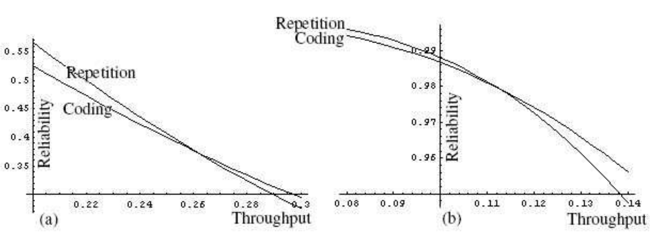

Fig. 11. Comparison of coding and repetition for (a) the multiple path network of Fig 6 and (b) generalized multiple path wireless network of Fig 10. Here $\text{snr} = 800,\ d = 8,\ k = 2.5,\ l = 1,\ p = 2,\ l_1 = 6$.

IV. Conclusion

We have investigated the benefits of network coding for throughput and reliability using a wireless network model where link capacity depends on the network geometry and the signal to interference and noise ratio. In particular, we compare strategies with and without network coding on three networks: a multicast network with and without fading, and two multiple path networks with fading. For the multicast network without fading, we find that the network geometry and snr affect which schemes attain higher throughput. For the case with fading, we compare the throughput-outage probability curves. We see that the relative performance of the schemes we consider depend on the network geometry, the ratio of signal to noise power, whether multiple simultaneous transmissions can be combined, and the operating point on the throughput-outage probability curve. Our results suggest that for the kinds of transmission schemes we consider, where each link transmits at a common network dependent rate R, schemes with higher overall throughput relative to R are generally more useful in the higher throughput/lower reliability range, and less useful in the lower throughput/higher reliability range. Further work remains to characterize these relationships more precisely. In any case, network coding enlarges the set of possible transmission schemes and corresponding throughput-reliability trade-offs, and it is of interest to develop techniques for predicting the usefulness of different network coding schemes for given network scenarios.

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