Network Coding from a Network Flow Perspective

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Abstract — **We make precise connections between algebraic network coding and network flows. Our combinatorial formulations offer new insights, mathematical simplicity, and lead to a substantially tighter upper bound on the coding field size required for a given connection problem than that in [5].**

We build on the algebraic network coding framework in [5], giving alternative interpretations in terms of network flows.

Following [5], we model a delay-free network as an acyclic graph with ν unit capacity directed links, and one or more source nodes, at which $r \geq 1$ discrete, independent, unit entropy rate random processes are observable.[1] In a linear network code, the signal $Y(j)$ on a link j is a linear combination of processes X_i generated at node $v = \text{tail}(j)$ and signals $Y(l)$ on incident incoming links l:

$$Y(j) = \sum_{\{i\,:\,X_i \text{ generated at } v\}} a_{i,j} X_i + \sum_{\{l\,:\,\text{head}(l)=v\}} f_{l,j} Y(l)$$

and an output process $Z(\beta, i)$ at receiver node β is a linear combination of signals on its terminal links:

$$Z(\beta, i) = \sum_{\{l\,:\,\text{head}(l)=\beta\}} b_{\beta_{i,l}} Y(l)$$

The coefficients $\{a_{i,j}, f_{l,j}, b_{\beta_{i,l}} \in \mathbb{F}_{2^u}\}$ can be collected into $r \times \nu$ matrices $A = (a_{i,j})$ and $B_\beta = (b_{\beta_{i,j}})$, and the $\nu \times \nu$ matrix $F = (f_{l,j})$, whose structure is constrained by the network.

Reference [5] gives the following necessary and sufficient condition for a multicast connection problem to be feasible (or for a particular set of network coefficients to be a valid solution): that for each receiver β, the transfer matrix $A(I-F)^{-1}B_\beta^T$ has nonzero determinant. This condition is equivalent to the max-flow min-cut condition of [1].

We present two alternative formulations related to network flows, which are easier to work with mathematically as they do not involve matrix products and inversions. These lead to new results on randomized distributed transmission and compression of information in networks, presented in our companion paper [3], and an upper bound on required coding field size that substantially tightens the bound given in [5].

We denote a path $\mathcal{E}$ traversing links $\{e_1, \ldots, e_k\}$ by the ordered set $\{e_1, \ldots, e_k\}$, where $e_1 < \ldots < e_k$, and define the product of gains along the path as

$$g(\mathcal{E}) = \begin{cases} f_{e_1,e_2} f_{e_2,e_3} \cdots f_{e_{k-1},e_k} & \text{if } k > 1 \\ 1 & \text{if } k = 1 \end{cases}$$

Theorem 1 *A multicast connection problem is feasible (or a particular (A, F) can be part of a valid solution) if and only if each receiver β has a set $\mathcal{H}_\beta$ of r incident incoming links for which*

$$P_{\mathcal{H}_\beta} = \sum_{\substack{\{\text{disjoint paths } \mathcal{E}_1, \ldots, \mathcal{E}_r\,: \\ \mathcal{E}_i \text{ from outgoing link} \\ l_i \text{ of source } i \text{ to } h_i \in \mathcal{H}_\beta\}}} \left|A_{\{l_1,\ldots,l_r\}}\right| \prod_{j=1}^{r} g(\mathcal{E}_j) \neq 0$$

where $A_{\{l_1,\ldots,l_r\}}$ is the submatrix of A consisting of columns corresponding to links $\{l_1, \ldots, l_r\}$. The sum is over all flows that transmit all source processes to links in $\mathcal{H}_\beta$, each flow being a set of r disjoint paths each connecting a different source to a different link in $\mathcal{H}_\beta$. □

From this we obtain the result below on the equivalence of the transfer matrix formulation of [5] and the Edmonds matrix formulation for checking if a bipartite graph has a perfect matching, which is a classical reduction of the problem of checking the feasibility of an $s-t$ flow [4]. The equivalence in $\mathbb{F}_2$ and without coding is not so surprising given the similarity of the two network problems, but it is not obvious that the two formulations would be equivalent in higher order fields and with coding.

Theorem 2 *The determinant of the transfer matrix $M_1 = A(I-F)^{-1}B_\beta^T$ for receiver β in a network code can be calculated as*

$$|M_1| = (-1)^{r(\nu+1)} |M_2|$$

where $M_2 = \begin{bmatrix} A & 0 \\ I-F & B_\beta^T \end{bmatrix}$ is the corresponding Edmonds matrix with appropriately chosen coefficients. □

These results lead easily to the following bound on the size of the finite field needed for coding, which determines the complexity of a linear multicast code.

Theorem 3 *For a feasible multicast connection problem with d receivers, there exists a solution in a finite field $\mathbb{F}_q$ where $q > d$, or in a finite field $\mathbb{F}_{2^u}$ where $u \leq \lceil \log_2(d+1) \rceil$.* □

This result tightens the upper bound of $u \leq \lceil \log_2(rd+1) \rceil$ given in [5], where r is the number of processes being transmitted in the network. References [6], [2] independently and simultaneously derive a similar bound.

References

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[1]Our model admits parallel links and multiple sources at one node.